University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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Dragan Marušič Roman Nedela

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DRAGAN MARUŠIČ[1] ROMAN NEDELA[2]

IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1111 Ljubljana
Slovenija

Katedra Matematiky
Univerzita Mateja Bela
975 49 Banska Bystrica
Slovakia

Abstract

Transitive permutation groups having a non-self-paired suborbit of length 2 are investigated via the corresponding orbital graphs. If G is such a group and X is the orbital graph associated with a suborbit of length 2 of G, which is not self-paired, then X has valency 4 and admits a vertex- and edge- but not arc-transitive action of G. There is a natural balanced orientation of the edge set of X induced and preserved by G. An analysis of the properties of this oriented graph is performed using a variety of graph-theoretic tools, resulting in some partial results on the point stabilizer of G in the case when X is connected. In particular, the point stabilizer must be a 2-group generated by $h \geq 1$ involutions. Moreover, a characterization of the groups G is obtained in the case of point stabilizers of order 8 and in the case of abelian point stabilizers. Finally, a graphical realization of such actions is given, that is an infinite family of tetravalent graphs admitting a vertex and edge but not arc-transitive action with vertex stabilizers $\mathbb{Z}_2^h$, $h \geq 1$, as well as an infinite family of such graphs with vertex stabilizer D_8, the dihedral group of order 8, is constructed.

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1 Introduction

Throughout this paper by a *graph* we mean an ordered pair (V, E), where V is a finite nonempty set and E is a symmetric irreflexive relation on V whose transitive closure is the universal relation on V. Throughout this paper graphs are thus assumed to be finite and connected. By an *oriented graph* we mean an ordered pair (V, A), where V is a finite nonempty set and A, the set of *arcs*, is an asymmetric relation on V. Furthermore, all groups are assumed to be finite. For a graph X, we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\,X$ denote the respective sets of vertices, edges and arcs, and the automorphism group of X. For graph-theoretic and group-theoretic terms not defined here we refer the reader to [1, 2, 5, 13].

When investigating the structure of graphs admitting transitive actions of subgroups of automorphisms one often relies on results which are purely group-theoretic in nature. On the other hand, a graph-theoretic language may often provide useful insights in the study of various problems in permutation groups. A particular case of such a fruitful interplay of group-theoretic and graph-theoretic concepts is dealt with in this paper and is motivated by results on point stabilizers of transitive permutation groups having suborbits of small length [8, 10, 12, 14] and by recent research in vertex- and edge- but not arc-transitive graphs – see [6, 7, 11, 15].

It is easily seen that the class of transitive permutation group having a self-paired connected suborbit of length 2 consists of dihedral groups alone. On the other hand, we have the classical results on point stabilizers by Tutte [12] and Wong [14] in the case of self-paired suborbits of length 3, and by Sims [10] in the case of primitive permutation groups and self-paired suborbits of length 4. Some partial results for other lengths are also known (see [8]). From a group-theoretic point of view, the aim of this paper is to study the structure of transitive permutation groups having a non-self-paired suborbit of length 2 relative to which the corresponding orbital graph (see below) is connected, with the emphasis on their point stabilizers. From a graph-theoretic point of view this is equivalent to the study of the structure of graphs of valency 4 admitting a vertex- and edge- but not arc-transitive group action, in particular, the study of the corresponding vertex stabilizers.

Let G be a transitive permutation group acting on a set V and let $v \in V$. Note that there is a 1-1 correspondence between the set of orbits of the stabilizer G_v on V, that is the set of *suborbits* of G, and the set of *orbitals* of G, that is the set of orbits in the natural action of G on $V \times V$, with the trivial suborbit $\{v\}$ corresponding to the diagonal $\{(v, v) : v \in V\}$. The paired orbital Δ^t of an orbital Δ is the orbital $\{(v, w) : (w, v) \in \Delta\}$. If $\Delta^t = \Delta$ we say that Δ is a *self-paired orbital.* A *self-paired suborbit* of G is a suborbit which corresponds to a self-paired orbital. The *orbital graph* $X(G, \Delta)$ *of G relative to* Δ, is the graph with vertex set V and arc set Δ. The suborbit Δ is said to be *connected* if the underlying undirected graph of $X(G, \Delta)$ is connected. Of course, if $\Delta = \Delta^t$ is a self-paired orbital then $X(G, \Delta)$ can be viewed as an undirected graph which admits a vertex- and arc-transitive action of G. On the other hand, if $\Delta \neq \Delta^t$ is a non-self-paired orbital then $\Delta \cap \Delta^t = \emptyset$ and the underlying undirected graph of the orbital graph $X(G, \Delta)$ admits a vertex- and edge- but not arc-transitive action of G. Conversely, given an edge $[u, v]$ of a graph X of valency $2d$ admitting a vertex- and edge- but not arc-transitive action of some group $G \leq \mathrm{Aut}\, X$, the two arcs (u, v) and (v, u) give rise (via the action of G) to two oriented graphs, namely the orbital graphs of G relative to two paired orbitals, where the length of the corresponding two suborbits is d.

As already mentioned, a particular instance of this situation, the case $d = 2$, will be considered in this paper. We are going to study the structure of transitive permutation groups having a non-self-paired connected suborbit of length 2 via the corresponding orbital graphs (and the underlying undirected graphs). In other words, the objects of our interest are (finite connected) graphs of valency 4 admitting a vertex- and edge- but not arc-transitive group action. The commonly accepted term, at least among graph theorists, for such group actions is $\frac{1}{2}$-*transitive* group actions, a somewhat unhappy choice from a group-theoretic point of view. It follows from the comments of the previous paragraph that a subgroup G of the automorphism group of a graph X, acting vertex- and edge- but not arc-transitively, must necessarily have two orbits on the arc set, with each orbit containing an arc corresponding to each edge; that is, in the terminology of [13, p.24], it acts $\frac{1}{2}$-transitively on the arc set having two orbits (of equal length). We shall thus say, albeit in a slight dissonance with this usual meaning in the literature, that a graph X is $\frac{1}{2}$-*transitive* provided it is vertex- and edge- but not arc-transitive, that is, provided its automorphism group acts transitively on the sets of vertices and edges, but intransitively on the set of arcs. More generally, by a $\frac{1}{2}$-*transitive* action of a subgroup $G \leq \mathrm{Aut}\, X$ on X we shall mean a vertex- and edge- but not arc-transitive action on X. In this case we shall say that the graph X is

$(G, \frac{1}{2})$-*transitive*. In particular, since vertex stabilizers G_v, $v \in V(X)$, play an important role in this paper, we shall say that X is $(G, \frac{1}{2}, H)$-*transitive* provided $H = G_v$ for some $v \in V(X)$. Note that H must be a 2-group. Firstly, since X has valency 4, no prime greater than 3 may divide the order of H, and secondly, the existence of elements of order 3 in H is excluded in view of the $\frac{1}{2}$-transitivity of the action of G. We shall say that the G-*height* $h_G(X)$ of X is equal to h, where $|G_v| = 2^h$, $v \in V(X)$.

Throughtout this paper we shall, for each $n \geq 3$, use the symbol $\mathbb{Z}_2^n \rtimes \mathbb{Z}_n$ to denote the semidirect product where the generator 1 of $\mathbb{Z}_n$ cyclically permutes the coordinates of each vector in $\mathbb{Z}_2^n$. Let us now state the main results of this paper.

Theorem 1.1 *Let G be a finite transitive permutation group having a non-self-paired connected suborbit of length 2, and let H be a point stabilizer of G. Then there exists an integer $h \geq 1$ such that*

(i) H is generated by h involutions $\tau_1, \ldots, \tau_h$; and

(ii) for any $i \in \{0, \ldots, h-1\}$, and $j \in \{1, \ldots h-i\}$, the subgroup $\langle \tau_{i+1}, \ldots, \tau_{i+j} \rangle$ has order 2^j.

