University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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AUTOMATIC CONTINUITY OF HOMOMORPHISMS INTO SMOOTH NORMED ALGEBRAS

A. Cedilnik

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Automatic continuity of homomorphisms into smooth normed algebras

by

A. CEDILNIK (Ljubljana)

Abstract. The article treats the normed algebras with the property that all homomorphisms into such an algebra from any Banach algebra are continuous. We show that smooth normed algebras are of this kind.

By saying that a pair $(B, \|\cdot\|)$ is *normed algebra* over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ we mean that the normed space B is a (nonassociative) algebra and the norm is algebraic: $\forall x, y \in B \ : \ \|xy\| \leq \|x\| \ \|y\|$.

Suppose that $(B, \|\cdot\|_B)$ is such a normed algebra that for any complete normed algebra $(A, \|\cdot\|_A)$, each homomorphism $A \xrightarrow{\varphi} B$ is continuous; then $(B, \|\cdot\|_B)$ will be called *ACHR–algebra.* The abbreviation ACHR stands for <u>a</u>utomatic <u>c</u>ontinuity of <u>h</u>omomorphisms into the algebra on the <u>r</u>ight side of the arrow.

Firstly, we shall state some facts about ACHR–algebras which are either classical or easy to be proven.

(i) If $(B, \|\cdot\|_B)$ is an ACHR–algebra and $C \subset B$ a subalgebra with a norm $\|\cdot\|_C$ for which

$$\exists \omega > 0 \ \ \forall x \in C \ : \ \|x\|_C \leq \omega \|x\|_B \ ,$$

then $(C, \|\cdot\|_C)$ is also ACHR–algebra.

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(ii) If a normed algebra $(B, \|\cdot\|)$ has an isotropic element x (i.e.: $x \neq 0$, $x^2 = 0$), it is not ACHR–algebra.

(iii) Let a normed algebra $(B, \|\cdot\|)$ be algebraic of the first order: any element generates a subalgebra of dimension ≤ 1. $(B, \|\cdot\|)$ is ACHR–algebra iff $B = \{0\}$ or $B \cong \mathbb{F}$.

(iv) Any complex commutative associative semisimple Banach algebra is ACHR–algebra.

Let $(B, \|\cdot\|_B)$ be a normed algebra, $e \notin B$, $D := \mathbb{F}e \oplus B$. We make D unital algebra in the following way:

$$(\alpha e + x)(\beta e + y) := \alpha\beta e + (\alpha y + \beta x + xy)$$

and equip it with a suitable norm $\|\cdot\|_D$. From (i) it follows that if

$$\exists \omega > 0 \ \ \forall x \in B \ : \ \|x\|_B \leq \omega \|x\|_D$$

and if $(D, \|\cdot\|_D)$ is ACHR–algebra then so is $(B, \|\cdot\|_B)$.

This statement has the opposite proposition (v).

(v) If (assuming previous definitions of $(B, \|\cdot\|_B)$ and $(D, \|\cdot\|_D)$) $(B, \|\cdot\|_B)$ is ACHR–algebra, if

$$\exists \omega > 0 \ \ \forall x \in B \ : \ \|x\|_D \leq \omega \|x\|_B$$

and if the direct sum $\mathbb{F}e \oplus B$ is topological, then $(D, \|\cdot\|_D)$ is also ACHR–algebra.

An example of the norm $\|\cdot\|_D \ : \ \|\alpha e + x\|_D := |\alpha| + \|x\|_B \quad (\omega = 1)$.

Suppose that a normed algebra $(B, \|\cdot\|)$ is a direct sum of twosided ideals: $B = \bigoplus_{i \in I} B_i$. By (i), if $(B, \|\cdot\|)$ is ACHR–algebra, so is each of $(B_i, \|\cdot\|)$. This statement has the following reverse.

(vi) Let a normed algebra $(B, \|\cdot\|)$ be a finite direct sum of twosided ideals: $B = \bigoplus_{i=1}^{n} B_i$. If all $(B_i, \|\cdot\|)$ are ACHR–algebras, then $(B, \|\cdot\|)$ is too.

(vii) Suppose that a normed algebra $(B, \|\cdot\|)$ is ACHR–algebra and that there exists another norm $|||\cdot|||$ on B, such that $(B, |||\cdot|||)$ is complete normed algebra. Then the topology of the norm $\|\cdot\|$ is weaker than the one of $|||\cdot|||$:

$$\exists \omega > 0 \;\; \forall x \in B : \|x\| \leq \omega |||x||| \; .$$

If $(B, \|\cdot\|)$ is also complete, the topologies are homeomorphic and $(B, |||\cdot|||)$ is ACHR–algebra as well.

