University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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AN INFINITE FAMILY OF BIPRIMITIVE SEMISYMMETRIC GRAPHS

Shao-Fei Du Dragan Marušič

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AN INFINITE FAMILY OF BIPRIMITIVE SEMISYMMETRIC GRAPHS

SHAO-FEI DU[1]
Department of Mathematics
Shanxi University
Taiyuan, Shanxi 030006
Peoples Republic of China

DRAGAN MARUŠIČ[2]
IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1111 Ljubljana
Slovenija

Abstract

A regular and edge-transitive graph which is not vertex-transitive is said to be *semisymmetric*. Every semisymmetric graph is necessarily bipartite, with the two parts having equal size and the automorphism group acting transitively on each of these two parts. A semisymmetric graph is called *biprimitive* if its automorphism group acts primitively on each part. In this paper a classification of biprimitive semisymmetric graphs arising from the action of the group $PSL(2,p)$, $p \equiv \pm 1 \,(mod\, 8)$ a prime, acting on cosets of S_4 is given, resulting in several new infinite families of biprimitive semisymmetric graphs.

Keywords: primitive group, semisymmetric graph, biprimitive graph.

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1 Introduction

Throughout this paper graphs and groups are assumed to be finite. Also unless otherwise specified graphs are assumed to be simple and undirected. For the group-theoretic concepts and notation not defined here we refer the reader to [3, 7]. By p we shall always denote a prime number.

For a graph X we let $V(X)$, $E(X)$ and $\mathrm{Aut}\,X$ be, respectively, the vertex set, the edge set and the automorphism group of X. We say that X is *vertex-transitive*, *edge-transitive* and *symmetric*, if $\mathrm{Aut}\,X$ acts transitively on the set of vertices, edges, or arcs of X, respectively. Moreover, we say that X is *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. We remark that every semisymmetric graph is bipartite with the two parts of equal size and the automorphism group acting transitively on each of these two parts. The study of semisymmetric graphs was initiated by Folkman [6] who gave a construction of several infinite families of such graphs including, among others, a smallest semisymmetric graph on 20 vertices. At the end of his paper several problems were posed, most of which have already been solved (see [1, 2, 11, 14, 9, 8]). In all of the semisymmetric graphs given by Folkman [6] the automorphism group acts imprimitively on each of the two bipartition parts. A semisymmetric graph X is called *biprimitive* if $\mathrm{Aut}\,X$ acts primitively on each of the two parts of the bipartition. The first construction of a biprimitive graph is due to Iofinova and Ivanov who gave a classification of cubic biprimitive graphs [8]. It follows from their classification that only five such graphs exist. A first construction of an infinite family of biprimitive graphs was given in [4]. The graphs are associated with the group $PSL(2,p)$, $p \equiv 1\,(mod\,48)$ a prime, acting on cosets of S_4.

It is the main purpose of this paper to classify all biprimitve graphs arising from the action of the group $PSL(2,p)$, $p \equiv \pm 1\,(mod\,8)$ a prime, on cosets of S_4. This work is motivated by our long-term goal to classify all biprimitive graphs of order $2kp$, where $k < p$, p a prime. In fact, it may be seen that with the exception of the biprimitive graphs arising from the action of the group $PSL(2,p)$, $p \equiv \pm 1\,(mod\,8)$, on cosets of S_4 and possibly those arising from the action of the group $PSL(2,p)$, $p \equiv \pm 1\,(mod\,10)$, on cosets of A_5, there are only finitely many biprimitive graphs of order $2kp$, $k < p$, p a prime. Thus obtaining a classification of biprimitive graphs associated with the group $PSL(2,p)$, $p \equiv \pm 1\,(mod\,8)$, acting on cosets of S_4, is an important first step towards a complete classification of biprimitive graphs of order $2kp$,

$k < p$, p a prime.

We recall the general method for constructing semisymmetric graphs. Let G be a permutation group on a set V having two orbits U and W of the same cardinality and no other orbits. Furthermore let Δ be an orbit of the action of G on $U \times W$ and let X denote the bipartite graph with vertex set V and edges of the form uw, where $(u, w) \in \Delta$. Of course, X is regular and edge-transitive with bipartition $\{U, W\}$. Moreover, X is semisymmetric if and only if $\mathrm{Aut}\, X$ preserves the two orbits of G.

Conversely, every semisymmetric graph can be obtained in the way described above. Namely, let X be a semisymmetric graph with the automorphism group G and bipartition $\{U, W\}$ of its vertex set V. Take $u \in U$ and $w \in W$ and let $J = G_u$ and $L = G_w$. It may be easily seen that there is a one-to-one correspondence between the orbits of J on W (as well as the orbits of L on U) and the orbits of the action of G on the set $U \times W$, giving us precisely the situation of the previous paragraph.