(iii) for any $i \in \{0, \ldots, h-1\}$, $k \in \{0, \ldots, i\}$ and $j \in \{1, \ldots, h-i\}$, the subgroups $\langle \tau_{i+1}, \ldots, \tau_{i+j} \rangle$ and $\langle \tau_{k+1}, \ldots, \tau_{k+j} \rangle$ are isomorphic.

Theorem 1.2 *Let $h \geq 1$ be an integer and let G be a group generated by two non-involutory elements a and b such that*

(i) $(a^i b^{-i})^2 = 1$ and $a^i b^{-i} \neq 1$ for $i \in \{1, \ldots, h\}$;

(ii) the order of b is greater than $2h$; and

(iii) $ab^i \neq b^i a$ for $i \in \{1, \ldots, h-1\}$.

Then there exists a finite set V on which G acts in such a way that it has a non-self-paired connected suborbit of length 2. Moreover, $G_v \cong \mathbb{Z}_2^h$ for $v \in V$ unless $G \cong \mathbb{Z}_2^h \rtimes \mathbb{Z}_h$ when $G_v \cong \mathbb{Z}_2^{h-1}$.

Theorem 1.3 *Let G be a finite transitive permutation group on a set V having a non-self-paired connected suborbit of length 2, and let G_v, $v \in V$, be abelian. Then there exist an integer $h \geq 1$ such that $G_v \cong \mathbb{Z}_2^h$ and non-involutory generators a and b of G satisfying the following conditions*

(i) $(a^i b^{-i})^2 = 1$ *and* $a^i b^{-i} \neq 1$ *for* $i \in \{1, \dots h\}$;

(ii) the order of b is greater than h; and

(iii) $ab^i \neq b^i a$ *for* $i \in \{1, \dots, h-1\}$.

We remark that the above two theorems give an almost complete classification of finite transitive permutation groups having a non-self-paired connected suborbit and an abelian point stabilizer. The proofs of Theorems 1.1, 1.2 and 1.3 will be carried out in Section 3 and Section 4, respectively, by first restating these theorems in graph-theoretic terms (Theorems 3.5 and 4.3). Furtherome, for each $h \geq 1$, we shall provide a graphical realization of group actions with non-self-paired connected suborbits of length 2 and abelian point stabilizer, by giving a construction of an infinite family of tetravalent graphs admitting a $\frac{1}{2}$-transitive group action with vertex stabilizer $\mathbb{Z}_2^h$ in Section 4. Also, an infinite family of tetravalent graphs admitting a $\frac{1}{2}$-transitive group action with vertex stabilizer D_8 will be constructed in Section 5, thus showing the existence of transitive permutation groups with non-self-paired suborbits of length 2 and having nonabelian point stabilizers.

In order to accomplish these tasks, certain graph-theoretic tools will be needed. Two paired operators Pl and Al acting on graphs, introduced in Section 2, are of particular importance. They will enable us to associate with each graph of valency 4 admitting a $\frac{1}{2}$-transitive group action of height $h \geq 1$, a particular Cayley graph on a group generated by two elements. The study of the properties of these Cayley graphs is the principal content of Section 3 and provides the machinery needed in the proofs of the above three theorems.

2 Pl and Al operators on oriented graphs

Let X be an oriented graph. A path P of X is called *transitive* if every vertex of P of valency 2 is the tail of one and the head of the other of its two incident arcs. A *transitive cycle* in X is a cycle in which a removal of an arc results in a transitive path. An even length cycle C in X is an *alternating cycle* if every other vertex of C is the tail and every other vertex of C is the head of its two incident arcs. By an *edge* of an oriented graph we mean an edge of the underlying undirected graph.

We are now going to introduce two operators on oriented graphs which will play a crucial role in our study of $\frac{1}{2}$-transitive group actions on tetravalent graphs. Let $X = (V, A)$ be an arbitrary oriented graph. The operator Pl is defined as follows. We let the *partial line graph* $Y = Pl(X)$ of X be the oriented graph with vertex set A such that there is an arc in Y from $x \in A$ to $y \in A$ in Y if and only if xy is a transitive 2-path in X.

If the arc set of Y decomposes into alternating 4-cycles no two of which intersect in more than one vertex, and if the maximum valency of Y is 4 (and so every vertex in Y has valency 2 or 4), we may also introduce the inverse operator Al. We let the vertex set of $Al(Y)$ be the set of alternating cycles (of length 4) in Y, with two such cycles adjacent in $Al(Y)$ if and only if they have a common vertex in Y. The orientation of the edges of $Al(Y)$ is inherited from that of the edges of Y in a natural way. Letting C_v and C_w be the two alternating 4-cycles in Y, corresponding to two adjacent vertices v and w in $Al(Y)$, we orient the edge $[v, w]$ in $Al(Y)$ from v to w if and only if the two arcs in Y with the tail in $u \in C_v \cap C_w$ have heads on C_w. Observe that $Al(Pl(X) = X$ for every balanced oriented graph X of valency 4. Moreover, $Pl(Al(Y)) = Y$ as long as the graph Y has the above assumed properties.

Let us remark that there are instances in this paper when these two operators are also applied to (undirected) graphs. This will occur whenever an accompanying oriented graph is (perhaps tacitly) associated with the undirected graph in question. A typical situation is presented by a tetravalent graph admitting a $\frac{1}{2}$-transitive group action and its two accompanying balanced oriented graphs, or by a Cayley graph arising from a set of non-involutory generators, for each of which one of the two possible orientation is prescribed.

Proposition 2.1 *If X is a balanced oriented 4-valent graph, then $Aut\, Pl(X) = Aut\, X$. Conversely, let Y be a balanced oriented graph of valency 4 such that the alternating cycles have length 4, no two intersect in more than one vertex, and they decompose the edge set. Then $Aut\, Al(Y) = Aut\, Y$.*

PROOF. Every automorphism of X permutes the edges of X (that is the edges of the underlying undirected graph) and thus it can be viewed as a permutation of the vertices of $Pl(X)$. Moreover it maps a transitive 2-path onto a transitive 2-path and thus it preserves the adjacency in $Pl(X)$. Hence $Aut\,X \leq Aut\,Pl(X)$. To see that also the reverse inclusion holds, observe that every automorphism of $Pl(X)$ permutes the alternating 4-cycles of $Pl(X)$ – which correspond to vertices of X – preserving, of course, the adjacency of these cycles. Besides, every automorphism of $Pl(X)$ preserves the arcs of $PL(X)$, that is the transitive 2-paths of X, and hence also the arcs of X. Therefore it must be induced by an automorphism of X. Hence $Aut\,Pl(X) = \mathrm{Aut}\ X$. To see that the second statement of Proposition 2.1 holds, we only need to take into account the fact that operators Al and Pl are inverses of each other. ■

Given a balanced oriented 4-valent graph X, the four arcs incident with a vertex in X give rise to an alternating 4-cycle C in $Pl(X)$, which may be thought of as the image of that vertex under Pl. As the next step we consider the second image of that vertex to be the subgraph $Pl(\bar{C})$ of $Pl^2(X)$, where $\bar{C}$ is the alternating 4-cycle C together with all the incident arcs in $Pl(X)$. To formalize the notion of the n-th image of a vertex it is more convenient to use the Al operator. The *n-th image of a vertex $v \in V(X)$* is the subgraph $U_n = U_n(v)$ of $Pl^n(X)$ for which $v = Al^n(U_n)$. It is easy to see that for a fixed n the graph U_n is uniquely determined, that is, it does not depend on X or on the choice of the particular vertex v. An alternative definition of U_n by means of Pl operator reads as follows. Set $U_0 = K_1$. For a given U_n let $\bar{U}_n$ denote the graph formed by adding two ingoing pendant arcs to each vertex of U_n with indegree 0 and by adding two outgoing arcs to each vertex of U_n with outdegree 0 . Then $U_{n+1} = Pl(\bar{U}_n)$.