(viii) Let $(B, \|\cdot\|_B)$ be a real normed algebra and $C := B \otimes \mathbb{C}$ its complexification with a norm $\|\cdot\|_C$. If $(C, \|\cdot\|_C)$ is ACHR–algebra, then also is $(B, \|\cdot\|_B)$.

(ix) Let $(B, \|\cdot\|)$ be a complex normed algebra and B_R the same algebra, viewed as a real normed algebra. If $(B_R, \|\cdot\|)$ is ACHR–algebra, then also is $(B, \|\cdot\|)$.

An algebra B is *quadratic* if it has a unit $e \notin 0$ and is algebraic of the second order (every element generates a subalgebra of dimension ≤ 2). It is well known that each element x fulfils an equation $x^2 - 2\tau(x)x + \nu(x)e = 0$, where τ is linear and ν quadratic form.

(x) Let $(B, \|\cdot\|)$ be a complex normed quadratic algebra. The following statements are equivalent:

(a) B is ACHR–algebra;

(b) B has no isotropic elements;

(c) $B \cong \mathbb{C}$ or $B \cong \mathbb{C} \oplus \mathbb{C}$ (= the direct sum of two onedimensional ideals).

Exceptionally, we will prove the next statement.

(xi) A twodimensional algebra with the multiplication table

$\cdot$	e	a
e	e	a
a	a	$\alpha e+\beta a$

equipped with a suitable norm is ACHR–algebra iff $\beta^2+4\alpha\neq 0$.

P r o o f. If $\beta^2+4\alpha=0$ then $(2a-\beta e)^2=0$ and because of (ii) B is not ACHR–algebra. Therefore, take $\beta^2+4\alpha\neq 0$. Then the base (e,a) can be replaced by a base (e,b), for which it is $b^2=\delta b,\quad \delta=-1$ (for $\mathbb{F}=\mathbb{C}$) or $\delta=\pm 1$ (for $\mathbb{F}=\mathbb{R}$). In the case $\mathbb{F}=\mathbb{C}$, (xi) follows from (x). In the case $\mathbb{F}=\mathbb{R}$ we complexify B and find out that $B\otimes\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C}$, which is enough for proof, regarding (x) and (viii). □

A special case: $(\mathbb{C},|\cdot|)$ as a twodimensional real algebra is ACHR–algebra. This well known theorem was once again proven in [5] and the method of this new proof will be used here for proving a much wider theorem.

Smooth normed algebra is a normed algebra $(B,|\cdot|)$ with the unit e, such that $|e|=1$ and the closed unit ball is smooth in e. In [2] and in [6] it was (with different methods) proven that such an algebra is quadratic and also noncommutative Jordan. The only complex smooth algebra is $\mathbb{C}$.

In a real smooth algebra which can have any, even infinite dimension, the norm is generated by an inner product: $|x|^2=\langle x,x\rangle$. If $p\perp e$, then $p^2=-\langle p,p\rangle e$. Suppose that $x=\alpha e+p$ in accordance with the orthogonal sum $B=\mathbb{R}e\oplus e^{\perp}$; then we define a conjugation $x\mapsto\overline{x}:=\alpha e-p$. For this map the identity $\langle y,xz\rangle=\langle\overline{x}y,z\rangle$ is valid.

If a real smooth algebra is alternative, it is exactly one of $\mathbb{R},\mathbb{C},\mathbb{H}$ and $\mathbb{O}$.

THEOREM. *Any smooth algebra is ACHR–algebra.*

P r o o f. Let $(B,|\cdot|)$ be a smooth algebra. If $\dim B=1$, the theorem follows from (iii). Therefore, from now on: $\dim B>1$, $\mathbb{F}=\mathbb{R}$. Let $(A,\|\cdot\|)$ be a complete normed algebra. In case A does not have a unit, we can add it to A in the following way. We put $A'=\mathbb{F}u\oplus A$ for some $u\notin A$ and in conformity with this direct sum we define a multiplication

$$(\alpha u + x)(\beta u + y) := \alpha\beta u + (\alpha y + \beta x + xy)$$

and a norm

$$\|\alpha u + x\|' := |\alpha| + \|x\| \ .$$

Any homomorphism $\varphi : A \to B$ can be extended to a homomorphism $\varphi' : A' \to B$ in the natural way:

$$\varphi'(\alpha u + x) := \alpha e + \varphi(x) \ .$$

The restriction of any homomorphism $\varphi' : A' \to B$ to A is a homomorphism from A into B. It is obvious that a homomorphism $\varphi : A \to B$ and its extension $\varphi' : A' \to B$ are mutually continuous. Hence, we may suppose that A itself has a unit, say u.

