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MULTIBASIC AND MIXED GOSPER'S ALGORITHM

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Abstract

Gosper's summation algorithm finds a hypergeometric closed form of an indefinite sum of hypergeometric terms, if such a closed form exists. We generalize his algorithm to the case when the terms are simultaneously hypergeometric and multibasic q-hypergeometric. We also provide algorithms for computing hypergeometric canonical forms of rational functions and for finding polynomial solutions of recurrences in the multibasic and mixed case.

1 Introduction and notation

Let $\mathbb{F}$ be a field of characteristic zero and $\langle t_n \rangle_{n=0}^{\infty}$ a sequence of elements from $\mathbb{F}$ which is eventually non-zero. Call t_n:

- *hypergeometric*, if there are polynomials $p_1, p_2 \in \mathbb{F}[x]$ such that $p_1(n)t_{n+1} = p_2(n)t_n$ for all n;
- *q-hypergeometric* or *basic hypergeometric*, if there are polynomials $p_1, p_2 \in \mathbb{F}[x]$ such that $p_1(q^n)t_{n+1} = p_2(q^n)t_n$ for all n, where $q \in \mathbb{F}$ is a constant called the *base*;
- *multibasic hypergeometric*, if there are polynomials $p_1, p_2 \in \mathbb{F}[y_1, \ldots, y_m]$ such that $p_1(q_1^n, \ldots, q_m^n)t_{n+1} = p_2(q_1^n, \ldots, q_m^n)t_n$ for all n, where $q_1, \ldots, q_m \in \mathbb{F}$ are constants called the *bases*;
- *multibasic and mixed hypergeometric* (*mmHS*, for short), if there are polynomials $p_1, p_2 \in \mathbb{F}[x, y_1, \ldots, y_m]$ such that $p_1(n, q_1^n, \ldots, q_m^n)t_{n+1} = p_2(n, q_1^n, \ldots, q_m^n)t_n$ for all n.

The celebrated *Gosper's algorithm* [7, 8] finds hypergeometric solutions f_n of the inhomogeneous first-order recurrence

$$f_{n+1} - f_n = t_n$$

where t_n is a given hypergeometric sequence. Besides its obvious use for indefinite hypergeometric summation, it also plays a crucial role in the algorithms for definite hypergeometric summation, construction of annihilating recurrences, and automated verification of identities [17, 18]. Therefore it is not surprising that analogous algorithms have been designed for many other settings, e.g., integration of hyperexponential functions [4], basic [16, 15] and bibasic hypergeometric summation [13].

We present in Section 6 an analogue of Gosper's algorithm for the multibasic and mixed hypergeometric case. Our algorithm `m&m-Gosper`[1] is a common generalization of algorithms presented in [8, 16, 13]. Sections 2 and 3 give the required algebraic and algorithmic preliminaries, while in Section 5 we develop the multibasic and mixed hypergeometric canonical form of rational functions. Although in Gosper's

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[1]available as `GosperSum` in *Mathematica* package `gosper.m` at `http://www.cis.upenn.edu/~wilf/AeqB.html`

algorithm only first-order recurrences are checked for polynomial solutions, we provide in Section 4 algorithm m&m-Poly[2] which finds all polynomial solutions of inhomogeneous, parametric, multibasic and mixed recurrences with polynomial coefficients.

The set of integers is denoted by $\mathbb{Z}$, the set of nonnegative integers by $\mathbb{N}_0$, and the field of rational numbers by $\mathbb{Q}$.

If $n, m \in \mathbb{N}_0$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ are m-tuples of elements of a ring, we write $\mathbf{ab}$ for the componentwise product $(a_1b_1, a_2b_2, \ldots, a_mb_m)$, and $\mathbf{a}^n$ for the componentwise power $(a_1^n, a_2^n, \ldots, a_m^n)$. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m$ then we write $\mathbf{a}^{\boldsymbol{\alpha}}$ for the power product $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m}$.

We say that two multivariate polynomials over a field are *coprime* if they do not have a common non-constant factor. When a and b are coprime, we write $a \perp b$. When $\mathcal{S}$ is a set of polynomials and $a \perp b$ for all $b \in \mathcal{S}$, we write $a \perp \mathcal{S}$.

2 Algebraic preliminaries

Let $\mathbb{F}$ be a field of characteristic zero. Let $q_1, \ldots, q_m \in \mathbb{F} \setminus \{0\}$, and suppose that for any integers $k_1, \ldots, k_m \in \mathbb{Z}$

$$q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m} = 1 \implies k_1 = k_2 = \cdots = k_m = 0. \tag{1}$$

This seems to be the right generalization of the condition that q is not a root of unity in the q-hypergeometric case (see [3]). For example, if $\mathbb{F} = \mathbb{R}$, $q_1 = \sqrt[3]{2}$ and $q_2 = \sqrt[5]{2}$, then $q_1^3 q_2^{-5} = 1$, and we should have chosen $q = \sqrt[15]{2}$ in the first place. We denote $\mathbf{q} = (q_1, \ldots, q_m)$.

Let $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ be an m-tuple of variables, $\mathbb{F}[x, \mathbf{y}]$ the ring of polynomials over $\mathbb{F}$ in x and $\mathbf{y}$, and $\mathbb{F}(x, \mathbf{y})$ the corresponding rational function field. We define a difference operator $\mathbf{E}$ on $\mathbb{F}(x, \mathbf{y})$ by stipulating that $\mathbf{E}$ be fixed on $\mathbb{F}$, that $\mathbf{E}x = x + 1$, and that $\mathbf{E}y_k = q_k y_k$ for $k = 1, \ldots, m$. Note that $\mathbf{E}$ is inversive, that $\mathbb{F}(x, \mathbf{y})$ is a difference field, and that $\mathbb{F}[x, \mathbf{y}]$ is a difference subring of $\mathbb{F}(x, \mathbf{y})$ (see [6] for terminology).

Let $\mathcal{M}$ be the set of power products in $y_1, y_2, \ldots, y_m$:

$$\mathcal{M} = \{y_1^{k_1} y_2^{k_2} \cdots y_m^{k_m} \mid k_i \in \mathbb{N}_0 \text{ for } i = 1, \ldots, m\}.$$

If $u = y_1^{k_1} y_2^{k_2} \cdots y_m^{k_m} \in \mathcal{M}$, we write $u(\mathbf{q})$ for the corresponding power product of the bases $q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m}$. Note that $\mathbf{E}u = u(\mathbf{q})u$ for all $u \in \mathcal{M}$.

As a multiplicative monoid, $\mathcal{M}$ is obviously isomorphic to $\mathbb{N}_0^m$, the direct product of m copies of the additive monoid $\mathbb{N}_0$. We denote by $\preceq$ an *admissible term order* in $\mathbb{N}_0^m$, which is a total order satisfying

1. $\mathbf{0} \preceq \boldsymbol{\alpha}$,
2. $\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \Rightarrow \boldsymbol{\alpha} + \boldsymbol{\gamma} \preceq \boldsymbol{\beta} + \boldsymbol{\gamma}$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}_0^m$. An example of an admissible term order is the *lexicographic order* $\preceq_{\text{lex}}$, with $\boldsymbol{\alpha} \prec_{\text{lex}} \boldsymbol{\beta}$ when $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ and $\alpha_k < \beta_k$ where $k = \min\{i;\ \alpha_i \neq \beta_i\}$.

Definition 2.1 *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^m$. Then we write*

$$\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$$

whenever $\alpha_i \leq \beta_i$ for all i between 1 and m.

Clearly, $(\mathbb{N}_0^m, \subseteq)$ is a partial order isomorphic to $(\mathcal{M}, |)$ where $|$ denotes divisibility of power products, and is contained in any admissible term order:

$$\boldsymbol{\alpha} \subseteq \boldsymbol{\beta} \Rightarrow \boldsymbol{\alpha} \preceq \boldsymbol{\beta}, \quad \text{for all } \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^m.$$

We adjoin to $\mathbb{N}_0^m$ an absorbing bottom element, $\perp$, such that for all $\boldsymbol{\alpha} \in \mathbb{N}_0^m$

$$\begin{aligned} &\perp \prec \boldsymbol{\alpha}, \\ &\perp + \boldsymbol{\alpha} = \boldsymbol{\alpha} + \perp = \perp. \end{aligned}$$

Definition 2.2 *Let $p \in \mathbb{F}[x, \mathbf{y}]$. Write*

$$p(x, \mathbf{y}) = \sum_{\boldsymbol{\alpha} \in N_0^m} p_{\boldsymbol{\alpha}}(x) \mathbf{y}^{\boldsymbol{\alpha}} = \sum_{i \in N_0} c_i(\mathbf{y}) x^i \tag{2}$$

where only finitely many $p_{\boldsymbol{\alpha}} \in \mathbb{F}[x]$ and $c_i \in \mathbb{F}[\mathbf{y}]$ are non-zero.