The lemma below establishes some important properties of the graphs U_n, $n \in \mathbb{Z}^+ \cup \{0\}$. By an *$m$-alternating cycle of reduced length $l = 2n$* we mean a closed walk W in X of the form $P_1Q_1^{-1}P_2Q_2^{-1}\dots P_nQ_n^{-1}$, where P_i and Q_i are transitive paths of length m. In particular, 1-alternating cycles are precisely the alternating cycles in X. For a subgraph X' of X we let $V^+(X')$ denote the set of all vertices of X' with indegree 0, and similarly, by $V^-(X')$ we denote the set of all vertices of X' with outdegree 0.

Lemma 2.2 *Let $n \geq 1$ be a positive integer. Then the oriented graph U_n satisfies the following properties:*

(i) for each $j \in \{0, 1, \ldots, n\}$, U_n contains U_j as an induced subgraph;

(ii) there are no transitive cycles in U_n;

(iii) there are n-alternating cycles in U_n of reduced length 4;

(iv) $|V^+(U_n)| = 2^n = |V^-(U_n)|$ and $|V(U_n)| = (n+1)2^n$;

(v) $U_n - V^+(U_n) \cong 2U_{n-1} \cong U_n - V^-(U_n)$.

PROOF. To prove (i) it is sufficient to observe that the graphs U_j appear in U_n as j-th images of vertices of U_{n-j}.

Next, (ii) follows from the fact that the Al operator produces from a transitive cycle in U_j $(j = 1, \ldots, n)$ a transitive cycle in U_{j-1}. But then since U_0 contains no transitive cycles, there are no transitive cycles in U_n either.

To see that (iii) holds, we use induction on n. The statement is clearly valid for $n = 1$. Also it is easy to verify that an n-alternating cycle of reduced length 4 in U_n gives rise to a collection of $(n+1)$-alternating cycles of reduced length 4 in U_{n+1}.

To prove (iv), let $v(n) = |V(U_n)|$, $v^+(n) = |V^+(U_n)|$ and $v^-(n) = |V^-(U_n)|$. Since $U_{n+1} = Pl(\bar{U}_n)$, the vertices of outdegree 0 in U_{n+1} correspond to pendant outgoing arcs in $\bar{U}_n$. These arcs originate in $V^-(U_n)$, and hence $v^-(n+1) = v^-(n)$. Consequently, $v^-(n) = 2^n$. A similar argument yields $v^+(n) = 2^n$. Since $U_n = Al(U_{n+1})$, the vertices of U_n are in a 1-1 correspondence with the alternating 4-cycles in U_{n+1}. Counting the arcs in U_{n+1} we get

$$4v(n) = 2v^+(n+1) + 2(v(n+1) - v^+(n+1) - v^-(n+1)) = 2(v(n+1) - v^-(n+1)).$$

Taking into account that $v^-(n+1) = 2^{n+1}$ we obtain

$$v(n+1) = 2v(n) + 2^{n+1} \quad \text{and} \quad v(0) = 1.$$

These two equations determine the function $v(n)$ uniquely. On the other hand, the function $f(n) = 2^n(n+1)$ satisfies the above recursive equation and so $v(n) = f(n)$.

Finally, we show that $U_{n+1} - F \cong 2U_n$, for $F = V^-(U_{n+1})$. (The case when $F = V^+(U_{n+1})$ is left to the reader.) We use induction on n. Clearly, the statement holds true for $n = 1$. By the induction hypothesis $U_n - F$ consists of two copies of U_{n-1}. The arc set of U_n decomposes into two sets formed by arcs with origins in the two respective copies U^i_{n-1} $(i = 1, 2)$ of U_{n-1}. Recall that $U_{n+1} = Pl(\bar{U}_n)$. Clearly, $\bar{U}^i_{n-1} \subset \bar{U}_n$ and the arc sets of these two subgraphs are disjoint. Moreover, every arc in $\bar{U}^i_{n-1}$ terminates at a vertex of outdegree 2 in $\bar{U}_n$. Hence the corresponding vertices in $U^i_n = Pl(\bar{U}^i_{n-1})$ also have outdegree 2. Thus $U^1_n \cup U^2_n \subseteq U_{n+1} - V^-(U_{n+1})$. On the other hand, by Lemma 2.2 (iv), we have $|V(U^1_n)| + |V(U^2_n)| = |V(U_{n+1})| - |V^-(U_{n+1})|$. Therefore $U^1_n \cup U^2_n = U_{n+1} - V^-(U_{n+1})$, completing the proof of Lemma 2.2. ■

3 Characterizing $\frac{1}{2}$-transitive actions relative to their height

Concepts peculiar to oriented graphs, such as transitive and alternating cycles, may be extended to graphs admitting $\frac{1}{2}$-transitive group actions, via the orientation of the edge set induced by the corresponding group of automorphisms in the following way.

A $(G, \frac{1}{2})$-transitive graph X of valency 4, where $G \leq \mathrm{Aut}\, X$, gives rise to two oriented graphs – with X as their underlying graph – namely, the two orbital graphs of the action of G on $V(X)$ relative to two paired orbitals of length 2. Let $D_G(X)$ be one of these two graphs fixed from now on. For $u, v \in V(X)$ and shall say that u is the *tail* of (u, v), and that v is the *head* of (u, v) if (u, v) is an arc in $D_G(X)$. We remark that by the *G-orientation* of the edges of X, that is by the orientation induced by the $\frac{1}{2}$-transitive action of G, we shall always mean the corresponding orientation of the edges in $D_G(X)$. A path P in X is a *G-transitive path* if it is a transitive path in $D_G(X)$. A cycle of X is a *G-transitive cycle*, and a *G-alternating cycle*, respectively, provided it is a transitive cycle, and an alternating cycle in $D_G(X)$. When the particular group G is clear from the context, the symbol G will sometimes be omitted in the above concepts. It transpires that all G-alternating cycles in X have the same length and form a decomposition of the edge set of X [6, Proposition 2.4]; half of this length is denoted by $r_G(X)$ and is called the *G-radius* of X. Moreover, any two adjacent G-alternating cycles of X intersect in the same number of vertices. This number, called the *G-attachment number* of X, divides $2r_G(X)$ [6, Proposition 2.6].

For a group G and a generating set S of G such that $1 \notin S = S^{-1}$, the *Cayley graph* $Cay(G,S)$ *of* G *relative to* S has vertex set G and edges of the form $[g, gs], g \in G, s \in S$. Note that the group G acts on $Cay(G,S)$ by left regular action as a regular subgroup of $\mathrm{Aut}\, Cay(G,S)$. In this context we will throughout this paper always identify G and any of its subgroups with its left regular action. The accompanied right translation of any subgroup H of G on itself will be denoted by H^*. Let $S = \{a, a^{-1}, b, b^{-1}\}$, where a and b are non-involutory elements of G. We let $Cay(G; a, b)$ denote the (undirected) graph $Cay(G,S)$ together with the implicit orientation inherited from the oriented Cayley graph $Cay(G, \{a, b\})$.

Let X be a graph together with an inhereted orientation given via an oriented graph X', whose underlying graph it is. Then we let the *partial line graph* $Y = Pl(X)$ of X be the underlying graph of $Pl(X')$. In a similar fashion, also the operator Al may be extended to graphs possessing an implicit orientation of their edge sets. Again, these two operators are inverses of each other also for graphs.