Consider now the special case of the group $G \cong PSL(2, p)$, where $p \equiv \pm 1\, (mod\, 8)$, acting semisymmetrically on X where both U and W are identified with the set of right cosets of some (not necessarily the same) subgroup isomorphic to S_4. Let $A = PGL(2, p)$. Clearly, $A = G : \langle \sigma \rangle$ for some involution σ. If the actions of G on U and W are equivalent then there is, in view of the above comments, a one-to-one correspondence between the orbits of G on $U \times W$ and the suborbits of G acting on U (as well as those on W), which may also be identified with the suborbits of the group $G \times Z_2$ in its transitive action on V. If $S = S_\Delta$ is the suborbit of $G \times Z_2$ corresponding to an orbit Δ of G acting on $U \times W$ we let $X(G, V, S)$ denote the graph with vertex set V and edges of the form uw, where $(u, w) \in \Delta$. On the other hand, if the actions of G on U and W are inequivalent, then there is a one-to-one correspondence between the orbits of, say, J on W and the suborbits of A in its transitive action on V. Consequently, there is a one-to-one correspondence between the orbits of G on $U \times W$ and the suborbits of A acting on V. If $S = S_\Delta$ is the suborbit of A corresponding to an orbit Δ of G acting on $U \times W$ we shall use the symbol $X(G, V, S)$ to denote the graph with vertex set V and edges of the form uw, where $(u, w) \in \Delta$. (The fact that the same notation is used for equivalent and inequivalent actions should cause no confusion as it will be clear from the context which of the two cases one refers to.)

We may now state the main result of this paper.

Theorem 1.1 *Let X be a biprimitive semisymmetric graph with bipartition $\{U, W\}$ of its vertex set V and a subgroup $G \cong PSL(2, p)$, $p \equiv \pm 1 \, (mod\, 8)$, of* Aut X *acting edge-transitively on X and having S_4 as a vertex stabilizer. Then*

(i) X is isomorphic to some $X(G, V, S)$ where S is a non-self-paired suborbit of $G \times Z_2$ or A on V, respectively, depending on whether G acts equivalently or inequivalently on U and W;

(ii) two semisymmetric graphs $X(G, V, S_1)$ and $X(G, V, S_2)$ are isomorphic if and only if S_1 and S_2 are paired suborbits;

(iii) The total numbers of nonisomorphic semisymmetric graphs $X(G, V, S)$ for each of the congruence classes of p are given in Table 1, where the columns under the headings $G \times Z_2$ and A, respectively, give the numbers of such graphs associated with the groups $G \times Z_2$ and A.

TABLE 1.

p	valency 12		valency 24	
(mod 48)	$G \times Z_2$	A	$G \times Z_2$	A
1	$\frac{p-17}{16}$	$\frac{p-1}{16}$	$\frac{p^3-24p^2+93p-70}{2304}$	$\frac{p^3-24p^2+141p-108}{2304}$
-1	$\frac{p-15}{16}$	$\frac{p+1}{16}$	$\frac{p^3-24p^2+141p+16670}{2304}$	$\frac{p^3-24p^2+93p+118}{2304}$
7	$\frac{p-7}{16}$	$\frac{p-7}{16}$	$\frac{p^3-24p^2+141p-154}{2304}$	$\frac{p^3-24p^2+93p+182}{2304}$
-7	$\frac{p-9}{16}$	$\frac{p-9}{16}$	$\frac{p^3-24p^2+93p-134}{2304}$	$\frac{p^3-24p^2+141p+202}{2304}$
17	$\frac{p-17}{16}$	$\frac{p-1}{16}$	$\frac{p^3-24p^2+93p+442}{2304}$	$\frac{p^3-24p^2+141p-374}{2304}$
-17	$\frac{p-15}{16}$	$\frac{p+1}{16}$	$\frac{p^3-24p^2+141p+422}{2304}$	$\frac{p^3-24p^2+93p-394}{2304}$
23	$\frac{p-7}{16}$	$\frac{p-7}{16}$	$\frac{p^3-24p^2+141p-410}{2304}$	$\frac{p^3-24p^2+93p+694}{2304}$
-23	$\frac{p-9}{16}$	$\frac{p-9}{16}$	$\frac{p^3-24p^2+93p-646}{2304}$	$\frac{p^3-24p^2+141p+458}{2304}$

Some group-theoretic results are presented in Section 2 and used in Section 3 to analyze the suborbit structure of some permutation groups (Lemma 3.1). This lemma plays a crucial role in the proof of Theorem 1.1 in Section 4.