[2]available as MixedPoly in *Mathematica* package gosper.m at http://www.cis.upenn.edu/~wilf/AeqB.html

1. *We define the* multidegree *of p in $\mathbf{y}$ as*

$$\deg_{\mathbf{y}} p(x,\mathbf{y}) = \begin{cases} \max_{\preceq}\{\boldsymbol{\alpha} \in \mathbb{N}_0^m;\ p_{\boldsymbol{\alpha}} \neq 0\}, & p \neq 0, \\ \bot, & p = 0. \end{cases}$$

2. *We write $[\mathbf{y}^{\boldsymbol{\alpha}}]p(x,\mathbf{y})$ for $p_{\boldsymbol{\alpha}}(x)$ and $[x^i]p(x,\mathbf{y})$ for $c_i(\mathbf{y})$ in (2).*
3. *When $\deg_{\mathbf{y}} p \prec \boldsymbol{\alpha}$ we write $p = o(\mathbf{y}^{\boldsymbol{\alpha}})$.*
4. *Let $\boldsymbol{\delta} = deg_{\mathbf{y}} p$. We call p* mixed monic *when $[\mathbf{y}^{\boldsymbol{\delta}}]p(x,\mathbf{y})$ is monic as a univariate polynomial in x.*

Note that the concepts of multidegree and mixed monicity are relative to the chosen term order $\preceq$. By convention, $\gcd(a,b)$ always denotes a mixed monic greatest common divisor of $a, b \in \mathbb{F}[x,\mathbf{y}]$.

Lemma 2.3 *Let $\mathbb{F}$ be a field of characteristic zero and $r_1,\ldots,r_k \in \mathbb{F}\setminus\{0\}$, with $r_i \neq r_j$ for $i \neq j$. Let $d_1, d_2, \ldots, d_k \in \mathbb{N}_0$ and $d = d_1 + d_2 + \cdots + d_k$. Then the d functions $g_{ij}\colon \mathbb{N}_0 \to \mathbb{F}$, defined by $g_{ij}(n) = n^j r_i^n$, for $i = 1,\ldots,k$, $j = 0,\ldots,d_i - 1$, are linearly independent in the vector space $\mathbb{N}_0 \to \mathbb{F}$ over $\mathbb{F}$.*

This is well known, and can be proved by evaluating the *generalized Vandermonde determinant* composed of the values of $g_{ij}(n)$ for $n = 0, 1, \ldots, d-1$. We give an alternative short proof using generating functions.

Proof: Define subspaces L_1, L_2 of $\mathbb{N}_0 \to \mathbb{F}$. First, let L_1 be the subspace generated by the g_{ij}, $L_1 = \{\lambda n. \sum_{i=1}^r P_i(n) r_i^n \mid P_i \in \mathbb{F}[n], \deg P_i < d_i\}$. As L_1 is generated by d elements, $\dim L_1 \le d$. Second, let $q \in \mathbb{F}[x]$ be the polynomial

$$q(x) = \prod_{i=1}^{k} (1 - r_i x)^{d_i},$$

and let L_2 be the space of sequences whose generating function is a proper fraction with fixed denominator $q(x)$, $L_2 = \{f\colon \mathbb{N}_0 \to \mathbb{F} \mid \sum_{n=0}^{\infty} f(n)x^n = p(x)/q(x), p \in \mathbb{F}[x], \deg p < d\}$. Clearly, L_2 is isomorphic to the polynomial space $\{p \in \mathbb{F}[x];\ \deg p < d\}$, therefore $\dim L_2 = d$. Using partial fraction decomposition

$$\frac{p(x)}{q(x)} = \frac{p(x)}{\prod_{i=1}^{r}(1 - r_i x)^{d_i}} = \sum_{i=1}^{r}\sum_{j=1}^{d_i} \frac{c_{ij}}{(1 - r_i x)^j}$$

and the binomial series expansion

$$(1 - r_i x)^{-j} = \sum_{n=0}^{\infty} \binom{-j}{n} (-r_i x)^n = \sum_{n=0}^{\infty} \binom{n+j-1}{j-1} r_i^n x^n,$$

we find that

$$\frac{p(x)}{q(x)} = \sum_{n=0}^{\infty} f(n) x^n = \sum_{n=0}^{\infty} x^n \sum_{i=1}^{r}\sum_{j=1}^{d_i} c_{ij} \binom{n+j-1}{j-1} r_i^n.$$

Thus for any $f \in L_2$ we have $f(n) = \sum_{i=1}^r P_i(n) r_i^n$, where $P_i(n) = \sum_{j=1}^{d_i} c_{ij}\binom{n+j-1}{j-1}$ is a polynomial in n of degree at most $d_i - 1$. It follows that $f \in L_1$, and so $L_2 \subseteq L_1$. As $\dim L_2 = d$ and $\dim L_1 \le d$, we conclude that $\dim L_1 = d$ as well. Therefore the d generators g_{ij}, $i = 1,\ldots,k$, $j = 0,\ldots,d_i - 1$, of L_1 are linearly independent. ■

The ring of functions $\mathbb{N}_0 \to \mathbb{F}$ contains a subring $\mathbb{F}[\lambda n.n, \lambda n.q_1^n, \ldots, \lambda n.q_m^n]$ (or shorter $\mathbb{F}[n, \mathbf{q}^n]$) of functions that are polynomials in n and $q_1^n, \ldots, q_m^n$.

Theorem 2.4 *The ring of functions $\mathbb{F}[n, \mathbf{q}^n]$ is isomorphic to the ring of polynomials $\mathbb{F}[x, \mathbf{y}]$. The isomorphism $\Phi\colon \mathbb{F}[x,\mathbf{y}] \to \mathbb{F}[n, \mathbf{q}^n]$ maps $x \mapsto \lambda n.n$ and $y_i \mapsto \lambda n.q_i^n$.*

Proof: It is obvious that Φ is an epimorphism. We show that it is a monomorphism. Let $f \in \mathbb{F}[x, \mathbf{y}]$. Write f as

$$f = \sum_{i=0}^{k} p_i u_i,$$

where $p_1, \ldots, p_k \in \mathbb{F}[x]$, $u_1, \ldots, u_k \in \mathcal{M}$, and $u_i \neq u_j$ for $i \neq j$. Suppose $\Phi f = 0$:

$$0 = \Phi f = \sum_{i=0}^{k} p_i(n) u_i(\mathbf{q})^n.$$

Because $q_1, \ldots, q_n$ satisfy condition (1), $u_i(\mathbf{q}) \neq u_j(\mathbf{q})$ for $i \neq j$. The result now follows from Lemma 2.3. ∎

As a consequence, $\mathbb{F}[n, \mathbf{q}^n]$ is an integral domain, and its field of fractions $\mathbb{F}(n, \mathbf{q}^n)$ is isomorphic to the rational function field $\mathbb{F}(x, \mathbf{y})$. The map $\Phi\colon \mathbb{F}[x, \mathbf{y}] \to \mathbb{F}[n, \mathbf{q}^n]$ defined in Theorem 2.4 can be naturally extended to a map from $\mathbb{F}(x, \mathbf{y})$ to $\mathbb{F}(n, \mathbf{q}^n)$.

We define a difference operator $\mathbf{S}$ on $\mathbb{N}_0 \to \mathbb{F}$ by $\mathbf{S}\colon \lambda n.r(n) \mapsto \lambda n.r(n+1)$. This makes $\mathbb{F}(n, \mathbf{q}^n)$ a difference field and $\mathbb{F}[n, \mathbf{q}^n]$ a difference subring of $\mathbb{F}(n, \mathbf{q}^n)$. The difference operator $\mathbf{S}$ is inversive.

As $\Phi \circ \mathbf{E} = \mathbf{S} \circ \Phi$, we see that Φ is a difference isomorphism of the two fields $\mathbb{F}(x, \mathbf{y})$ and $\mathbb{F}(n, \mathbf{q}^n)$, as well as of the two rings $\mathbb{F}[x, \mathbf{y}]$ and $\mathbb{F}[n, \mathbf{q}^n]$.