Proposition 3.1 *Let X, Y be graphs of valency 4. Then*

(i) If X is $(G, \frac{1}{2}, H)$-transitive for some $H \leq G \leq Aut X$ and $|H| > 2$, then $Pl(X)$ is $(G, \frac{1}{2}, K)$-transitive with G-radius 2 for some $K \leq H$ of index 2 in H. Conversely, if Y is $(G, \frac{1}{2}, K)$-transitive with G-radius 2 and G-attachment number 1 for some $K \leq G \leq Aut Y$ such that $|K| \geq 2$, then $X = Al(Y)$ is $(G, \frac{1}{2}, H)$-transitive for some $H \leq G$ such that $[H : K] = 2$ and thus $|H| > 2$.

(ii) If X is $(G, \frac{1}{2}, \mathbb{Z}_2)$-transitive for some nonabelian subgroup $G \leq Aut X$, then there are non-involutory generators a and b of G such that $(ab^{-1})^2 = 1$ and $Pl(X) \cong Cay(G; a, b)$. Conversely, if Y is such a Cayley graph then $Al(Y)$ is $(G, \frac{1}{2}, \mathbb{Z}_2)$ transitive.

Proof. To prove the first part of (i) assume that X is $(G, \frac{1}{2}, H)$-transitive with $H \leq G \leq \mathrm{Aut}\, X$ and $|H| > 2$. It follows from Proposition 2.1 that G is a subgroup of the automorphism group of $Y = Pl(X)$ and that it preserves the orientation of Y induced by X. Since G acts transitively on edges of X, it acts transitively on vertices of Y. Let xy and xz be two transitive 2-paths in X with the arc x in common. To see that G acts transitively on edges of Y it is sufficient to realize that the assumption $|H| > 2$

implies the existence of an element of G fixing the arc x and interchanging the arcs y and z in X. Since Y has twice as many vertices as X, it follows that $[H:K]=2$, where K is a vertex stabilizer in the action of G on Y.

To prove the second part of (i) assume that $Y = Pl(X)$ is $(G, \frac{1}{2}, K)$ transitive for some $K \leq G \leq \mathrm{Aut}\, Y$. It follows from Proposition 2.1 that $G \leq Aut\, X$ and that G preserves the G-orientation of the edges of X induced by Y. Since G acts transitively on G-alternating 4-cycles and vertices of Y, it acts transitively on vertices and edges of X, respectively. Thus X is $(G, \frac{1}{2})$-transitive. Since X has half the number of vertices of Y, a vertex stabilizer of the action of G must have twice as many elements as K.

To prove the first part of (ii), we conclude as in (i) that G acts on $Y = Pl(X)$ as a group of automorphisms and preserves the orientation of the edges in Y induced by X. Since G acts transitively on edges of X, the action of G on Y is transitive on vertices. Since Y has twice as many vertices as X and since the stabilizer of the action on X is isomorphic to $\mathbb{Z}_2$, we conclude that G acts regularly on the set of vertices of Y and so Y is a Cayley graph of G. Let xy and xz be two transitive 2-paths in X. The $\frac{1}{2}$-transitivity implies the existence of automorphisms a, b mapping the arc x onto y and z, respectively. Clearly, both a and b are non-involutory. Thus $Y \cong Cay(G; a, b)$. Besides, ab^{-1} fixes the head of x. Hence, $(ab^{-1})^2 = 1$.

To prove the converse statement, let $Y = Cay(G; a, b)$ satisfy the assumptions. The relation $(ab^{-1})^2 = 1$ gives rise to a decomposition of the set of edges into G-alternating 4-cycles. Moreover, the existence of two G-alternating 4-cycles in Y intersecting in two vertices forces G to be abelian, contradicting the assumption. Thus $X = Al(Y)$ is well defined. As above we deduce that G acts on X as an orientation preserving group of automorphisms. Since the action of G on Y is transitive on G-alternating 4-cycles and on vertices, the action on X is transitive on vertices and edges of X, in other words X is $(G, \frac{1}{2})$-transitive. Since X has half the number of vertices of Y, it follows that a vertex stabilizer of the action of G on X is isomorphic to $\mathbb{Z}_2$. ■

Let X be the Cayley graph $X = Cay(G; a, b)$ of a group G generated by two non-involutory generators a and b. We say that X *satisfies the property* $Cyc(h)$ for some integer $h \geq 1$, if a system of h irreducible relations of the form $T_i U_i^{-1} V_i W_i^{-1} = 1$, $i = 1, 2, \ldots, h$ is satisfied in G, where T_i, U_i, V_i, W_i are words of length i consisting of letters a and b but not containing their inverses a^{-1} and b^{-1}.

The lexicographic products $C_n[K_2^c]$, $n \geq 3$, are the only 4-valent graphs admitting a $\frac{1}{2}$-transitive group action with respect to which the radius and the attachment number both equal 2. Thus the condition on the G-attachment number in Proposition 3.1 (i) can be replaced by the condition $Y \ncong C_n[K_2^c]$, $n \geq 3$. In view of this fact any graph of the form $Pl^j(C_n[K_2])$, where $n \geq 3$ and $0 \leq j \leq n-1$, will be called *degenerate*.

Theorem 3.2 *(i) Let X be a $(G, \frac{1}{2})$-transitive 4-valent graph with G-height $h \geq 1$ for some subgroup $G \leq Aut X$. Then there exist non-involutory generators a and b of G such that $Pl^h(X) \cong Cay(G; a, b)$ and $Pl^h(X)$ contains U_h as an induced subgraph, and in particular $Pl^h(X)$ satisfies the property $Cyc(h)$.*

(ii) Conversely, let G be a group generated by two non-involutory elements a and b such that $Y = Cay(G; a, b)$ is not degenerate and contains U_h as an induced subgraph for some positive integer h. Then $Al^h(Y)$ is a $(G, \frac{1}{2})$-transitive graph of G-heigth h.

PROOF. To prove (i) we first apply the operator Pl on the graph X successively $h-1$ times. By Proposition 3.1 (i), the graph $Pl^{h-1}(X)$ is $(G, \frac{1}{2}, \mathbb{Z}_2)$-transitive. By Proposition 3.1 (ii), there exist non-involutory generators a and b of G such that the graph $Y = Pl^h(X)$ is isomorphic to the Cayley graph $Cay(G; a, b)$. Moreover, Y (being equal to $Pl^h(X)$) contains an n-th image of every vertex of X. Finally, Lemma 2.2 (iii) implies that the condition $Cyc(h)$ is satisfied in Y.

We now prove (ii). Since Y contains U_h as an induced subgraph for some $h \geq 1$, its edge set decomposes into G-alternating 4-cycles. Since Y is not degenerate, Proposition 3.1 (ii) implies that $Al(Y)$ is a $(G, \frac{1}{2}, \mathbb{Z}_2)$-transitive graph. Since $Al(Y)$ contains U_{h-1} as an induced subgraph and is not degenerate we can repeat the procedure provided $h > 1$. Using Proposition 3.1 (i) at each step we derive that $Al^h(Y)$ is $(G, \frac{1}{2})$-transitive of height h. ■