2 Preliminaries

For a vertex v of a graph X we let $N(v)$ denote the set of neighbors of v in X. The following proposition is extracted from [5] and gives some sufficient conditions for a bipartite regular edge-transitive graph to be vertex-transitive.

Proposition 2.1 *([5, Lemma 2.4]) Let X be a regular bipartite graph with bipartition (U, W) (such that $|U| = |W|$) of its vertex set V and let G be a subgroup of* $\mathrm{Aut}\, X$ *with orbits U and W. Let $u \in U$, $w \in W$, $J = G_u$, $L = G_w$ and $D = \{g \in G \mid w^g \in N(u)\}$. If there exists an element $\sigma \in \mathrm{Aut}\, G$ such that $J^\sigma = L$, $L^\sigma = J$ and $D^\sigma = D^{-1}$ then X is vertex-transitive. In particular,*

(i) if G is abelian and acts regularly on each of U and W, then X is vertex-transitive;

(ii) if the lengths of the orbits of J on W (or the orbits of L on U) are all distinct then X is vertex-transitive;

(iii) if the two representations of G on U and W are equivalent and all the suborbits of G acting on $\{Jg \mid g \in G\}$ by right multiplication are self-paired, then X is vertex-transitive.

Proposition 2.2 *([12, Proposition ?.?]) Let G be a transitive group on a set V, let $H = G_v$ for some $v \in V$ and let K be a subgroup of H. If the set of G-conjugates of K which are contained in H form t conjugacy classes of H with representatives K_1, K_2, $\cdots$, K_t, then K fixes*

$$\sum_{i=1}^{t} |N_G(K_i) : N_H(K_i)|$$

points of V .

The next two lemmas will be used in Section 3 to analyze the self-pairedness of suborbits of permutation groups.

Proposition 2.3 *([13, Lemma 2.2]) Let $D = \langle a, b\rangle \cong D_{2n}$, $n \geq 2$, be a permutation group on $V = \{1, 2, \cdots, n\}$, where*

$$a = (1, 2, \cdots, n) \quad \textit{and} \quad b = (1)(2, n)(3, n-1) \cdots (i, n+2-i) \cdots$$

Then the nontrivial orbitals of D are $\Delta_i = (1,i)^D = (1,n+2-i)^D$, for $2 \leq i \leq \frac{1}{2}(n+2)$. Each of these orbitals is self-paired. Moreover, for all points i,j, with $i \neq j$, there is an involution in D which interchanges i and j.

Proposition 2.4 *([13, Lemma 2.4]) Let G be a transitive group on V and let $H = G_v$ for some $v \in V$. Assume that G has t conjugacy classes of involutions, say $C_1, C_2, \ldots, C_t$. Take a representative $\sigma_j \in C_j$. Assume that σ_j has N_j cycles of length 2. For a nontrivial self-paired orbital $\Delta = \Delta(v)$ and a pair $(v,w) \in \Delta$, let $\mathrm{inv}(\Delta)$ be the number of involutions in G with a 2-cycle (v,w). Then*

$$\sum_{j=1}^{t} \frac{N_j}{c_j} = \frac{1}{2|H|} \sum_{\Delta=\Delta'} |\Delta| \,\mathrm{inv}(\Delta),$$

where c_j is the order of the centralizer of σ_j and Δ' denotes the paired suborbit of Δ.

For the rest of this paper we let $K_1, \ldots, K_7$ denote the subgroups D_8, D_6, D_4, Z_4, Z_3, Z_2^A and Z_2^B of S_4, where $D_4 \leq D_8$, $Z_4 \leq D_8$, $Z_3 \leq D_6$, $Z_2^A = D_4 \cap Z_4$ and $Z_2^B = D_6 \cap D_4$. The following result is extracted from [4].

Proposition 2.5 *([4, Lemma 3.2]) Let G be a transitive permutation group on a set V with the point stabilizer S_4. With the above notation, suppose that for each $i \in \{1, \ldots, 7\}$ the group K_i fixes k_i points in V and that G has x_i suborbits with the point stabilizer K_i. Then*

$$x_1 = k_1 - 1, \quad x_2 = k_2 - 1, \quad x_3 = \frac{1}{2}(k_3 - k_1), \quad x_4 = \frac{1}{2}(k_4 - k_1),$$

$$x_5 = \frac{1}{2}(k_5 - k_2), \quad x_6 = \frac{1}{4}(k_6 - k_1 - k_3 - k_4 + 2), \quad x_7 = \frac{1}{2}(k_7 - k_4 - 2k_2 + 2)$$

3 Group-theoretic computations

We fix the following notation. Assume that $p \equiv \pm 1 \pmod 8$ is a prime. Let $G = PSL(2,p) \leq A = PGL(2,p)$ and let $H \cong S_4$ be a maximal subgroup of G. Assume that $V = \{Hg \mid g \in A\}$, $V_1 = \{Hg \mid g \in G\}$ and $V_2 = V \setminus V_1 = \{H\sigma g \mid g \in G\}$, where $\sigma \in A \setminus G$ is an involution. Let $K_1, \ldots, K_7$ have the meaning described in the previous section and let $K_8 = 1$. We now consider the right multiplication actions of G and A on V_1 and V respectively.