Lemma 2.5 *Let $p \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$ and $\mu \in \mathbb{F}$. Then $\mathbf{E}p = \mu p$ if and only if $p = ru$ for some $r \in \mathbb{F}$, $u \in \mathcal{M}$, and $\mu = u(\mathbf{q})$.*

Proof: Sufficiency is obvious. Suppose $\mathbf{E}p = \mu p$. Write p as

$$p(x, \mathbf{y}) = \sum_{i=1}^{n} p_i(x)\, u_i(\mathbf{y}),$$

where $u_1, \ldots, u_n \in \mathcal{M}$ are pairwise different and $p_1, \ldots, p_n \in \mathbb{F}[x] \setminus \{0\}$. It follows that

$$\sum_i \mu\, p_i(x)\, u_i(\mathbf{y}) = \mu p = \mathbf{E}p = \sum_i p_i(x+1)\, u_i(\mathbf{q})\, u_i(\mathbf{y}).$$

Hence, for all $i = 0, \ldots, n$

$$\mu\, p_i(x) = u_i(\mathbf{q})\, p_i(x+1).$$

By looking at the leading coefficients in the above equation, we conclude that $\mu = u_i(\mathbf{q})$ for all $i = 1, \ldots, n$. However, if it were the case that $u_i(\mathbf{q}) = u_j(\mathbf{q})$ for some $i \neq j$, condition (1) would be violated. It follows that $n = 1$, and $p(x, \mathbf{y}) = r(x)\, u(\mathbf{y})$ for some $r \in \mathbb{F}[x] \setminus \{0\}$ and $u \in \mathcal{M}$. From $\mathbf{E}p = \mu p$ we get $r(x+1) = r(x)$, which is only possible if r is a constant. ∎

Definition 2.6 *For $1 \leq i \leq m$, we denote by $\boldsymbol{\pi}_i$ the endomorphism of $\mathbb{F}[x, \mathbf{y}]$ which substitutes 0 for y_i.*

Lemma 2.7 *The endomorphisms $\boldsymbol{\pi}_i$, $1 \leq i \leq m$, commute with $\mathbf{E}$ and $\mathbf{E}^{-1}$.*

Proof: Let $S = \{y_1, y_2, \ldots, y_m\} \setminus \{y_i\}$ and $p \in \mathbb{F}[x, \mathbf{y}]$. Consider p to be a polynomial in $\mathbb{F}[x, S][y_i]$. It is easy to check that $\mathbf{E}\boldsymbol{\pi}_i p = \boldsymbol{\pi}_i \mathbf{E}p$ and $\mathbf{E}^{-1}\boldsymbol{\pi}_i p = \boldsymbol{\pi}_i \mathbf{E}^{-1} p$. ∎

3 Algorithmic preliminaries

For polynomials $a, b \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$ such that $a, b \perp \mathcal{M}$, we will need to compute the set

$$D(a, b) = \{n \in \mathbb{N}_0;\ a \not\perp \mathbf{E}^n b\}$$

of all nonnegative integers n such that a and $\mathbf{E}^n b$ have a non-constant common divisor. Define polynomials $R_1, R_2, \ldots, R_m, R \in \mathbb{F}[x, \mathbf{y}][\xi, \boldsymbol{\eta}]$ as polynomial resultants

$$\begin{aligned} R_i(\xi, \boldsymbol{\eta}) &= \mathrm{Res}_{y_i}(a(x, \mathbf{y}), b(x+\xi, \boldsymbol{\eta}\mathbf{y})) \quad (1 \leq i \leq m), \\ R(\xi, \boldsymbol{\eta}) &= \mathrm{Res}_x(a(x, \mathbf{y}), b(x+\xi, \boldsymbol{\eta}\mathbf{y})). \end{aligned}$$

Here ξ is a variable and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_m)$ is an m-tuple of variables. Let

$$P(\xi, \boldsymbol{\eta}) = R(\xi, \boldsymbol{\eta}) \prod_{i=1}^{m} R_i(\xi, \boldsymbol{\eta}). \tag{3}$$

The following lemma leads to an algorithm for computing $D(a, b)$:

Lemma 3.1 $D(a, b) = \{n \in \mathbb{N}_0;\ P(n, \mathbf{q}^n) = 0\}$.

Proof: For $n \in \mathbb{N}_0$, let $\phi_n : \mathbb{F}[x, \mathbf{y}, \xi, \boldsymbol{\eta}] \to \mathbb{F}[x, \mathbf{y}]$ be the evaluation homomorphism which substitutes n for ξ and $\mathbf{q}^n$ for $\boldsymbol{\eta}$. It is easy to see that for any non-zero polynomial $p \in \mathbb{F}[x, \mathbf{y}]$, the homomorphic image $\phi_n(p(x + \xi, \boldsymbol{\eta}\mathbf{y})) = p(x + n, \mathbf{q}^n\mathbf{y}) = \mathbf{E}^n p(x, \mathbf{y})$ is non-zero. Therefore, by the Homomorphism Lemma for resultants (see, e.g., [9, Lemma 7.3.1]),

$$\begin{aligned} R_i(n, \mathbf{q}^n) &= \phi_n(R_i(\xi, \boldsymbol{\eta})) = \mathrm{Res}_{y_i}(\phi_n(a(x, \mathbf{y})), \phi_n(b(x + \xi, \boldsymbol{\eta}\mathbf{y}))) = \mathrm{Res}_{y_i}(a, \mathbf{E}^n b) \quad (1 \le i \le m), \\ R(n, \mathbf{q}^n) &= \phi_n(R(\xi, \boldsymbol{\eta})) = \mathrm{Res}_x(\phi_n(a(x, \mathbf{y})), \phi_n(b(x + \xi, \boldsymbol{\eta}\mathbf{y}))) = \mathrm{Res}_x(a, \mathbf{E}^n b). \end{aligned}$$

Thus we have the following chain of equivalences:

$$\begin{aligned} n \in D(a, b) &\iff \text{one of } \deg_x \gcd(a, \mathbf{E}^n b), \deg_{y_i} \gcd(a, \mathbf{E}^n b) \text{ is positive} \\ &\iff \text{one of } \mathrm{Res}_x(a, \mathbf{E}^n b), \mathrm{Res}_{y_i}(a, \mathbf{E}^n b) \text{ vanishes} \\ &\iff \text{one of } R(n, \mathbf{q}^n), R_i(n, \mathbf{q}^n) \text{ vanishes} \\ &\iff R(n, \mathbf{q}^n) \prod_{i=1}^{m} R_i(n, \mathbf{q}^n) = 0 \\ &\iff P(n, \mathbf{q}^n) = 0. \end{aligned} \tag{4}$$

The second equivalence above follows from the well-known properties of polynomial resultants. ∎

Next we show how to find integral solutions n of equation (4) in two special cases.

3.1 Transcendental bases

Let $\mathbb{F} = \mathbb{Q}(q_1, \ldots, q_m)$ where $q_1, \ldots, q_m$ are algebraically independent over $\mathbb{Q}$. Let $p \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$. We look for $n \in \mathbb{N}_0$ such that

$$p(n, q_1^n, \ldots, q_m^n) = 0. \tag{5}$$

We present a recursive algorithm for finding an upper bound for n. Once the bound is known, all integers between zero and the bound can be checked.

In equation (5), the coefficients are elements of $\mathbb{F}$, which are rational functions of $q_1, \ldots, q_m$. We can clear the denominators and obtain an equation in which q_i occur polynomially:

$$r(n, q_1, \ldots, q_m, q_1^n, \ldots, q_m^n) = 0, \tag{6}$$

where $r \in \mathbb{F}[x, z_1, \ldots, z_m, y_1, \ldots, y_m] \setminus \{0\}$. We show how to reduce recursively the problem of finding an upper bound for solutions of (6). Consider all terms of r with lowest degree of y_m, and let that degree be j. Among these terms, consider the one with the lowest degree of z_m, and let d be that degree. The term has the form $s z_m^d y_m^j$ for some $s \in \mathbb{F}[x, z_1, \ldots, z_{m-1}, y_1, \ldots, y_{m-1}] \setminus \{0\}$. Let M be an upper bound on natural solutions of equation

$$s(n, q_1, \ldots, q_{m-1}, q_1^n, \ldots, q_{m-1}^n) = 0, \tag{7}$$

which we can get recursively. Then $\max(M, d)$ is an upper bound for solutions of (6). Suppose $n > \max(M, d)$. Then n is not a solution of (7), and the lowest power of q_m that occurs in (6) is $d + nj$. Since this power occurs only in the term $s(n) q_m^d q_m^{nj}$, the term does not cancel, and n is not a solution of (6).

The base case of the recursion is an equation $r(n) = 0$, where $r \in \mathbb{Q}[x] \setminus \{0\}$. This can be handled easily, since any natural solution of this equation must divide the constant term (after we have cleared the denominators).