Let us discuss the degenerate graphs in more detail. The graph $X_n = C_n[K_2^c]$, $n \geq 3$, may be thought of as being formed from a cycle C of length n by replacing each vertex v of C by two vertices v_0 and v_1 and joining u_i to v_j by an edge if and only if the vertices u and v are adjacent in C. If we choose one of the two transitive orientations of C then this orientation induces an orientation of X_n in the obvious way. Clearly, the stabilizer of a vertex u in X_n contains all the transpositions $\tau_v = (v_0, v_1)$, where $v \neq u$. Moreover, these transpositions preserve the prescribed orientation and they generate a group $H = \langle\{\tau_v, v \in V(C) - u\}\rangle$ is isomorphic to $\mathbb{Z}_2^{n-1}$. Furthemore, there is a rotary automorphism ρ of X_n mapping every vertex to one of the two of its successors. Now, it is easy to realize that X_n is a $(G_n, \frac{1}{2}, \mathbb{Z}_2^{n-1})$-transitive graph. By Proposition 3.1 the graph $Pl^j(X_n)$ is $(G_n, \frac{1}{2}, \mathbb{Z}_2^{n-1-j})$-transitive for $0 \leq j \leq n-2$. The automorphisms in G_n can be identified with the elements of the semidirect product $H_n = \mathbb{Z}_2^n \rtimes \mathbb{Z}_n$. It may be checked that H_n can be generated by two non-involutory elements a and b satisfying the relations $(a^i b^{-i})^2 = 1$, for $i = 1, \ldots, n-1$. Indeed, if we set $a = (\epsilon_1, 1)$ and $b = (\epsilon_2, 1)$, where ϵ_2 is the image of ϵ_1 under the action of $1 \in \mathbb{Z}_n$ on $\mathbb{Z}_2^n$, then a direct computation yields the required relations. Set $Y = Cay(H_n; a, b)$. By Proposition 3.1 the graph $Z = Al^j(Y)$, $j \in \{1, \ldots, n-1\}$, is a $(H_n, \frac{1}{2}, \mathbb{Z}_2^j)$-transitive graph. In fact, one can prove that Z is isomorphic to $Pl^{n-1-j}(X_n)$.

When dealing with examples of tetravalent graphs admitting a $\frac{1}{2}$-transitive group action it helps to have a simple criterion for recognizing that a particular graph is not degenerate. The following lemma may be useful for that purpose. The proof is straightforward.

Lemma 3.3 *Let $X \cong Pl^h(C_n[K_2^c])$ for some $h \geq 0$ and $n \geq 3$. Then X contains two transitive paths of length $h+2$ with common heads and tails and having no other vertex in common.*

We may now give a characterization of $\frac{1}{2}$-transitive group actions of height $h \in \{1, 2, 3\}$.

Corollary 3.4 *Let X be a $(G, \frac{1}{2})$-transitive graph of height $1 \leq h \leq 3$. Then there exist non-involutory generators a and b such that $Pl^h(X) \cong Cay(G; a, b)$ and a, b satisfy the following relations:*

(i) $ab^{-1}ab^{-1} = 1$ *if* $h = 1$*;*

(ii) $ab^{-1}ab^{-1} = a^2b^{-2}a^2b^{-2} = 1$ *if* $h = 2$*;*

(iii) $ab^{-1}ab^{-1} = a^2b^{-2}a^2b^{-2} = a^3b^{-3}a^3b^{-3} = 1$ *or*
$ab^{-1}ab^{-1} = a^2b^{-2}a^2b^{-2} = a^3b^{-3}a^3b^{-1}a^{-1}b^{-1} = 1$, *if* $h = 3$.

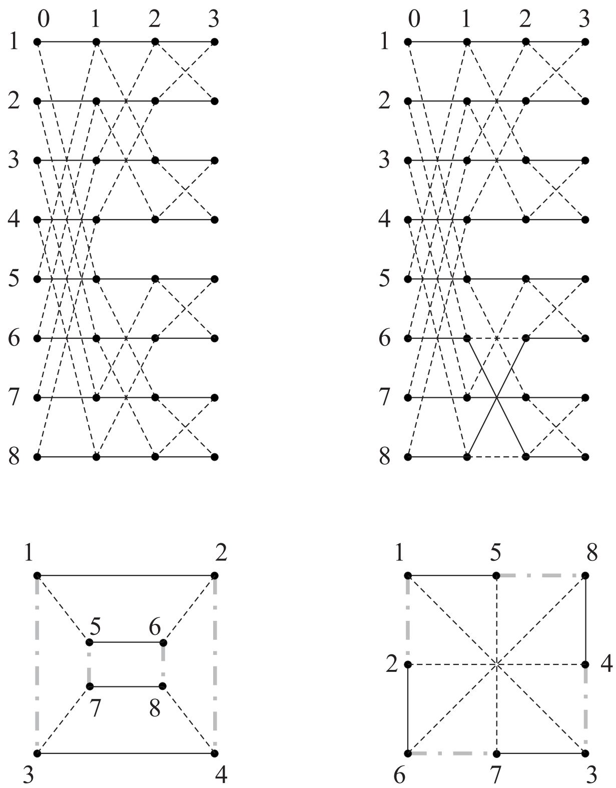

Figure 1: Two colorings of U_3 and the associated Cayley graphs– the edges are oriented from 'left' to 'right'.

PROOF. In view of Theorem 3.2 (i) it is sufficient to prove that if the Cayley graph $Y = Cay(G; a, b)$, where a and b are non-involutory generators of G, contains U_h, $h \in \{1, 2, 3\}$, as a subgraph, then a and b satisfy relations (i), (ii) and (iii). The case $h = 1$ is trivial. To prove the cases $h = 2$ and $h = 3$ we ispect all possible assignments of arcs of the graphs U_2 and U_3 by elements a and b. Taking into account that every G-alternating 4-cycle can be colored by a and b in a unique way (up to interchanging a with b), and using the automorphisms of U_2 and U_3 we get a unique coloring of U_2, and two colorings of U_3 (see Figure 1). The relations can be now derived from the colored graphs. ■

We are now going to prove Theorem 1.1 by restating it first in a graph-theoretic language.

Theorem 3.5 *Let G be a group acting $\frac{1}{2}$-transitively on a 4-valent graph X with vertex stabilizer H. Then there exists an integer $h \geq 1$ such that*

(i) H is generated by h involutions $\tau_1, \ldots, \tau_h$;

(ii) for each $i \in \{0, \ldots, h-1\}$ and each $j \in \{1, \ldots, h-i\}$, the subgroup $\langle \tau_{i+1}, \ldots, \tau_{i+j} \rangle$ has order 2^j.

(iii) for any $i \in \{0, \ldots, h-1\}$, $k \in \{0, \ldots, i\}$ and $j \in \{1, \ldots, h-i\}$, the subgroups $\langle \tau_{i+1}, \ldots, \tau_{i+j} \rangle$ and $\langle \tau_{k+1}, \ldots, \tau_{k+j} \rangle$ are isomorphic.

PROOF. Clearly, because of $\frac{1}{2}$-transitivity, H is a 2-group, say of order 2^h for some integer $h \geq 1$. By Theorem 3.2 (i) there are non-involutory generators a and b of G such that the graph $Pl^h(X)$ is isomorphic to the Cayley graph $Cay(G, a, b)$. In what follows we shall restrict our analysis to a copy of U_h arising as an h-th image of a given vertex v in X such that $H = G_v$. Apart from the oriented structure in U_h we also have a coloring of its arcs induced by the two generators a and b. The respective colors will be called *red* and *blue*.