For each $i \in \{1, \ldots, 8\}$ let k_i and k_{i1} denote the respective numbers of fixed points of K_i on V and V_1. Let x_i, x_{i1} and $x_{i2} = x_i - x_{1i}$ denote the numbers of suborbits of A acting on V, V_1 and V_2, respectively. Furthermore let y_i, y_{i1}, $y_{i2} = y_i - y_{i1}$ denote the numbers of self-paired suborbits of A acting on V, V_1 and V_2, respectively. The symbols $h_{i1} = x_{i1} - y_{i1}$ and $h_{i2} = x_{i2} - y_{i2}$ will then denote the numbers of non-self-paired suborbits of A acting on V_1 and V_2, respectively.

Lemma 3.1 *For each $i \in \{1, \ldots, 8\}$ the values of y_{i1}, y_{i2}, h_{i1} and h_{i2} are given in Table 2.*

PROOF. Noting that $p \equiv \pm 1 \pmod 8$, we may divide the proof into eight cases according to whether $p \equiv \pm 1\,(mod\,48)$, $p \equiv \pm 7\,(mod\,48)$, $p \equiv \pm 17\,(mod 48)$, or $p \equiv \pm 23\,(mod\,48)$. We only give the details of the proof for $p \equiv \pm 1 \pmod{48}$. The proofs for the other cases are analogous and are therefore omitted.

We start by determining the values of x_i, x_{i1} and x_{i2}. For $i \in \{1, \ldots, 7\}$ these values are given in Table 2 below and are obtained in the following way. (We omit the technical details of the actual computations.) After having determined the respective normalizers of each K_i in H and in G we apply Proposition 2.2 to calculate k_{1i}. We then use Proposition 2.5 to compute x_{1i}. The values k_i and x_i are obtained in an analogous way by replacing G with A. Finally, x_{i2} is obtained as the difference $x_i - x_{i1}$ (see Table 3 below).

TABLE 2.