3.2 Rational bases

Suppose $q_1, \ldots, q_m \in \mathbb{Q}$. Let $p \in \mathbb{Q}[x, \mathbf{y}] \setminus \{0\}$. We consider the problem of finding $n \in \mathbb{N}_0$ such that

$$p(n, q_1^n, \ldots, q_m^n) = 0. \tag{8}$$

Write p as

$$p = \sum_{i=1}^{k} p_i u_i,$$

where $p_1, \ldots, p_k \in \mathbb{Q}[x] \setminus \{0\}$, $u_1, \ldots, u_k \in \mathcal{M}$, and $u_i \neq u_j$ for $i \neq j$. Equation (8) can be written as

$$\sum_{i=1}^{k} p_i(n) u_i(\mathbf{q})^n = 0 \tag{9}$$

Because bases $q_1, \ldots, q_m$ satisfy condition (1), $|u_i(q)| \neq |u_j(q)|$ for $i \neq j$. Let $s_i = u_i(\mathbf{q})$ for $i = 1, \ldots, k$. We may assume that $|s_1| < |s_2| < \cdots < |s_k|$. Suppose $p_k(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$. Equation (9) is equivalent to

$$a_d n^d s_k^n + \sum_{i=0}^{d-1} a_i n^i s_k^n + \sum_{i=1}^{k-1} p_i(n)\, s_i^n = 0 \tag{10}$$

The first term in (10) dominates the sum of the others. We only need a lower bound on n, such that the absolute value of the first term is greater than the absolute value of the sum of the other terms. Then we can check all integers between zero and the lower bound.

Let $\operatorname{dom}(a, b, k)$ be a function which gives an integer lower bound, such that for all $n \geq \operatorname{dom}(a, b, k)$ it is true that $a^n > b n^k$. Here $a > 1$, $b > 0$ and $k \in \mathbb{Z}$.

Let $\delta = 1/(d + k)$. For $i = 0, \ldots, d - 1$, define

$$M_i = \left\lceil \left| \frac{a_i}{a_d \delta} \right|^{\frac{1}{d-i}} \right\rceil$$

Let K_i be the maximum absolute value of the coefficients of p_i. For $i = 1, \ldots, k - 1$, define

$$N_i = \operatorname{dom}\left(\left| \frac{s_k}{s_i} \right|, \left| \frac{2K_i}{\delta a_d} \right|, \deg p_i - d \right)$$

Let $N = \max(2, M_0, \ldots, M_{d-1}, N_1, \ldots, N_{k-1})$. The choice of M_i ensures that

$$|\delta a_d n^d s_k^n| > |a_i n^i s_k^n|$$

for all $n \geq N$. The choice of K_i ensures that $|p_i(n)| < 2K_i n^{\deg p_i}$ for all $n \geq 2$. Therefore,

$$|\delta a_d n^d s_k^n| > |p_i(n) s_i^n|$$

for all $n > N$. This means that equation (8) does not have any solutions larger than N. We can find all solutions of (8) by checking all integers between 0 and N.

4 Polynomial solutions

In this section we present an algorithm for finding all polynomial solutions $f \in \mathbb{F}[x, \mathbf{y}]$ of parametric inhomogeneous equations of the form

$$\mathbf{L} f = g + \sum_{j=1}^{s} \lambda_j h_j \tag{11}$$

where

$$\mathbf{L} = \sum_{i=0}^{\rho} p_i \mathbf{E}^i \tag{12}$$

is a linear recurrence operator with polynomial coefficients $p_i \in \mathbb{F}[x, \mathbf{y}]$, λ_j are free parameters to be determined, and $g, h_j \in \mathbb{F}[x, \mathbf{y}]$ are given polynomials. More precisely, the problem is to compute a basis of the affine space $\mathbf{L}_p^{-1}(g)$ where $\mathbf{L}_p : \mathbb{F}[x, \mathbf{y}] \oplus \mathbb{F}^s \to \mathbb{F}[x, \mathbf{y}]$ and $\mathbf{L}_p : (f, \boldsymbol{\lambda}) \mapsto \mathbf{L} f - \sum_{j=1}^{s} \lambda_j h_j$ for

$f \in \mathbb{F}[x, \mathbf{y}]$ and $\boldsymbol{\lambda} \in \mathbb{F}^s$. Thus by a *solution* of (11) we mean a pair $(f, \boldsymbol{\lambda})$ with $f \in \mathbb{F}[x, \mathbf{y}]$ and $\boldsymbol{\lambda} \in \mathbb{F}^s$ such that (11) is satisfied.

As a special case, (11) includes inhomogeneous equations without parameters (when all $h_j = 0$) as well as homogeneous equations (when also $g = 0$). The ability to solve parametric inhomogeneous equations is crucial if one wants to apply Zeilberger's Creative Telescoping algorithm [18] in the mixed multibasic case. Another reason for allowing linear parameters in the equation is the nature of our algorithm which finds the terms of the solution one by one, introducing new free parameters into the right hand side at each step.

Let $f(x, \mathbf{y})$ be a polynomial solution of (11). Write

$$\begin{aligned}
\boldsymbol{\alpha} &= \max_{0 \le i \le \rho} \deg_{\mathbf{y}} p_i, & (13)\\
p_{i,\boldsymbol{\alpha}}(x) &= [\mathbf{y}^{\boldsymbol{\alpha}}]\, p_i, & (14)\\
d &= \max_{0 \le i \le \rho} \deg_x p_{i,\boldsymbol{\alpha}}(x), & (15)\\
p_{i,\boldsymbol{\alpha},d} &= [x^d]\, p_{i,\boldsymbol{\alpha}}(x), & (16)\\
\mathrm{rhs}(\boldsymbol{\lambda}) &= g + \sum_{j=1}^{s} \lambda_j h_j, & (17)\\
\boldsymbol{\varphi} &= \deg_{\mathbf{y}} f(x, \mathbf{y}), & (18)\\
t(x) &= [\mathbf{y}^{\boldsymbol{\varphi}}]\, f(x, \mathbf{y}), & (19)\\
\boldsymbol{\beta} &= \deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}), & (20)\\
r_{\boldsymbol{\beta}} &= [\mathbf{y}^{\boldsymbol{\beta}}]\, \mathrm{rhs}(\boldsymbol{\lambda}), & (21)
\end{aligned}$$

where $t \in \mathbb{F}[x] \setminus \{0\}$, $p_{i,\boldsymbol{\alpha}} \in \mathbb{F}[x]$ and $p_{i,\boldsymbol{\alpha},d} \in \mathbb{F}$. In (20) we regard λ_j's as variables over $\mathbb{F}(x)$, and $\mathrm{rhs}(\boldsymbol{\lambda})$ as belonging to $\mathbb{F}(x, \boldsymbol{\lambda})[\mathbf{y}]$. Note that when the parameters λ_j are given specific values from $\mathbb{F}$, the multidegree of $\mathrm{rhs}(\boldsymbol{\lambda}) = g + \sum_{j=1}^{s} \lambda_j h_j$ in $\mathbf{y}$ can be lower than $\boldsymbol{\beta}$.

Lemma 4.1 *Let* $\mathbf{L}, \boldsymbol{\alpha}, p_{i,\boldsymbol{\alpha},d}$ *and* $\boldsymbol{\varphi}$ *be as given in (12)–(18). If* $\deg_{\mathbf{y}} \mathbf{L}f \prec \boldsymbol{\alpha} + \boldsymbol{\varphi}$, *then* $\boldsymbol{\varphi} = \deg_y f$ *satisfies* $P(\mathbf{q}^{\boldsymbol{\varphi}}) = 0$ *where*

$$P(x) = \sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha},d}\, x^i \tag{22}$$

is the characteristic polynomial *of* $\mathbf{L}$.

Proof: From (19), $\mathbf{E}^i f = t(x+i)\mathbf{q}^{i\boldsymbol{\varphi}}\mathbf{y}^{\boldsymbol{\varphi}} + o(\mathbf{y}^{\boldsymbol{\varphi}})$, so $\mathbf{L}f = T(x)\mathbf{y}^{\boldsymbol{\alpha}+\boldsymbol{\varphi}} + o(\mathbf{y}^{\boldsymbol{\alpha}+\boldsymbol{\varphi}})$ where

$$T(x) = \sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha}}(x)\mathbf{q}^{i\boldsymbol{\varphi}} t(x+i). \tag{23}$$

If $\deg_{\mathbf{y}} \mathbf{L}f \prec \boldsymbol{\alpha} + \boldsymbol{\varphi}$ then $T = 0$. This is an ordinary recurrence relation with non-zero polynomial solution $t(x)$. As the coefficient of $x^{d+\deg_x t}$ in $T(x)$ must vanish, $\sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha},d}\mathbf{q}^{i\boldsymbol{\varphi}} = 0$ as claimed. ∎

Let $\mathcal{R}$ denote the set of exponents of those roots of the characteristic polynomial (22) (if any) which are power products of the bases:

$$\mathcal{R} = \{\boldsymbol{\sigma} \in \mathbb{N}_0^m;\ P(\mathbf{q}^{\boldsymbol{\sigma}}) = 0\}. \tag{24}$$

When $\mathcal{R}$ is empty we take $\max \mathcal{R} = \perp$. The following lemma gives rise to an algorithm for finding all polynomial solutions of equation (11).