Since $H = G_v$ it follows that H fixes U_h setwise. In view of Lemma 2.2 (ii), every monochromatic connected component in U_h is a (transitive) path joining a vertex in $V^+ = V^+(U_h)$ with a vertex in $V^- = V^-(U_h)$. Let $\mathcal{B}$ and $\mathcal{R}$ denote the respective sets of all such blue and red paths. Clearly H permutes elements of $\mathcal{B}$ as well as those of $\mathcal{R}$. Since H is a subgroup of the group G acting regularly on $Pl^h(X)$, the action of H on each of these two sets is semiregular. But $|H| = 2^h = |V^+| = |\mathcal{B}| = |\mathcal{R}|$, and so both of these actions are regular. We may think of the vertices of U_h as arranged in a $(2^h, h+1)$-array with the rows corresponding to the 2^h blue paths and the $h+1$ columns $\mathcal{C}_i$, $i = 0, 1, \ldots, h$, consisting of all the vertices at distance i along transitive monochromatic paths from the set $\mathcal{C}_0 = V^+$. Clearly, the sets $\mathcal{C}_i$ are precisely the orbits of H on $V(U_h)$. We will now study the regular action of H on $\mathcal{B}$ (rows of the array) in detail. In order to do that we define an associated graph Y as follows. The vertex set of Y is $\mathcal{B}$ with two blue paths in $\mathcal{B}$ being adjacent in Y if and only if there is a red arc in U_h joining them in U_h. Note that if x is a red arc joining a blue path P to a blue path

Q then there is a unique red arc y joining Q to P. The corresponding four vertices form a G-alternating 4-cycle in U_h. (Clearly, on each G-alternating 4-cycle the colors alternate too, see Figure 2 for $h = 3$.) Thus each edge in Y is associated with a pair of red arcs whose heads are in the same column $\mathcal{C}_i$ for some $i \in \{1, 2, \ldots, h\}$. We may thus define a coloring of the edge set of Y by assigning color $i \in \{1, 2, \ldots, h\}$ to all of the edges in Y arising from pairs of red arcs with heads in $\mathcal{C}_i$. Clearly, each vertex of Y is incident with an edge of each color, thus giving rise to anedge decomposition of Y. Recall that the group H is regular on the vertex set of Y. Besides, it preserves G-alternating 4-cycles in U_h and so it preserves adjacency of vertices in Y. Moreover, since the columns $\mathcal{C}_i$ are orbits of the action of H on U_h, colors of the edges in Y are also preserved by H. In fact, the orbits of the action of H on $E(Y)$ are precisely the color classes. Note that since U_h is connected, the graph Y is connected too. It follows that Y is a Cayley graph of a group isomorphic to H, with respect to the generators τ_i, $i \in \{1, \ldots, h\}$, given by the coloring of the edges of Y. More precisely, $\tau_i(u) = w$ if and only if $[u, w]$ is an edge in Y colored by i. In particular, all of these generators are distinct involutions. For if $\tau_i = \tau_j$ for some $i < j$ then by the definition of τ_i and τ_j we have the identity $ab^{j-i} = b^{j-i}a$. However, this is impossible for the existence of U_h in $Pl^h(X) \cong Cay(G; a, b)$ implies, by Lemma 2.2 (v), that all the words in a and b (not containing their inverses) of length at most h are distinct. This proves (i).

The proof of (ii) is now easily at hand. Applying Lemma 2.2 (v) it may be seen that a removal of the column $\mathcal{C}_h$ from U_h gives rise to two copies of U_{h-1}. In the graph Y this operation corresponds to the removal of all the edges colored with color h, yielding two connected components. Thus there must exist a subgroup of index 2 in H which is generated by elements $\tau_1, \ldots, \tau_{h-1}$. This subgroup acts regularly on each of the two copies of U_{h-1} above. By repeating this operation $h - (i + j)$ times we get $|\langle \tau_1, \ldots \tau_{i+j} \rangle| = 2^{i+j}$. Let us consider a connectivity component Z of $U_h - \{C_{i+j+1}, \ldots, C_h\}$, that is the graph obtained from U_h by a removal of the columns $C_{i+j+1}, \ldots C_h$. Note that Z is isomorphic to U_{i+j}. Now let Y' be the subgraph of Y corresponding to Z. It follows that the subgroup $\langle \tau_1, \ldots \tau_{i+j} \rangle$ acts regularly on Y'. Applying Lemma 2.2 (v) we again have that $U_{i+j} - C_1$ consists of two copies of U_{i+j-1}. In Y' the corresponding operation consists of removing all edges colored with color 1. Consequently, Y' splits into isomorphic connectivity components. By repeating this operation i times we obtain $|\langle \tau_{i+1}, \ldots \tau_{i+j} \rangle| = 2^j$.

Finally, to prove (iii) let us denote by $H_{i,j}$ and by $H_{k,j}$ the respective subgroups $\langle\tau_{i+1},\ldots,\tau_{i+j}\rangle$ and $\langle\tau_{k+1},\ldots,\tau_{k+j}\rangle$ and prove that $H_{i,j}\cong H_{k,j}$. Note that Lemma 2.2 (v) implies

$$U_h-\{C_1,\ldots,C_i,C_{i+j+1},\ldots,C_h\}\cong 2^{h-j}U_j\cong U_h-\{C_1,\ldots,C_k,C_{k+j+1},\ldots,C_h\}.$$

Let us choose connectivity components $U'_j\subseteq U_h-\{C_1,\ldots,C_i,C_{i+j+1},\ldots,C_h\}$ and $U''_j\subseteq U_h-\{C_1,\ldots,C_k,C_{k+j+1},\ldots,C_h\}$. Then U'_j and U''_j are j-th images of vertices u' and u'' of the $(G,\frac{1}{2})$-transitive graph $Al^j(Pl^h(X))$. As above we derive that the groups $H_{i,j}$ and $H_{k,j}$ act regularly on the first columns U'_j and U''_j, respectively. Thus $H_{i,j}\cong G_{u'}\cong G_{u''}\cong H_{k,j}$, completing the proof of Theorem 3.5. ∎

In view of Theorem 3.5 let us now revisit the case of height $h\in\{1,2,3\}$ discussed in Corollary 3.4. Clearly, $\mathbb{Z}_2$ and $\mathbb{Z}_2\times\mathbb{Z}_2$ are the only groups satisfying conditions (i) and (ii) in Theorem 3.5 for $h=1$ and $h=2$, respectively. Let $h=3$. If G_v is abelian then clearly $G_v\cong\mathbb{Z}_2^3$. If G_v is not abelian then, by [4, p.134], it is either dihedral or the group of quaternions. However, Theorem 3.5 excludes the latter possibility, because the group of quaternions does not contain three involutions. On the other hand, condition (ii) is satisfied for $\tau_1=(1,3)$, $\tau_2=(1,3)(2,4)$ and $\tau_3=(1,2)(3,4)$. Thus the group $\langle\tau_1,\tau_2,\tau_3\rangle\cong D_8$ satisfies the assumptions of Theorem 3.5. We conclude that either $G_v\cong\mathbb{Z}_2^3$ or $G_v\cong D_8$. The constructions given in Sections 4 and 5 show that both possibilities do occur. To summarize we have the folowing corollary.

Corollary 3.6 *Let X be a $(G,\frac{1}{2})$-transitive graph for some $G\le Aut X$ of G-height $h\in\{1,2,3\}$ and let $v\in V(X)$. Then $G_v\cong\mathbb{Z}_2$ if $h=1$, $G_v\cong\mathbb{Z}_2^2$ if $h=2$, and $G_v\cong\mathbb{Z}_2^3$ or $G_v\cong D_8$ if $h=3$.*

4 Abelian case

The lemma below follows from the general theory of group actions. We omit the proof.

Lemma 4.1 *Let $X=Cay(G,Q)$ be a Cayley graph. Then O is an orbit of G on $V(X)$ if and only if O is an orbit of a subgroup $H^*\le G^*$ in the right* *regular action of G on $V(X)$. In particular, the set of words in G induced by the walks originating and terminating in O forms a subgroup of G isomorphic to H.*

Proposition 4.2 *Let X be a 4-valent graph admitting a $\frac{1}{2}$-transitive action of $G \le Aut\, X$ having height $h = h_G(X) \ge 1$ and let $v \in V(X)$. Then there exist non-involutory elements $a, b \in G$ generating G such that the following statements are equivalent.*

(i) G_v *is abelian;*

(ii) $G_v \cong \mathbb{Z}_2^h$;

(iii) $(a^i b^{-i})^2 = 1$ *for* $i \in \{1, \ldots, h\}$.

Proof. To see that (i) implies (ii), note that $|G_v| = 2^h$ and that, in view of Theorem 3.5, G_v is generated by h involutions. But then clearly G_v is elementary abelian.

