

UNIVERSITY OF LJUBLJANA
INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
DEPARTMENT OF MATHEMATICS
JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

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Dragan Marušič

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DRAGAN MARUŠIČ¹
IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1111 Ljubljana
Slovenija

Abstract

An infinite family of cubic edge-transitive but not vertex-transitive graph graphs with edge stabilizer isomorphic to \mathbb{Z}_2 is constructed.

1 Introduction

Throughout this paper graphs are assumed to be finite, simple and, unless specified otherwise, undirected and connected. For the group-theoretic concepts and notation not defined here we refer the reader to [5, 11].

Given a graph X we let $V(X)$, $E(X)$ and $\text{Aut } X$ be the vertex set, the edge set and the automorphism group of X , respectively. We say that X is *vertex-transitive* and *edge-transitive* if $\text{Aut } X$ acts transitively on $V(X)$ and $E(X)$, respectively. It is easily seen that an edge- but not vertex-transitive graph X is necessarily bipartite, where the two parts of the bipartition are orbits of $\text{Aut } X$. Moreover, if X is regular then these two parts have equal cardinality. The study of edge- but not vertex-transitive graphs was initiated by Folkman [7] who gave a construction of several infinite families of such graphs including, among others, a smallest edge- but not vertex-transitive graph on 20 vertices. At the end of his paper several problems were posed, most of which have already been solved (see [1, 2, 8, 9, 10]).

This paper deals with cubic edge- but not vertex-transitive graphs. The first example of such a graph, the so called Gray graph on 54 vertices, was

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described in [1], thus answering an open problem from [7] about the existence of edge- but not vertex-transitive graphs with prime valency. It is believed that no smaller cubic edge- but not vertex-transitive graph exists. In [8] the classification of all cubic biprimitive graphs, that is cubic edge- but not vertex-transitive graphs for which the automorphism group acts primitively on each of the two bipartition parts, is given. It follows that only five such graphs exist. In the known examples of cubic edge- but not vertex-transitive graphs [1, 8] the edge stabilizer is a 2-group of order at least 8. For example, the edge stabilizer of the Gray graph is a group of order 16. It is the object of this paper to give a construction of an infinite family of cubic edge- but not vertex-transitive graphs with edge stabilizer isomorphic to \mathbb{Z}_2 . It would be interesting to see a construction of cubic edge- but not vertex-transitive graphs with the automorphism group acting regularly on the edge set.

For a group G and a generating set Q of G such that $1 \notin Q = Q^{-1}$, the *Cayley graph* $\text{Cay}(G, Q)$ of G relative to Q has vertex set G and edges of the form $[g, gq]$, $g \in G, q \in Q$. If $Y = L(X)$ is the line graph of a graph X and is not the line graph of any graph not isomorphic to X , we shall say that $X = L^{-1}(Y)$ is the *anti-line graph* of Y . We shall use the symbol \wp_n to denote the set $\{1, 2, \dots, n\}$ of the first n positive integers. A cycle in a graph is said to be *induced* if it has no chords. An *n-cycle* is a cycle of length n .

The following is the main result of this paper.

Theorem 1.1 *Let $n = 3k \geq 9$ and let $a = a_n$ and $b = b_n$ be permutations of \wp_n mapping according to the following rules:*

$$a = (1, 2, 3)(4, 5, 6)(7, 8, 9) \dots (3k - 2, 3k - 1, 3k)$$

and

$$b = (1, 2, 6)(4, 5, 9)(7, 8, 12) \dots (3k - 5, 3k - 4, 3k)(3)(3k - 2)(3k - 1).$$

Then $\langle a, b \rangle = A_n$ and the anti-line graph X_n of the Cayley graph $Y_n = \text{Cay}(A_n, Q_{a,b})$, where $Q_{a,b} = \{a, b, a^{-1}, b^{-1}\}$, is a cubic edge- but not vertex-transitive graph with edge stabilizer isomorphic to \mathbb{Z}_2 . Moreover, $\text{Aut } X_n = \text{Aut } Y_n$ is isomorphic to $A_n \times \mathbb{Z}_2$ for n even and to S_n for n odd.