Lemma 4.2 *Let* $(f, \boldsymbol{\lambda}')$ *be a solution of (11) with* $f \in \mathbb{F}[x, \mathbf{y}]$ *and* $\boldsymbol{\lambda}' \in \mathbb{F}^s$.

1. *If* $\boldsymbol{\alpha} + \max \mathcal{R} \succ \boldsymbol{\beta}$ *then* $\deg_{\mathbf{y}} f \preceq \max \mathcal{R}$.
2. *Let* $\boldsymbol{\alpha} + \max \mathcal{R} \preceq \boldsymbol{\beta}$.
   (a) *If* $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$ *then* $\deg_{\mathbf{y}} f \preceq \boldsymbol{\beta} - \boldsymbol{\alpha}$.
   (b) *If* $\boldsymbol{\alpha} \not\subseteq \boldsymbol{\beta}$ *then* $\deg_{\mathbf{y}} rhs(\boldsymbol{\lambda}') \prec \boldsymbol{\beta}$.

Proof: Let $\boldsymbol{\varphi} = \deg_{\mathbf{y}} f$. Let T be as in (23).

1. $\boldsymbol{\alpha} + \max \mathcal{R} \succ \boldsymbol{\beta}$
   If $T = 0$ then $\boldsymbol{\varphi} \in \mathcal{R}$, by Lemma 4.1. If $T \neq 0$ then $\deg_{\mathbf{y}} \mathbf{L}f = \boldsymbol{\alpha} + \boldsymbol{\varphi}$. As $\deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}') \preceq \boldsymbol{\beta}$, it follows that $\boldsymbol{\alpha} + \boldsymbol{\varphi} \preceq \boldsymbol{\beta} \prec \boldsymbol{\alpha} + \max \mathcal{R}$, so $\boldsymbol{\varphi} \prec \max \mathcal{R}$. In either case, $\boldsymbol{\varphi} \preceq \max \mathcal{R}$ as claimed.
2. $\boldsymbol{\alpha} + \max \mathcal{R} \preceq \boldsymbol{\beta}$
   (a) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$
   If $T = 0$ then $\boldsymbol{\varphi} \in \mathcal{R}$ by Lemma 4.1, so $\boldsymbol{\varphi} \preceq \max \mathcal{R}$ and therefore $\boldsymbol{\alpha} + \boldsymbol{\varphi} \preceq \boldsymbol{\beta}$. If $T \neq 0$ then $\deg_{\mathbf{y}} \mathbf{L}f = \boldsymbol{\alpha} + \boldsymbol{\varphi}$. As $\deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}') \preceq \boldsymbol{\beta}$, it follows that $\boldsymbol{\alpha} + \boldsymbol{\varphi} \preceq \boldsymbol{\beta}$. In either case, $\boldsymbol{\varphi} \preceq \boldsymbol{\beta} - \boldsymbol{\alpha}$ as claimed.
   (b) $\boldsymbol{\alpha} \not\subseteq \boldsymbol{\beta}$
   Assume that $\deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}') = \boldsymbol{\beta}$. If $T = 0$ then by Lemma 4.1, $\boldsymbol{\varphi} \in \mathcal{R}$ and $\deg_{\mathbf{y}} \mathbf{L}f \prec \boldsymbol{\alpha} + \boldsymbol{\varphi} \preceq \boldsymbol{\alpha} + \max \mathcal{R} \preceq \boldsymbol{\beta} = \deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}')$, a contradiction. If $T \neq 0$ then $\boldsymbol{\alpha} + \boldsymbol{\varphi} = \boldsymbol{\beta}$ which implies that $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$, contrary to the assumption. Both cases lead to contradiction, so $\deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}') \prec \boldsymbol{\beta}$ as claimed. ∎

Based on Lemma 4.2, we can find the general solution $(f, \boldsymbol{\lambda})$ of equation (11) as follows: First compute the set $\mathcal{R}$ as given in (24). Then distinguish three cases:

1. $\boldsymbol{\alpha} + \max \mathcal{R} \succ \boldsymbol{\beta}$
   Set $\boldsymbol{\varphi} = \max \mathcal{R}$ and look for f in the form
   $$f = t(x)\mathbf{y}^{\boldsymbol{\varphi}} + f_1 \tag{25}$$
   where $f_1 = o(\mathbf{y}^{\boldsymbol{\varphi}})$. To find $t(x)$, apply the algorithm of [2] to $T = 0$ (an ordinary homogeneous recurrence relation). Then remove $\max \mathcal{R}$ from $\mathcal{R}$ and find f_1 recursively by solving
   $$\mathbf{L}f_1 = \mathrm{rhs}(\boldsymbol{\lambda}) - \mathbf{L}\left(t(x)\mathbf{y}^{\boldsymbol{\varphi}}\right). \tag{26}$$
2. $\boldsymbol{\alpha} + \max \mathcal{R} \preceq \boldsymbol{\beta}$
   (a) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$
   Set $\boldsymbol{\varphi} = \boldsymbol{\beta} - \boldsymbol{\alpha}$ and look for f in the form (25). To find $t(x)$, apply the algorithm of [2] to $T = r_{\boldsymbol{\beta}}$ (an ordinary parametric inhomogeneous recurrence relation). Then remove $\max \mathcal{R}$ from $\mathcal{R}$ (only in case that $\boldsymbol{\alpha} + \max \mathcal{R} = \boldsymbol{\beta}$), and find f_1 recursively by solving (26).
   (b) $\boldsymbol{\alpha} \not\subseteq \boldsymbol{\beta}$
   Let $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ be the solution of the system of linear algebraic equations for the free parameters $\boldsymbol{\lambda}$ obtained by equating the coefficients of powers of x in $r_{\boldsymbol{\beta}}$ to zero. Then find f recursively by solving $\mathbf{L}f = \mathrm{rhs}(\boldsymbol{\lambda}')$.

Remarks:

1. Note that in steps 1 and 2(a), $t(x)$ can contain new free parameters which are then joined with the existing ones. This explains the need for allowing parameters in the right hand side of the equation.
2. In step 2(b), the number of free parameters will drop by the rank of the linear system to be solved.
3. If the ordinary recurrence in steps 1 or 2(a) has no polynomial solution, or the linear system in step 2(b) is unsolvable, then the original parametric recurrence has no polynomial solution, and the algorithm terminates unsuccessfully.
4. At each step, either the cardinality of the set $\mathcal{R}$ drops, or the multidegree $\boldsymbol{\beta} = \deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda})$ decreases in the admissible term order $\prec$. As every admissible term order is a well-order, this assures termination of the algorithm.
5. Unless the algorithm terminates unsuccessfully, eventually $\mathcal{R}$ becomes empty and $\mathrm{rhs}(\boldsymbol{\lambda})$ becomes 0. Then the only polynomial solution of (11) is $f = 0$.

An iterative version of this tail-recursive algorithm called `m&m-Poly` is given in appendix A.

5 A canonical form

Let $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$. Write r as

$$r = \frac{u}{v} \cdot \frac{a_0}{b_0},$$

where $u, v \in \mathcal{M}$, $a_0, b_0 \in \mathbb{F}[x, \mathbf{y}]$, $a_0 b_0 \perp \mathcal{M}$, $u a_0 \perp v b_0$, and b_0 is mixed monic (Def. 2.2).

There are finitely many $h \in \mathbb{N}_0$ such that $a_0 \not\perp \mathbf{E}^h b_0$. Let $0 \le h_1 < h_2 < \cdots < h_N$ be those natural numbers for which a_0 and $\mathbf{E}^{h_i} b_0$ have a common non-constant factor. According to Lemma 3.1, they are the natural roots of $p(h) = P(h, \mathbf{q}^h)$ where P is the polynomial given in (3), and can be found as described in section 3.

Lemma 5.1 *Consider the algorithm* `CanonicalForm` *in appendix B. Define $h_{N+1} = \infty$, and let $0 \le k \le i, j \le N$, $h \in \mathbb{N}_0$ and $h < h_{k+1}$. Then $a_i \perp \mathbf{E}^h b_j$.*

Proof: Let $S = \{h_1, \ldots, h_N\}$. Suppose $h \notin S$. Since $a_i \mid a_0$ and $b_j \mid b_0$ and $a_0 \perp \mathbf{E}^h b_0$, it follows that $a_i \perp \mathbf{E}^h b_j$.