	i	1	2	3	4	5	6	7	8
	K_i	D_8	D_6	Z_4	D_4	Z_3	Z_2^A	Z_2^B	1
$p \equiv 1 \pmod{48}$	y_{i1}	1	1	$\frac{p-17}{16}$	1	$\frac{p-13}{12}$	0	$\frac{p-17}{8}$	$\frac{p^2-12p+107}{48}$
	h_{i1}	0	0	0	0	0	$\frac{p-17}{16}$	$\frac{p-17}{16}$	$\frac{p^3-24p^2+93p-70}{1152}$
	y_{i2}	0	0	$\frac{p-1}{16}$	0	$\frac{p-1}{12}$	0	$\frac{p-1}{8}$	$\frac{p^2-14p+13}{48}$
	h_{i2}	0	0	0	0	0	$\frac{p-1}{16}$	$\frac{p-1}{16}$	$\frac{p^3-24p^2+141p-108}{1152}$
$p \equiv -1 \pmod{48}$	y_{i1}	1	1	$\frac{p-15}{16}$	1	$\frac{p-11}{12}$	0	$\frac{p-15}{8}$	$\frac{p^2-14p+81}{48}$
	h_{i1}	0	0	0	0	0	$\frac{p-15}{16}$	$\frac{p-15}{16}$	$\frac{p^3-24p^2+141p+166}{1152}$
	y_{i2}	0	0	$\frac{p+1}{16}$	0	$\frac{p+1}{12}$	0	$\frac{p+1}{8}$	$\frac{p^2-12p-13}{48}$
	h_{i2}	0	0	0	0	0	$\frac{p+1}{16}$	$\frac{p+1}{16}$	$\frac{p^3-24p^2+93p+118}{1152}$
$p \equiv 7 \pmod{48}$	y_{i1}	0	0	$\frac{p-7}{16}$	1	$\frac{p-7}{12}$	0	$\frac{p-7}{8}$	$\frac{p^2-14p+49}{48}$
	h_{i1}	0	0	0	0	0	$\frac{p-7}{16}$	$\frac{p-7}{16}$	$\frac{p^3-24p^2+141p-154}{1152}$
	y_{i2}	1	1	$\frac{p-7}{16}$	0	$\frac{p-7}{12}$	0	$\frac{p-7}{8}$	$\frac{p^2-12p+35}{48}$
	h_{i2}	0	0	0	0	0	$\frac{p-7}{16}$	$\frac{p-7}{16}$	$\frac{p^3-24p^2+93p+182}{1152}$
$p \equiv -7 \pmod{48}$	y_{i1}	0	0	$\frac{p-9}{16}$	1	$\frac{p-7}{12}$	0	$\frac{p-9}{8}$	$\frac{p^2-12p+59}{48}$
	h_{i1}	0	0	0	0	0	$\frac{p-9}{16}$	$\frac{p-9}{16}$	$\frac{p^3-24p^2+93p-134}{1152}$
	y_{i2}	1	1	$\frac{p-9}{16}$	0	$\frac{p-5}{12}$	0	$\frac{p-9}{8}$	$\frac{p^2-14p+45}{48}$
	h_{i2}	0	0	0	0	0	$\frac{p-9}{16}$	$\frac{p-9}{16}$	$\frac{p^3-24p^2+141p+202}{1152}$
$p \equiv 17 \pmod{48}$	y_{i1}	1	0	$\frac{p-17}{16}$	1	$\frac{p-5}{12}$	0	$\frac{p-9}{8}$	$\frac{p^2-12p+59}{48}$
	h_{i1}	0	0	0	0	0	$\frac{p-17}{16}$	$\frac{p-17}{16}$	$\frac{p^3-24p^2+93p+442}{1152}$
	y_{i2}	0	1	$\frac{p-1}{16}$	0	$\frac{p-5}{12}$	0	$\frac{p-9}{8}$	$\frac{p^2-14p+45}{48}$
	h_{i2}	0	0	0	0	0	$\frac{p-1}{16}$	$\frac{p-1}{16}$	$\frac{p^3-24p^2+141p-374}{1152}$
$p \equiv -17 \pmod{48}$	y_{i1}	1	0	$\frac{p-15}{16}$	1	$\frac{p-9}{12}$	0	$\frac{p-7}{8}$	$\frac{p^2-14p+49}{48}$
	h_{i1}	0	0	0	0	0	$\frac{p-15}{16}$	$\frac{p-15}{16}$	$\frac{p^3-24p^2+141p+422}{1152}$
	y_{i2}	0	1	$\frac{p+1}{16}$	0	$\frac{p-7}{12}$	0	$\frac{p-7}{8}$	$\frac{p^2-12p+35}{48}$
	h_{i2}	0	0	0	0	0	$\frac{p+1}{16}$	$\frac{p+1}{16}$	$\frac{p^3-24p^2+93p-394}{1152}$
$p \equiv 23 \pmod{48}$	y_{i1}	0	1	$\frac{p-7}{16}$	1	$\frac{p-11}{12}$	0	$\frac{p-15}{8}$	$\frac{p^2-14p+81}{48}$
	h_{i1}	0	0	0	0	0	$\frac{p-7}{16}$	$\frac{p-7}{16}$	$\frac{p^3-24p^2+141p-410}{1152}$
	y_{i2}	1	0	$\frac{p-7}{16}$	0	$\frac{p+1}{12}$	0	$\frac{p+1}{8}$	$\frac{p^2-12p-13}{48}$
	h_{i2}	0	0	0	0	0	$\frac{p-7}{16}$	$\frac{p-7}{16}$	$\frac{p^3-24p^2+93p+694}{1152}$
$p \equiv -23 \pmod{48}$	y_{i1}	0	1	$\frac{p-9}{16}$	1	$\frac{p-13}{12}$	0	$\frac{p-17}{8}$	$\frac{p^2-12p+107}{48}$
	h_{i1}	0	0	0	0	0	$\frac{p-9}{16}$	$\frac{p-9}{16}$	$\frac{p^3-24p^2+93p-646}{1152}$
	y_{i2}	1	0	$\frac{p-9}{16}$	0	$\frac{p-1}{12}$	0	$\frac{p-1}{8}$	$\frac{p^2-14p+13}{48}$
	h_{i2}	0	0	0	0	0	$\frac{p-9}{16}$	$\frac{p-9}{16}$	$\frac{p^3-24p^2+141p+458}{1152}$

TABLE 3.