Let $\varphi : A \to B$ be a homomorphism and suppose that for some $a \in A$ we have $\|a\| < 1$ and $|\varphi(a)| = 1$. Because of $\varphi(a) = \varphi(au) = \varphi(a)\varphi(u)$ there is $\varphi(u) \neq 0$. Then $\varphi(u) = \lambda e + \mu p$, $p \perp e$, $p^2 = -e$.

$$\lambda e + \mu p = \varphi(u) = \varphi(u^2) = \varphi(u)^2 = (\lambda^2 - \mu^2)e + 2\lambda\mu p \ .$$

So we get a system $\lambda = \lambda^2 - \mu^2$, $\mu = 2\lambda\mu$. For $\mu \neq 0$ we get $\lambda = \frac{1}{2}$; but then $\mu^2 = -\frac{1}{4}$, which is impossible. Therefore, $\mu = 0$ and $\lambda = 1$, $\varphi(u) = e$.

Now suppose that $\varphi(a) = \lambda e + \mu p$, $p \perp e$, $p^2 = -e$, $|p| = 1$. Then $1 = |\varphi(a)|^2 = \lambda^2 + \mu^2$. Define $\vartheta \in [0, 2\pi)$, $\cos\vartheta = \lambda$, $\sin\vartheta = -\mu$. Denote $P_n(a) := a(a(\ldots(a(aa^2))\ldots))$ (for n factors in the product), $n = 1, 2, 3, \ldots$. Since $\|P_n(a)\| \leq \|a\|^n$, the next two series converge:

$$b := e + \sum_{n=1}^{\infty} P_n(a)\cos(n\vartheta) \ , \qquad c := \sum_{n=1}^{\infty} P_n(a) sin(n\vartheta) \ .$$

A straightforward computation shows that

$$\begin{aligned} u &= b - ab\cos\vartheta + ac\sin\vartheta \ , \\ 0 &= c - ab\sin\vartheta - ac\cos\vartheta \ . \end{aligned}$$

Implying φ to these equations and considering the expressions of $\varphi(a), \cos\vartheta$ and $\sin\vartheta$ we find:

$$
\begin{aligned}
e &= \varphi(b) - \lambda(\lambda e + \mu p)\varphi(b) - \mu(\lambda e + \mu p)\varphi(c) \ , \qquad (1)\\
0 &= \varphi(c) + \mu(\lambda e + \mu p)\varphi(b) - \lambda(\lambda e + \mu p)\varphi(c) \ . \qquad (2)
\end{aligned}
$$

If $\mu = 0$, then $\lambda = 1$ from (2), which gives a contradiction in (1). So, $\mu \neq 0$. Rearranging (1) and (2) we get

$$
\begin{aligned}
\frac{1}{\mu}e &= \mu\varphi(b) - \lambda p\varphi(b) - \lambda\varphi(c) - \mu p\varphi(c) \ , \qquad (3)\\
0 &= -\lambda\varphi(b) - \mu p\varphi(b) - \mu\varphi(c) + \lambda p\varphi(c) \ . \qquad (4)
\end{aligned}
$$

Implying $\langle e, \cdot\rangle$ to (3) and $\langle p, \cdot\rangle$ to (4) we find another two equations whose sum gives again a contradiction $\frac{1}{\mu} = 0$. □

From the proof of the theorem it follows also that if $(A, \|\cdot\|)$ is a complete normed algebra, $(B, |\cdot|)$ a smooth normed algebra and $\varphi : A \to B$ a homomorphism, then $\|\varphi\| \leq 1$; if A has a unit u, then also $\|\varphi\| \geq \frac{1}{\|u\|}$.

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Institute of Mathematics, Physics and Mechanics
University of Ljubljana
1111 Ljubljana, Jadranska 19, Slovenia
E-mail: anton.cedilnik@uni-lj.si