To prove the lemma for $h \in S$, we use induction on k. When $k = 0$, there is nothing to prove because there is no $h \in S$ such that $h < h_1$. Assume that the lemma holds for all $h \in S$, $h < h_k$. We show that it holds for $h = h_k$. Since $a_i \mid a_k$ and $b_j \mid b_k$, it follows that $\gcd(a_i, \mathbf{E}^{h_k} b_j)$ divides $\gcd(a_k, \mathbf{E}^{h_k} b_k)$. Furthermore,

$$\gcd(a_k, \mathbf{E}^{h_k} b_k) = \gcd(\frac{a_{k-1}}{s_k}, \frac{\mathbf{E}^{h_k} b_{k-1}}{s_k}) = 1$$

by the definition of a_k, b_k and s_k in algorithm `CanonicalForm`. This completes the proof. ∎

Theorem 5.2 *Let $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$. There exist polynomials $a, b, c \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$ such that*

1. *b, c are mixed monic,*
2. *$c \perp \mathcal{M}$,*
3. *$a \perp \mathbf{E}^k b$ for all $k \in \mathbb{N}_0$,*
4. *$a \perp c$,*
5. *$b \perp \mathbf{E}c$, and*

$$r = \frac{a}{b} \cdot \frac{\mathbf{E}c}{c}. \tag{27}$$

Proof: Let a, b, c be constructed by the algorithm `CanonicalForm` from appendix B. Conditions 1 and 2 are satisfied by construction, and condition 3 follows from Lemma 5.1 by taking $i = j = k = N$. Identity (27) is verified by:

$$\begin{aligned} \frac{a}{b} \cdot \frac{\mathbf{E}c}{c} &= \frac{u \cdot a_N}{v \cdot b_N} \cdot \prod_{i=1}^{N} \prod_{j=1}^{h_i} \frac{\mathbf{E}^{-j+1} s_i}{\mathbf{E}^{-j} s_i} = \\ &= \frac{u \cdot a_0}{\prod_{i=1}^{N} s_i} \cdot \frac{\prod_{i=1}^{N} \mathbf{E}^{-h_i} s_i}{v \cdot b_0} \cdot \prod_{i=1}^{N} \frac{s_i}{\mathbf{E}^{-h_i} s_i} = \frac{u \cdot a_0}{v \cdot b_0} = r \end{aligned}$$

Proof of 4: Suppose $a \not\perp c$. Then also $a_N \not\perp \mathbf{E}^{-j} s_i$ for some i and j such that $1 \le i \le N$ and $1 \le j \le h_i$. By definition $\mathbf{E}^{h_i - j} b_{i-1} = \mathbf{E}^{h_i - j} b_i \cdot \mathbf{E}^{-j} s_i$, so it follows that $a_N \not\perp \mathbf{E}^{h_i - j} b_{i-1}$. Since $h_i - j < h_i$, this contradicts Lemma 5.1.

Proof of 5: Suppose $b \not\perp \mathbf{E}c$. Then also $b_N \not\perp \mathbf{E}^{-j} s_i$ for some i and j such that $1 \le i \le N$ and $0 \le j \le h_i - 1$. By definition $\mathbf{E}^{-j} a_{i-1} = \mathbf{E}^{-j} a_i \cdot \mathbf{E}^{-j} s_i$, so it follows that $a_{i-1} \not\perp \mathbf{E}^j b_N$. Since $j < h_i$, this contradicts Lemma 5.1. ∎

Lemma 5.3 *Let $a, b, c, A, B, C \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$ be polynomials such that $a \perp c$, $b \perp \mathbf{E}c$, $c \perp \mathcal{M}$, and $A \perp \mathbf{E}^k B$ for all $k \in \mathbb{N}_0$. If*

$$\frac{a}{b} \cdot \frac{\mathbf{E}c}{c} = \frac{A}{B} \cdot \frac{\mathbf{E}C}{C}, \tag{28}$$

then c divides C.

Proof: Let $g = \gcd(c, C)$, $d = c/g$, and $D = C/g$. Then $d \perp D$, $a \perp d$, and $b \perp \mathbf{E}d$. Clear denominators in (28) and cancel $g \cdot \mathbf{E}g$ on both sides. The result

$$A \cdot b \cdot d \cdot \mathbf{E}D = a \cdot B \cdot D \cdot \mathbf{E}d$$

shows that $d \mid B \cdot \mathbf{E}d$ and $\mathbf{E}d \mid A \cdot d$. Using these two relations repeatedly, we find that

$$\begin{array}{lcl} d & \mid & B \cdot \mathbf{E}B \cdots \mathbf{E}^{k-1}B \cdot \mathbf{E}^k d \\ d & \mid & \mathbf{E}^{-1}A \cdot \mathbf{E}^{-2}A \cdots \mathbf{E}^{-k}A \cdot \mathbf{E}^{-k}d \end{array}$$

for all $k \in \mathbb{N}_0$. Because $d \perp \mathcal{M}$, and $\mathbb{F}$ has characteristic zero, $d \perp \mathbf{E}^k d$ and $d \perp \mathbf{E}^{-k}d$ for large enough k. It follows that d divides both $B \cdot \mathbf{E}B \cdots \mathbf{E}^{k-1}B$ and $\mathbf{E}^{-1}A \cdot \mathbf{E}^{-2}A \cdots \mathbf{E}^{-k}A$ for large enough k. But these two polynomials are coprime by assumption, so d must be a constant. Hence, c divides C. ∎

Theorem 5.4 *Let $r \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$. Then the factorization of r described in Theorem 5.2 is unique.*

Proof: Suppose that a, b, c and A, B, C are two factorizations of r, as described in Theorem 5.2. Then

$$r = \frac{a}{b} \cdot \frac{\mathbf{E}c}{c} = \frac{A}{B} \cdot \frac{\mathbf{E}C}{C}.$$

By Lemma 5.3, c divides C, and vice versa. As c and C are mixed monic they must be equal, hence $a/b = A/B$. As $a \perp b$, $A \perp B$, and b, B are mixed monic, it follows that $b = B$ and $a = A$ as well. ∎

The factorization of non-zero rational functions described in Theorem 5.2 is thus a canonical form. We introduce special notation for it.

Definition 5.5 *Let $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$ be a non-zero rational function. We write*

$$(a, b, c) = \text{C.f.}(r)$$

to denote the unique polynomials $a, b, c \in \mathbb{F}[x, \mathbf{y}]$ which satisfy the conditions of Theorem 5.2.

Theorem 5.6 *Let $a, b \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$, and $(A, B, C) = \text{C.f.}(b/a)$. The homogeneous first-order recurrence*

$$a \cdot \mathbf{E}f - bf = 0 \tag{29}$$

has a non-zero polynomial solution $f \in \mathbb{F}[x, \mathbf{y}]$ if and only if $A/B = u(\mathbf{q})$ for some $u \in \mathcal{M}$. In that case, $f = \lambda \cdot u \cdot C$ for some $\lambda \in \mathbb{F} \setminus \{0\}$.

Proof: Suppose (29) has a non-zero solution $f \in \mathbb{F}[x, \mathbf{y}]$. Write $f = \lambda \cdot u \cdot v$ where $\lambda \in \mathbb{F} \setminus \{0\}$, $u \in \mathcal{M}$ and $g \perp \mathcal{M}$ is mixed monic. Then $\text{C.f.}(\mathbf{E}f/f) = (u(\mathbf{q}), 1, g)$. Since

$$\frac{\mathbf{E}f}{f} = \frac{b}{a} = \frac{A}{B} \cdot \frac{\mathbf{E}C}{C},$$

$(A, B, C) = \text{C.f.}(\mathbf{E}f/f)$ as well. It follows that $A = u(\mathbf{q})$, $B = 1$ and $C = g$, so $A/B = u(\mathbf{q})$ and $f = \lambda \cdot u \cdot C$.

Conversely, if $A/B = u(\mathbf{q})$ for some $u \in \mathcal{M}$, then $f = u \cdot C$ is a non-zero solution of (29). ∎

6 Generalized Gosper's algorithm

Let $S_n = \sum_{k=0}^{n-1} t_k$. Clearly S_n satisfies the first-order recurrence

$$S_{n+1} - S_n = t_n. \tag{30}$$

We consider the following problem:

Given a sequence t_n, decide if equation (30) has an mmHS solution S_n, and if so, find it.