Next we prove that (ii) implies (iii). By Theorem 3.2 (i) there exist non-involutory generators a and b of G such that $Pl^h(X) \cong Cay(G; a, b)$. Let $\sigma_i = a^i b^{-i} \in G$, $i \in \{1, \ldots, h\}$. As in the proof of Theorem 3.5 let us consider an h-th image $U_h \subseteq Pl^h(X)$ of $v \in V(X)$. With the notation introduced in the proof of Theorem 3.5 we have that the walks in $Pl^h(X)$ associated with the elements σ_i originating in the column $\mathcal{C}_0 \subseteq U_h$ also terminate in $\mathcal{C}_0$. Thus, applying Lemma 4.1, we have that, for each i, the left translation corresponding to σ_i fixes $\mathcal{C}_0$ setwise. Recall, that the setwise stabilizer of $\mathcal{C}_0$ in the left action of G is isomorphic to the vertex stabilizer G_v. Therefore σ_i is an involution for each $i \in \{1, \ldots, h\}$.

Finally we prove that (iii) implies (i). Firstly, let us show that $H = \langle \sigma_1, \ldots, \sigma_h \rangle$ is abelian. Indeed,

$$b^{-i}\sigma_i\sigma_j b^j = a^{j-i}b^{i-j} = \sigma_{j-i} = b^{j-i}a^{i-j} = b^{-i}\sigma_j\sigma_i b^j$$

proves that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all i and j. Hence $|\langle \sigma_1, \ldots, \sigma_h \rangle| = 2^h$. Now σ_i considered as a right translation sends $\mathcal{C}_0$ onto $\mathcal{C}_0$. Therefore every element $w \in H^*$ stabilizes $\mathcal{C}_0$. But the action of H^* on C_0 is regular and so Lemma 4.1 implies that the corresponding group H of left translations has $\mathcal{C}_0$ as an orbit. But then $H = G_v$. The result follows. ■

We may now prove Theorems 1.2 and 1.3 by restating them in graph-theoretic terms.

Theorem 4.3 *Let $h \geq 1$ be an integer and let G be a finite group generated by non-involutory elements a and b satisfying the following conditions.*

(i) $(a^i b^{-i})^2 = 1$ and $a^i b^{-i} \neq 1$ for $1 \in \{1, \dots, h\}$;

(ii) the order of b is greater than $2h$;

(iii) $ab^i \neq b^i a$ for $h > 1$ and $i \in \{1, \dots, h-1\}$.

Then $X = Al^h(Cay(G; a, b))$ is a $(G, \frac{1}{2})$-transitive graph with vertex stabilizer in G isomorphic to $\mathbb{Z}_2^h$ or $Al^{h-1}(CayG; a, b) \cong C_h[K_2^c]$ and $G \cong \mathbb{Z}_2^h \rtimes \mathbb{Z}_h$ with vertex stabilizers in G isomorphic to $\mathbb{Z}_2^{h-1}$. Conversely let X be a $(G, \frac{1}{2}, H)$-transitive graph, where $H \leq G \leq Aut X$ and H is an abelian 2-group of order 2^h. Then G is generated by non-involutory generators a and b satisfying conditions (i) and (iii) above, the order of b is greater than h and $H \cong \mathbb{Z}_2^h$.

Proof. The essential part in the proof of the first part of Theorem 4.3 is to show that the Cayley graph $Cay(G; a, b)$ contains a copy of U_h. Then the statement of our theorem will follow combining Theorem 3.2 (ii) and Proposition 4.2. As in the proof of Theorem 3.5 the colors of the edegs of $Cay(G; a, b)$ induced by the generators a and b will be called red and blue.

Firstly, note that as in the proof of Proposition 4.2, we can show that the group $H = \langle \sigma_1, \dots, \sigma_h \rangle$, where $\sigma_i = a^i b^{-i}$, is elementary abelian and that $|\langle \sigma_1, \dots, \sigma_h \rangle| = 2^h$. For each $i \in \{0, \dots, h\}$ let $C_i = Hb^i$. We shall prove that the union of all C_i induces a subgraph, call it Z, isomorphic to U_h in $Cay(G; a, b)$. Note that in view of (ii) the sets C_i are all distinct for if $C_i = C_j$ then $b^{j-i} \in H$, which is impossible as the elements of H are involutions and the order of b is greater than $2h$.

Using (i) we have that $C_i a = C_{i+1} = C_i b$. Hence given any $x \in C_i$, $i \in \{0, \dots, h-1\}$, the vertex $xa \in C_{i+1}$ and of course they are joined by a red edge. Moreover, the vertex $xab^{-1} = xba^{-1} \in C_i$ is joined with a red edge to the vertex $xb \in C_{i+1}$. As in the proof of Theorem 3.5 we now study the action of H on the set of all blue paths of length h in Z by defining an auxiliary graph Y whose vertices are these blue paths with two such blue paths adjacent if and only if there is a red edge joining them in Z. We claim that Y is isomorphic to the h-dimensional cube Q_h. To see this note that H acts regularly on Y. Moreover, as in the proof of Theorem 3.5 each red perfect matching joining C_i to C_{i+1}, $i \in \{0, \dots, h-1\}$, gives rise to an involutory generator τ_i of H

giving rise to a perfect matching in Y. To show that $Y \cong Q_h$ we only need to see that these generators are all distinct. Assume that $\tau_i = \tau_j$ for some $i < j$. Then because of the definitions of τ_i and τ_j, this means that there is a cycle in Z giving rise to the identity $ab^{j-i}a^{-1}b^{-(j-i)} = 1$, contradicting (iii). Thus Y is a Cayley graph $Cay(H; \tau_1, \ldots, \tau_h)$ of the elementary abelian group $H \cong \mathbb{Z}_2^h$ and consequently $Y \cong Q_h$. Using this fact we can easily verify that $Z \cong U_h$. Combining this fact with Lemma 3.3 we deduce that $Cay(G; a, b)$ is not isomorphic to $Pl^k(C_{k+1}[K_2^c])$ for $k \in \{1, \ldots, h-2\}$. Thus the operator Al^h can indeed be applied to $Cay(G; a, b)$ unless $Pl^{h-1}(C_h[K_2^c]) cong Cay(G; a, b)$, for some $n \geq h$. This completes the proof of the first part of Theorem 4.3.

The proof of the second part is straightforward. Firstly, (i) is satisfied in view of Proposition 4.2. As for condition (iii), it folows from the fact that U_h is a subgraph of $Cay(G; a, b)$. Indeed, the existence of U_h implies, by Lemma 2.2 (v), that all the words in a and b (not containing their inverses) of length at most h are distinct. Similarly, the fact the order of b is greater than h isdeduced, completing the proof of Theorem 4.3. ■