i	1	2	3	4	5	6	7
K_i	D_8	D_6	Z_4	D_4	Z_3	Z_2^A	Z_2^B
$N_H(K_i)$	D_8	D_6	D_8	D_8	D_6	D_8	D_4^B
$N_G(K_i)$	D_{16}	D_{12}	D_{p-1}	S_4	D_{p-1}	D_{p-1}	D_{p-1}
k_{1i}	2	2	$\frac{1}{8}(p-1)$	4	$\frac{1}{6}(p-1)$	$\frac{3}{8}(p-1)$	$\frac{3}{8}(p-1)$
x_{i1}	1	1	$\frac{p-1}{6}-1$	1	$\frac{p-1}{12}-1$	$\frac{p-1}{16}-1$	$\frac{3(p-1)}{16}-3$
$N_A(K_i)$	D_{16}	D_{12}	$D_{2(p-1)}$	S_4	$D_{2(p-1)}$	$D_{2(p-1)}$	$D_{2(p-1)}$
k_i	2	2	$\frac{1}{4}(p-1)$	4	$\frac{1}{3}(p-1)$	$\frac{3}{4}(p-1)$	$\frac{3}{4}(p-1)$
x_i	1	1	$\frac{p-1}{8}-1$	1	$\frac{p-1}{6}-1$	$\frac{p-1}{8}-1$	$\frac{3(p-1)}{8}-3$
x_{i2}	0	0	$\frac{p-1}{16}$	0	$\frac{p-1}{12}$	$\frac{p-1}{16}$	$\frac{3(p-1)}{16}$

The calculation of x_{81} and x_{82} is now straightforward. We have

$$\begin{aligned} x_{81} &= \frac{1}{24}\{\frac{1}{48}(p^3-p) - [1+3\cdot 1+4\cdot 1+6\cdot(\frac{1}{16}(p-1)-1)+6\cdot 1 \\ &+ 8\cdot(\frac{1}{12}(p-1)-1)+12\cdot(\frac{1}{16}(p-1)-1)+12\cdot(\frac{3}{16}(p-1)-3)]\} \\ &= \frac{1}{1152}(p^3-195p+194)+2 \end{aligned}$$

and

$$\begin{aligned} x_{82} &= \frac{1}{24}\{\frac{1}{48}(p^3-p) - [6\cdot\frac{1}{16}(p-1)+8\cdot\frac{1}{12}(p-1) \\ &+ 12\cdot\frac{1}{16}(p-1)+12\cdot\frac{3}{16}(p-1)]\} = \frac{1}{1152}(p^3-195p+194). \end{aligned}$$

We now determine the values of y_i, y_{i1}, y_{i2}, h_{i1} and h_{i2} for all i. Let $v \in V$ be arbitrary. We claim that for $i = 1, \ldots, 5$, we must have $x_i = y_i$ and so $h_{i1} = 0 = h_{i2}$. Since $x_{i1} = 1$ for $i = 1, 2, 4$, we may assume that $i = 3$ or $i = 5$. The argument is the same in both cases. For instance, consider a suborbit $S = S(v)$ relative to a point v with the point stabilizer K_3. For $w \in S$, we have $A_{vw} = Z_4$ and $N_A(A_{vw}) = D_{2(p-1)}$. By Proposition 2.3 there exists an element in $D_{2(p-1)}$ which interchanges v and w. Hence S is self-paired and so $x_3 = y_3$ and $h_{31} = 0 = h_{32}$.

We claim that $y_6 = 0$ and so $h_{61} = x_{61}$ and $h_{62} = x_{62}$. In fact, suppose that $S = S(v)$ is a suborbit with the point stabilizer $K_6 = Z_2^A$ and that $b \in S$.

Then S is self-paired if and only if there exists $g \in G$ which interchanges v and w. But now g must be contained in $N_A(A_{vw}) = D_{2(p-1)}$. Consider the action of $N_A(A_{vw})$ on the set $\mathrm{Fix}\,_V(A_{vw})$ of all points in V fixed by A_{vw}. Note that $|\mathrm{Fix}\,_V(A_{vw})| = \frac{3}{4}(p-1)$. Since $N_A(A_{vw}) \cap H \cong D_8$, we have that v is contained in an $N_A(A_{vw})$-orbit Q of length $\frac{1}{4}(p-1)$ on V, in which every point is fixed by $Z_4 \leq H$. For any $w \in S$, we have $H_w \cong Z_2^A$. Hence $S \cap Q = \emptyset$. In other words, S is non-self-paired and so $y_6 = 0$.