Let S_n and t_n satisfy (30), with

$$\frac{S_{n+1}}{S_n} =: T_n \in \mathbb{F}(n, \mathbf{q}^n).$$

Then the two quotients

$$r_n := \frac{t_{n+1}}{t_n} = \frac{S_{n+2} - S_{n+1}}{S_{n+1} - S_n} = \frac{T_{n+1} - 1}{1 - 1/T_n}$$

and

$$R_n := \frac{S_n}{t_n} = \frac{S_n}{S_{n+1} - S_n} = \frac{1}{T_n - 1}$$

both belong to $\mathbb{F}(n, \mathbf{q}^n)$. So t_n must be an mmHS itself, and S_n is a rational multiple of t_n: $S_n = R_n t_n$. Using this, (30) yields a recurrence for the unknown rational function R_n,

$$r_n R_{n+1} - R_n = 1. \tag{31}$$

By Theorem 2.4, equation (31) is equivalent to

$$r \cdot \mathbf{E}R - R = 1, \tag{32}$$

where r and R are Φ-isomorphic images of r_n and R_n.

Next we show how to find rational solutions R of equation (32). The following theorem provides a multiple of the denominator and a divisor of the numerator of R. The missing factor in the numerator can then be found using algorithm `m&m-Poly` of Section 4.

Definition 6.1 *Let $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$ be a non-zero rational function, and $(a, b, c) = \text{C.f.}(r)$. For $1 \le i \le m$ define* exponents $e_i(r)$ *as follows: If $\boldsymbol{\pi}_i(a)\boldsymbol{\pi}_i(b) \neq 0$, let $(a_i, b_i, c_i) = \text{C.f.}(\boldsymbol{\pi}_i(b)/\boldsymbol{\pi}_i(a))$. If there are $v, w \in \mathcal{M}$ such that $v \perp w$ and $a_i/b_i = v(\mathbf{q})/w(\mathbf{q})$, then $e_i(r) = \deg_{y_i} w$. If not, or if $\boldsymbol{\pi}_i(a)\boldsymbol{\pi}_i(b) = 0$, then $e_i(r) = 0$.*

Theorem 6.2 *Let $R = f/(ug)$ be a rational solution of (32) with $f, g \in \mathbb{F}[x, \mathbf{y}]$, $u \in \mathcal{M}$, $g \perp \mathcal{M}$, and $f \perp ug$. Then*

1. *$g \mid c$ where $(a, b, c) = \text{C.f.}(r)$,*
2. *$\deg_{y_i} u \le e_i(r)$,*
3. *$\mathbf{E}^{-1}b \mid f$.*

Proof:

1. From (32),

$$r = \frac{R+1}{\mathbf{E}R} = \frac{(f + ug)u(\mathbf{q})}{\mathbf{E}f} \cdot \frac{\mathbf{E}g}{g}. \tag{33}$$

On the other hand, $(a, b, c) = \text{C.f.}(r)$, so

$$r = \frac{a}{b} \cdot \frac{\mathbf{E}c}{c}. \tag{34}$$

As $\mathbf{E}f \perp \mathbf{E}g$, $g \perp (f + ug)u(\mathbf{q})$, $g \perp \mathcal{M}$, and $a \perp \mathbf{E}^k b$ for all $k \in \mathbb{N}_0$, it follows by Lemma 5.3 that g divides c.

2. Write $F = fc/g \in \mathbb{F}[x, \mathbf{y}]$. Then $R = f/(ug) = F/(uc)$. Combining this with (33) and (34), we find that

$$(F + uc) \cdot u(\mathbf{q}) \cdot b = a \cdot \mathbf{E}F. \tag{35}$$

Now assume that $y_i \mid u$. Then setting $y_i \leftarrow 0$ in Eqn. (35) and rearranging yields

$$\boldsymbol{\pi}_i(a) \cdot \mathbf{E}\boldsymbol{\pi}_i(F) - u(\mathbf{q}) \cdot \boldsymbol{\pi}_i(b) \cdot \boldsymbol{\pi}_i(F) = 0. \tag{36}$$

Because $F \mid fc$, $f \perp u$ and $c \perp \mathcal{M}$, it follows that $y_i \nmid F$ and $\boldsymbol{\pi}_i(F) \neq 0$. If $y_i \mid a$ then from (35), $y_i \mid b \cdot F$. But $y_i \nmid b$ because $a \perp b$, so $y_i \mid F$. This contradiction shows that $\boldsymbol{\pi}_i(a) \neq 0$. In an analogous way we conclude that $\boldsymbol{\pi}_i(b) \neq 0$.

Let $(a_i, b_i, c_i) = \text{C.f.}(\boldsymbol{\pi}_i(b)/\boldsymbol{\pi}_i(a))$. Then $((u(\mathbf{q}) \cdot a_i, b_i, c_i) = \text{C.f.}(u(\mathbf{q}) \cdot \boldsymbol{\pi}_i(b)/\boldsymbol{\pi}_i(a))$. Since equation (36) has a non-zero polynomial solution $\boldsymbol{\pi}_i(F)$, it follows by Theorem 5.6 that there is $u_1 \in \mathcal{M}$ such that $u(\mathbf{q}) \cdot a_i/b_i = u_1(\mathbf{q})$, and that $\boldsymbol{\pi}_i(F) = \lambda u_1 c_i$ for some $\lambda \in \mathbb{F}$. Then $a_i/b_i = u_1(\mathbf{q})/u(\mathbf{q})$ is a quotient of two monomials. Write $u_1 = v \cdot t$ and $u = w \cdot t$ where $t, v, w \in \mathcal{M}$ and $v \perp w$. By Definiton 6.1, $e_i(r) = \deg_{y_i} w$. As $t \mid u_1 \mid \boldsymbol{\pi}_i(F)$ it follows that $t \perp y_i$, so

$$\deg_{y_i} u = \deg_{y_i} w = e_i(r).$$

We have shown that $\deg_{y_i} u$ is either 0 or $e_i(r)$, so in either case $\deg_{y_i} u \le e_i(r)$.

3. From (35) it follows that $b \mid a \cdot \mathbf{E}F$. As $a \perp b$, we have that $b \mid \mathbf{E}F \mid \mathbf{E}f\mathbf{E}c$. But $b \perp \mathbf{E}c$, so $\mathbf{E}^{-1}b \mid f$.

■

From Theorem 6.2 it follows that we can look for R in the form

$$R = \frac{\mathbf{E}^{-1}b \cdot p}{u \cdot c} \tag{37}$$

where $(a, b, c) = \text{C.f.}(r)$ and $u = \prod_{i=1}^{m} y_i^{e_i(r)}$ are known while $p \in \mathbb{F}[x, \mathbf{y}]$ is an unknown polynomial. Inserting (37) and (34) into (32) yields

$$a \cdot \mathbf{E}p - u(\mathbf{q}) \cdot \mathbf{E}^{-1}b \cdot p = u(\mathbf{q})u \cdot c, \tag{38}$$

an inhomogeneous first-order linear recurrence relation with polynomial coefficients satisfied by p. Algorithm `m&m-Poly` of Section 4 can now be applied to find a polynomial solution p of Eqn. (38).

The full algorithm is given in appendix C.

Example 6.1 For a simple mixed example, consider the sum

$$S_n = \sum_{k=0}^{n-1} (1 - q^{k+1}(k+1))(q;q)_k$$

where $(q;q)_n = \prod_{i=1}^{n}(1 - q^i)$. Following algorithm `m&m-Gosper` we have $t_n = (1 - q^{n+1}(n+1))(q;q)_n$ and, taking $q_1 = q$ and $y_1 = y$,

$$r = \Phi(t_{n+1}/t_n) = (1 - qy)\frac{1 - q^2(x+2)y}{1 - q(x+1)y}.$$

The canonical form of r is

$$(a, b, c) = (1 - qy, 1, (1 + x)y - 1/q)$$

and $u = 1$. The recurrence to be checked for polynomial solutions,

$$(1 - qy)\mathbf{E}p - p = (1 + x)y - 1/q$$

is satisfied by $p = -x/q$, so $R = x/(1 - qy(x+1))$ and we find that $S_{n+1} - S_n = t_n$ is satisfied by $S_n = n(q;q)_n + C$. Here C is an additive constant which, for our initial sum, equals $S_0 = 0$. ■

Example 6.2 For a multibasic example, we refer to the indefinite summation formula (too big to be reproduced here) proved in [14]. The formula contains an arbitrary number, k, of bases. Such formulæ cannot be proved by our algorithm. However, any specialization of this formula in which k is replaced by a specific natural number (such as 2 or 113), can be, at least in principle, not only proved, but also derived by `m&m-Gosper`. In [14], it is shown that several known basic and bibasic summation formulæ can be obtained as specializations of this k-basic master formula. ■

7 Concluding remarks

We have shown how to compute the hypergeometric canonical form of rational functions, how to find polynomial solutions of recurrences, and how to perform Gosper's algorithm, all in the multibasic and mixed case. With polynomial solutions, the usual approach – obtain a degree bound and then use undetermined coefficients – does not work because in the multivariate case, there may be infinitely many terms below a given one in the admissible term order. Our algorithm `m&m-Poly` works instead term-by-term, which guarantees termination because an admissible term order is also a well-order. In solving Gosper's equation (32), the hard part is obtaining the monomial factor in the denominator. It turns out that the degree of this factor in each variable can be found by projecting along that variable and using Theorem 5.6 on polynomial solutions of homogeneous first-order equations.