We end this section by giving an infinite family of tetravalent non-degenerate graphs admitting a $\frac{1}{2}$-transitive action with vertex stabilizers isomorphic to $\mathbb{Z}_2^h$, $h \geq 1$ an integer. Let $n = 2k + 1 \geq 2m + 3 \geq 5$ be an odd integer. Next set $a = a_m = (0, 1, \ldots, 2k)$, let $t = t_m = (0, m+1)$, let $b = b_m = a^t$ and let $G_{m,n} = \langle a, b \rangle$. (We remark that the group $G_{m,n}$ is isomorphic with A_n whenever $m+1$ is coprime with n. To see this observe that possible blocks of length at least 2 in the action of $G_{m,n}$ on the set $\{0, 1, \ldots, 2k\}$ must be of the form $\{i, i+d, i+2d, \ldots, i+n-d\}$ for some divisor d of n and some $i \in \{0, \ldots, d-1\}$. Also, every such block must clearly be fixed by $a^m b^{-m} = (0, -(m+1), m+1)$, forcing 0 and $m+1$ to be in the same block. But then applying a^{m+1} we have that this block coincides with the whole of $\{0, 1 \ldots, 2k\}$. Thus $G_{m,n}$ is a primitive group. The well known result of Marggraf [13, Theorem 13.8] then establishes the isomorphism with A_n.) Construct now the graph $X_{m,n}$ as the Cayley graph $Cay(G_{m,n}; a, b)$. It may be seen that $X_{m,n}$ admits a $\frac{1}{2}$-transitive action of height 1 of the group generated by the left regular action of $G_{m,n}$ and the permutation α_t which maps $x \in V(X_{n,m})$ into its conjugate x^t. Further, it may be checked that $a^i b^{-i} = (0, m+1)(-i, m+1-i)$ for $i = 1, \ldots, m$ and so the relations $(a^i b^{-i})^2 = 1$ are satisfied for each $i = 1, 2, \ldots, m$. It may also be checked that $ab^i \neq b^i a$ for each $i = 1, \ldots h-1$. Choose any $h \in \{1, \ldots, m\}$. Then the order of b is greater than $2h$ and so the assumptions of the first part of Theorem 4.3 are satisfied, implying that the graph $Y_{h,m,n} = Al^h(X_{m,n})$ admits a $\frac{1}{2}$-transitive action of $G_{m,n}$ of height h with vertex stabilizer $\mathbb{Z}_2^h$. (In fact it may be seen that it also admits a $\frac{1}{2}$-transitive action of $\langle G_{m,n}, \alpha_t \rangle$ of height $h+1$ with vertex stabilizer isomorphic to $\mathbb{Z}_2^{h+1}$.) For a fixed h we therefore have infinitely many graphs $Y_{h,m,n}$ with height h by letting m and n be arbitrary integers satisfying the above conditions.

Let us point out that the graphs $Y(h, m, n)$, where $n \geq 2^h$ and $(m+1, n) = 1$, in the above construction give a positive answer to the following question attributed to Godsil. Does there exist an infinite family of finite vertex-transitive graphs $\{X_n\}$ of fixed valency such that if G_n is a subgroup of $\mathrm{Aut}\, X_n$ of smallest possible order acting transitively on vertices of X_n then the order of the stabilizer in G_n of a vertex of X_n increases as $n \to \infty$? Families of such graphs were first constructed in [3] (but see also [9]).

5 An infinite family with nonabelian stabilizer

The aim of this section is construct an infinite family of $(G, \frac{1}{2}, D_8)$-transitive graphs. The construction is based on Theorem 3.2 (i) which reduces the problem to the task of constructing two non-involutory group elements a and b satisfying an appropriate set of relations. More precisely, for every $n \equiv 0\,(mod\,3)$ we shall construct a group G_n generated by two non-involutory elements a and b satisfying identities

$$(ab^{-1})^2 = 1, \quad (a^2b^{-2})^2 = 1, \quad a^3b^{-3}a^3b^{-1}a^{-1}b^{-1} = 1. \tag{1}$$

The elements a and b are chosen in such a way that $Y_n = Cay(G_n; a, b)$ contains a copy of U_3. By Theorem 3.2 (ii) we then have that the graph $X_n = Al^3(Y_n)$ is $(G_n, \frac{1}{2})$-transitive with G-height 3. Combining together Corollary 3.4 and Proposition 4.2 we have that the stabilizer in G_n of a vertex of X_n is isomorphic to D_8.

Let us now define the two elements a and b which generate the group G_n. We do this by representing them as permutations acting on the set $\mathbb{Z}_8 \times \mathbb{Z}_n$, where $n \equiv 0\,(mod\,3)$ in the following way:

$$(i,j)^a = (f_j(i), j+1) \quad \text{and} \quad (i,j)^b = (g_j(i), j+1), i \in \mathbb{Z}_8, j \in \mathbb{Z}_8,$$

where the functions $f_j(i)$ and $g_j(i)$ are defined in Tables 1 and 2 below.

TABLE 1: The function $f_j(i)$.

$f_j(i)$	i=0	i=1	i=2	i=3	i=4	i=5	i=6	i=7
$j \equiv 0\,(mod\,3)$	0	5	2	3	4	1	6	7
$j \equiv 1\,(mod\,3)$	0	1	2	3	6	5	4	7
$j \equiv 2\,(mod\,3)$	0	1	3	2	4	5	6	7

TABLE 2: The function $g_j(i)$.

$g_j(i)$	i=0	i=1	i=2	i=3	i=4	i=5	i=6	i=7
$j \equiv 0\,(mod\,3)$	4	1	6	7	0	5	2	3
$j \equiv 1\,(mod\,3)$	2	3	0	1	4	7	6	5
$j \equiv 2\,(mod\,3)$	1	0	2	3	5	4	7	6

Since $(i,j)^a = (i,j+3)^a$ and $(i,j)^b = (i,j+3)^b$, in order to prove that the identities (1) hold true, it is sufficient to verify that the permutations $\alpha = (ab^{-1})^2$, $\beta = (a^2b^{-2})^2$, and $\gamma = a^3b^{-3}a^3b^{-1}a^{-1}b^{-1}$ fix each of the 24 ordered pairs (i,j), $i \in \mathbb{Z}_8$ and $j \in \mathbb{Z}_3$. Indeed, this is the case and therefore the group G_n satisfies the required properties.

Let us remark that the above definition of the elements a and b as permutations on $\mathbb{Z}_8 \times \mathbb{Z}_n$ comes from an appropriate coloring of arcs of the

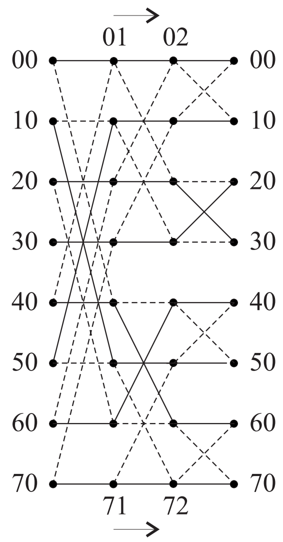

Figure 2: The coloring of U_3 defining the group $G_3 = \langle a, b : (ab^{-1})^2 = (a^2b^{-2}) = (a^3b^{-3}) = 1, \ldots \rangle$

oriented graph U_3 (see Figure 2). Define a (colored and oriented) base graph B_n with vertex set $\mathbb{Z}_8 \times \mathbb{Z}_n$ and with arcs joining (i, j) to $(f_j(i), j+1)$ and to $(g_j(i), j+1)$. In other words the graph B_n is obtained by consecutive glueing of $\frac{n}{3}$ copies of U_3 colored as in Figure 2. It may be seen that the Cayley graph Y_n covers B_n in such a way that the coloring is preserved. To prove that U_3 is a subgraph of Y_n it suffices to show that an arbitrary copy of U_3 in B_n lifts to disjoint unions of U_3 in Y_n. We omit the details.

The smallest group G_3 in the above family is generated by the following two permutations

$$a = (00, 01, 02)(10, 51, 52, 50, 11, 12)(20, 21, 22, 30, 31, 32)(40, 41, 62, 60, 61, 42)(70, 71, 72),$$

$$b = (00, 41, 42, 50, 51, 72, 60, 21, 02, 10, 11, 32, 30, 71, 52, 40, 01, 22, 20, 61, 62, 70, 31, 12),$$

where (i, j) is identified with ij for all $(i, j) \in \mathbb{Z}_8 \times \mathbb{Z}_3$. The group G_3 has 1008 elements and hence the associated 4-valent $(G_3, \frac{1}{2}, D_8)$-transitive graph $Al^3(Y_3)$ has 126 vertices.

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Contact address:

DRAGAN MARUŠIČ
IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1111 Ljubljana
Slovenija

tel. 386-61-1892320
fax 386-61-347997
email dragan.marusic@uni-lj.si