We now consider the case $i = 7$. Suppose that $S = S(v)$ is a suborbit with the point stabilizer $K_7 = Z_2^B$. Consider the action of $N_A(Z_2^B) \cong D_{2(p-1)}$ on $\mathrm{Fix}\,_V(A_{vw})$, a set of length $\frac{3}{4}(p-1)$. Then v is contained in a $N_G(Z_2^B)$-orbit Q of length $\frac{1}{2}(p-1)$. It is easy to see that there are only two points in S which are fixed by Z_2^B, and they either both belong to Q or none of them does. It may be seen that this fact implies the following two inequalities:

$$2y_7 \leq \frac{1}{2}(p-1) - 2$$

and

$$2((\frac{3}{8}(p-1) - 3) - y_7) \leq \frac{p-1}{4},$$

giving us $y_7 = \frac{1}{4}(p-1) - 2$. Now using the same argument as above with A replaced by G, we get that $y_{71} = \frac{1}{8}(p-1) - 2$, forcing $y_{72} = \frac{1}{8}(p-1)$. As a consequence $h_{71} = \frac{1}{6}(p-1) = h_{72}$.

We are now left with the case $i = 8$, the most tedious of all. In Table 4 below we let the symbol l_i, $i = 1, 2, 3, 4, 5, 7, 8$, denote the length of self-paired suborbits S with point stabilizer K_i, whereas $\mathrm{inv}(S)$ equals $\mathrm{inv}(\Delta)$ (defined in Proposition 2.4), where Δ is the orbital of G corresponding to S in the one-to-one correspondence between orbits of G on $V \times V$ and suborbits of G on V.

TABLE 4.

i	l_i	y_{i1}	y_i	$G_{vw} = A_{vw}$	$G_{\{v,w\}} = A_{\{v,w\}}$	$\mathrm{inv}(S)$
1	3	1	1	D_8	D_{16}	4
2	4	1	1	D_6	D_{12}	4
3	6	$\frac{1}{16}(p-1)-1$	$\frac{1}{8}(p-1)-1$	Z_4	D_8	4
4	6	1	1	D_4	D_8	2
5	8	$\frac{1}{12}(p-1)-1$	$\frac{1}{6}(p-1)-1$	$\frac{1}{6}(p-1)-1$	D_6	3
7	12	$\frac{1}{8}(p-1)-2$	$\frac{1}{4}(p-1)-2$	Z_2	D_4	2
8	24	y_{81}	y_8	1	Z_2	1

Note that G has only one conjugacy class of involutions and that each involution has precisley $\frac{3}{8}(p-1)$ fixed points in V_1, and therefore it has

$$N = \frac{1}{2}[(\frac{p^3-p}{48}) - \frac{3}{8}(p-1)] = \frac{(p-1)(p^2+p-18)}{96}$$

cycles of length 2. The centralizer of an involution has order $c = p-1$ and so, combining together Proposition 2.4 and Table 4, we get

$$\begin{aligned}\frac{(p-1)(p^2+p-18)}{96(p-1)} &= \frac{1}{2\cdot 24}\{1\cdot 3\cdot 4 + 1\cdot 4\cdot 4 + [\frac{1}{16}(p-1)-1]\cdot 6\cdot 4 \\ + \quad & 1\cdot 6\cdot 2 + [\frac{1}{12}(p-1)-1]\cdot 8\cdot 3 + +[\frac{1}{8}(p-1)-2]\cdot 12\cdot 2 + y_{81}\cdot 24\cdot 1\}.\end{aligned}$$

Hence

$$y_{81} = \frac{1}{48}(p^2 - 12p + 107)$$

and

$$h_{81} = x_{81} - y_{81} = \frac{p^3 - 24p^2 + 93p - 70}{1152}.$$

We know turn to the group A. It has two conjugacy classes of involutions. A representative of the first class, say $a_1 \in G$, fixes $\frac{3}{4}(p-1)$ points and so a_1 contains $N_1 = \frac{(p-1)(p^2+p-18)}{48}$ cycles of length 2. Besides, $N_A(a_1)$ has order $c_1 = 2(p-1)$. A representative of the second class, say $a_2 \in A \setminus G$, has no fixed point and so a_2 contains $N_2 = \frac{p(p-1)(p+1)}{48}$ cycles of length 2. Also, $N_A(a_2)$ has order $c_2 = 2(p+1)$. Combining together Proposition 2.4 and Table4, we have

$$\begin{aligned} & \frac{(p-1)(p^2+p-18)}{96(p-1)} + \frac{p(p-1)(p+1)}{96(p+1)} \\ = \quad & \frac{1}{2\cdot 24}\{1\cdot 3\cdot 4 + 1\cdot 4\cdot 4 + [\frac{1}{8}(p-1)-1]\cdot 6\cdot 4 + \quad 1\cdot 6\cdot 2 \\ + \quad & [\frac{1}{6}(p-1)-1]\cdot 8\cdot 3 + +[\frac{1}{4}(p-1)-2]\cdot 12\cdot 2 + y_8\cdot 24\cdot 1\}.\end{aligned}$$