What remains to be done on the theoretical plane is to generalize the important concept of *greatest factorial factorization* of polynomials [10] which seems to play a fundamental role in symbolic summation, from the basic [11] and bibasic [13] cases to the multibasic and mixed one. On the algorithmic plane, multibasic and mixed generalizations of algorithms for finding rational [1] and hypergeometric [12, 3] solutions of recurrences, and also of algorithms for factorization of recurrence operators [5] should be developed.

A Algorithm m&m-Poly

```
INPUT: p_0, ..., p_ρ, g, h_1, ..., h_s ∈ F[x, y],  p_0, p_ρ ≠ 0
```

INPUT: $p_0, \ldots, p_\rho, g, h_1, \ldots, h_s \in \mathbb{F}[x, \mathbf{y}]$, $p_0, p_\rho \neq 0$
OUTPUT: general solution $(f, \boldsymbol{\lambda}) \in \mathbb{F}[x, \mathbf{y}] \times \mathbb{F}^s$ of $\mathbf{L}f = g + \sum_{j=1}^{s} \lambda_j h_j$ where $\mathbf{L} = \sum_{i=0}^{\rho} p_i \cdot \mathbf{E}^i$
EXTERNAL ALGORITHMS USED:

Poly$(e, t, \boldsymbol{\lambda})$ returns general solution $(t, \boldsymbol{\lambda})$ of the parametric inhomogeneous ordinary recurrence e (see [2])

LinSolve$(e, x, \boldsymbol{\lambda})$ returns general solution $\boldsymbol{\lambda}$ of the linear algebraic equations resulting from equating the coefficients of like powers of x on both sides of the polynomial equation e

$\boldsymbol{\alpha} := \max_{0 \le i \le \rho} \deg_{\mathbf{y}} p_i$
$p_{i,\boldsymbol{\alpha}}(x) := [\mathbf{y}^{\boldsymbol{\alpha}}]\, p_i$
$d := \max_{0 \le i \le \rho} \deg_x p_{i,\boldsymbol{\alpha}}(x)$
$p_{i,\boldsymbol{\alpha},d} := [x^d]\, p_{i,\boldsymbol{\alpha}}(x)$
$P(x) := \sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha},d} x^i$
$\mathcal{R} := \{\boldsymbol{\sigma} \in \mathbb{N}_0^m;\ P(\mathbf{q}^{\boldsymbol{\sigma}}) = 0\}$
$\text{rhs} := g + \sum_{j=1}^{s} \lambda_j h_j$
$f := 0$
while $\mathcal{R} \neq \emptyset$ or $\text{rhs} \neq 0$ do
  if $\mathcal{R} \neq \emptyset$ then $\boldsymbol{\mu} := \max_{\prec} \mathcal{R}$ else $\boldsymbol{\mu} := \perp$
  if $\text{rhs} \neq 0$ then $\boldsymbol{\beta} := \deg_{\mathbf{y}} \text{rhs}$ else $\boldsymbol{\beta} := \perp$
  if $\boldsymbol{\alpha} + \boldsymbol{\mu} \succ \boldsymbol{\beta}$ then
    $\boldsymbol{\varphi} := \boldsymbol{\mu}$
    $(t', \boldsymbol{\lambda}') :=$ Poly$(\sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha}}(x) \mathbf{q}^{i\boldsymbol{\varphi}} t(x+i) = 0, t, \boldsymbol{\lambda})$
    $f := f + t' \mathbf{y}^{\boldsymbol{\varphi}}$
    $\text{rhs} := \text{rhs}|_{\boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda}'} - \mathbf{L}\,(t' \mathbf{y}^{\boldsymbol{\varphi}})$
    $\mathcal{R} := \mathcal{R} \setminus \{\boldsymbol{\varphi}\}$
  end
  else if $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$ then
    $\boldsymbol{\varphi} := \boldsymbol{\beta} - \boldsymbol{\alpha}$
    $(t', \boldsymbol{\lambda}') :=$ Poly$(\sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha}}(x) \mathbf{q}^{i\boldsymbol{\varphi}} t(x+i) = [\mathbf{y}^{\boldsymbol{\beta}}]\, \text{rhs}, t, \boldsymbol{\lambda})$
    $f := f + t' \mathbf{y}^{\boldsymbol{\varphi}}$
    $\text{rhs} := \text{rhs}|_{\boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda}'} - \mathbf{L}\,(t' \mathbf{y}^{\boldsymbol{\varphi}})$
    if $\boldsymbol{\varphi} = \boldsymbol{\mu}$ then $\mathcal{R} := \mathcal{R} \setminus \{\boldsymbol{\varphi}\}$
  end
  else
    $\boldsymbol{\lambda}' :=$ LinSolve$([\mathbf{y}^{\boldsymbol{\beta}}]\, \text{rhs} = 0, x, \boldsymbol{\lambda})$
    $\text{rhs} := \text{rhs}|_{\boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda}'}$
  end
end
return f.

NB: If either Poly or LinSolve fails then m&m-Poly fails as well.

B Algorithm CanonicalForm

INPUT: $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$
OUTPUT: canonical form of r
EXTERNAL ALGORITHMS USED:

Resultant(a, b, x) returns the resultant of polynomials a, b wrt. x

GCD(a, b) returns the mixed monic gcd of polynomials a, b

let $r = (u/v) \cdot (a_0/b_0)$ where $a_0 b_0 \perp \mathcal{M}$, $a_0 u \perp b_0 v$, $u, v \in \mathcal{M}$, and b_0 is mixed monic
$P(\xi, \boldsymbol{\eta}) :=$ Resultant$(a(x, \mathbf{y}), b(x + \xi, \boldsymbol{\eta}\mathbf{y}), x)$
for $i = 1, \ldots, m$ do
  $P(\xi, \boldsymbol{\eta}) := P(\xi, \boldsymbol{\eta}) \cdot$Resultant$(a(x, \mathbf{y}), b(x + \xi, \boldsymbol{\eta}\mathbf{y}), y_i)$

```
end
let $h_1 < h_2 < \cdots < h_N$ be the roots $h \in \mathbb{N}_0$ of $p(h) = P(h, \mathbf{q}^h)$
$c_0 := 1$
for $i = 1, \ldots, N$ do
    $s_i :=$ GCD$(a_{i-1}, \mathbf{E}^{h_i} b_{i-1})$
    $a_i := a_{i-1}/s_i$
    $b_i := b_{i-1}/\mathbf{E}^{-h_i} s_i$
    $c_i := c_{i-1} \prod_{j=1}^{h_i} \mathbf{E}^{-j} s_i$
end
$a := u \cdot a_N$
$b := v \cdot b_N$
$c := c_N$
return $(a, b, c)$.
```

C Algorithm m&m-Gosper

```
INPUT: mmHS $t_n$
OUTPUT: mmHS $S_n$ such that $S_{n+1} - S_n = t_n$, if it exists

$r := \Phi(t_{n+1}/t_n)$ (see theorem 2.4)
$(a, b, c) :=$ CanonicalForm$(r)$
$u := 1$
for $i = 1$ to $m$ do
    if $\boldsymbol{\pi}_i(ab) = 0$
    then
        $e_i := 0$
    else
        $(a_i, b_i, c_i) :=$ CanonicalForm$(\boldsymbol{\pi}_i(b/a))$
        if $\exists v, w \in \mathcal{M} : (v \perp w$ and $a_i/b_i = v(\mathbf{q})/w(\mathbf{q}))$
        then
          $e_i := \deg_{y_i} w$
        else $e_i := 0$
    end
    $u := u \cdot y_i^{e_i}$
end
$(p, \_) :=$ m&m-Poly$(a \cdot \mathbf{E}p - u(\mathbf{q}) \cdot \mathbf{E}^{-1} b \cdot p = u(\mathbf{q}) u \cdot c, p, \_)$
$R := (p \cdot \mathbf{E}^{-1} b)/(u \cdot c)$
return $\Phi^{-1}(R) \cdot t_n$.
```

NB: If m&m-Poly fails then m&m-Gosper fails as well.

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