It follows that

$$y_8 = \frac{1}{24}(p^2 - 13p + 60)$$

and

$$y_{82} = y_8 - y_{81} = \frac{1}{48}(p^2 - 14p + 13),$$

and so

$$h_{82} = x_{82} - y_{82} = \frac{p^3 - 24p^2 + 141p - 108}{1152},$$

completing the proof of Lemma 3.1. ■

4 Proof of Theorem 1.1

As in the proof of Lemma 3.1 we only give the details of the proof for $p \equiv 1 \pmod{48}$. The proofs for the other cases are analogous and are therefore omitted. We shall distinguish two cases according to whether the actions of $G = PSL(2, p)$ on the two parts U and W of X are equivalent or inequivalent.

Case 1: The actions of G on U and W are equivalent.

In view of our comments preceding the statement of Theorem 1.1 it follows that X may be identified with the graph $X(G, V, S)$ for some suborbit S of of $G \times Z_2$ in its transitive action on V. Moreover, applying Proposition 2.1 we have that S is non-self-paired.

To prove the converse let S be be a non-self-paired suborbit of $G \times Z_2$ on V. (Note that S may also be viewed either as a suborbit of G on U or G on W.) We show that $X = X(G, V, S)$ is semisymmetric. Since G is a maximal simply primitive group of degree $\frac{p^3-p}{48}$ (see [10]), it follows that $(\mathrm{Aut}\, X)_U = G = (\mathrm{Aut}\, X)_W$ and so $|\mathrm{Aut}\, X : G| \leq 2$. Suppose that $|\mathrm{Aut}\, X : G| = 2$. Then, since the actions of G on U and W are equivalent, there exists some involution $\tau \in \mathrm{Aut}\, X$ such that $\mathrm{Aut}\, X = G \times \langle \tau \rangle \cong G \times Z_2$, We may then choose $u \in U$ and $w \in W$ in such a way that u^g and w^g are interchanged by τ for all $g \in G$. It follows that $u^g w^{g'}$ is an edge of X if and only if $w^g u^{g'}$ is an edge of X, forcing S to be self-paired, a contradiction.

Finally, let $X_1 = X(G, V, S_1)$ $X_2 = X(G, V, S_2)$ for two suborbits S_1 and S_2 of $G \times Z_2$ in its transitive action on V (which as above mayalso be interpreted either as suborbits of G on U or of G on W). Let σ be an isomorphism from X_1 to X_2. We show that S_1 and S_2 are paired. We have $\sigma^{-1} G \sigma \leq \mathrm{Aut}\, X_2 = G$ and so $\sigma^{-1} G \sigma = G$. Hence σ belongs to the normalizer of G in S_V and so σ interchanges U and W. Consider the edge-disjoint union $X_1 \cup X_2$ of X_1 and X_2. Applying the arguments of the previous paragraph we have that $\mathrm{Aut}\,(X_1 \cup X_2) = G \times Z_2$ as well as that $S_1 \cup S_2$ is self-paired. This forces S_1 and S_2 to be paired. Conversely, assume that S_1 and S_2 are paired. In view of the equivalence of the actions of G on U and W we may choose $u \in U$ and $w \in W$ such that $G_u = G_w$. Let τ be the mapping on V which interchanges u^g and w^g for all $g \in G$. It may be easily shown that τ is an isomorphism between X_1 and X_2.

Case 2: The actions of G on U and W are inequivalent.

Following almost step by step the arguments used in Case 1 with $G \times Z_2$ replaced by A, we may now prove that, firstly, the graph X is semisymmetric if and only if it is isomorphic to the graph $X(G, V, S)$ for some non-self-paired-suborbit S of the action of A on V, and secondly, that two graphs $X(G, V, S_1)$ and $X(G, V, S_2)$ are isomorphic if and only if the two suborbits S_1 and S_2 of A on V are paired.

Finally, the numbers of nonisomorphic semisymmetric graphs in each of the two cases are then extracted from Table 2 in the following way. For valency 12 the number of such nonisomorphic graphs equals $\frac{h_{61}+h_{71}}{2}$ and $\frac{h_{62}+h_{72}}{2}$ for the equivalent and inequivalent cases, respectively. Similarly, for valency 24 the number of such nonisomorphic graphs equals $\frac{h_{81}}{2}$ and $\frac{h_{82}}{2}$ for the equivalent and inequivalent cases, respectively. ■

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