Theorem 1.1 will be proved in Section 3 following a series of lemmas dealing with the cycle structure of the graphs Y_n , in particular the structure of induced 24-cycles.

2 Preliminaries

Let a group G act (on the right) on a set V and let Q be a nonempty subset of G . We define the *action digraph* $Act(G, V, Q)$ to be the digraph with vertex set V and arcs of the form (v, vq) , $v \in V$, $q \in Q$. Throughout this paper we shall be assuming that the action of G is transitive and that Q is a generating set of G , thus forcing $Act(G, V, Q)$ to be (weakly) connected.

For a group G and a subset $Q \subseteq G$ we let $\text{Aut}(G, Q) = \{\alpha \in \text{Aut}(G) : \alpha(Q) = Q\}$. Next, by a Q -sequence and a Q -relation in G we shall mean a word on symbols from $Q \cup Q^{-1}$ which corresponds, respectively, to a simple path and to a simple cycle in $\text{Cay}(G, Q \cup Q^{-1})$. In other words, by a Q -relation we mean a primitive Q -relation and by a Q -sequence a reduced word on symbols from $Q \cup Q^{-1}$ such that no proper subword is a relation in G . (We remark that given elements $q_1, q_2, \dots, q_m \in Q$, the product symbol $q_1 q_2 \dots q_m$ will also be used to denote a Q -sequence in G with terms $q_1, q_2, \dots, q_m \in Q$. This should cause no confusion.) We say that two Q -sequences are *equivalent* if one can be obtained from the other by a finite series of transformations of the following three types: a cyclic rotation, taking the Q -sequence in the reverse order with all terms inverted (that is the inverse Q -sequence), or substituting each term in the Q -sequence by its image under an element of $\text{Aut}(G, Q \cup Q^{-1})$. Note that the corresponding equivalence relation on Q -sequences distinguishes between relations and nonrelations in G . To each Q -sequence in a group G acting on a set V and a vertex v of $Act(G, V, Q)$, we may associate in a natural way a walk originating in v . Furthermore, if the action of G on V is faithful, then a Q -sequence in G is a relation if and only if it represents a closed walk at every vertex of $Act(G, V, Q)$. In this sense the action digraph is a useful geometric tool for testing whether a given sequence is a group relation or not.

Let X be a graph, G be a subgroup of $\text{Aut } X$, \mathcal{C} be a G -orbit of cycles in X , that is an orbit of the action of G on cycles in X , and let \mathcal{P} be a G -orbit of paths of X . We define the \mathcal{C} -frequency $\nu_{\mathcal{C}}(\mathcal{P})$ of \mathcal{P} to be the number of cycles in \mathcal{C} containing a given path in \mathcal{P} . Similarly, we define the \mathcal{P} -frequency $\mu_{\mathcal{P}}(\mathcal{C})$ of \mathcal{C} as the number of 2-paths from \mathcal{P} contained on a given cycle in \mathcal{C} . A straightforward counting argument gives us the following lemma.

Lemma 2.1 *Let \mathcal{C} and \mathcal{P} be G -orbits of cycles and paths in X respectively.*

Then

$$|\mathcal{P}| \nu_{\mathcal{C}}(\mathcal{P}) = |\mathcal{C}| \mu_{\mathcal{P}}(\mathcal{C}).$$

Given a group G and an element $g \in G$, we let α_g and λ_g denote the conjugation and the left multiplication of elements in G by g , and we let $L_G = \{\lambda_g : g \in G\}$ denote the left regular representation of G .

The proof of the next result is straightforward.

Lemma 2.2 *Let Q be a generating set of a group G such that $1 \notin Q = Q^{-1}$ and let $X = \text{Cay}(G, Q)$. Then $\text{Aut } X \geq L_G \cdot \text{Aut}(G, Q)$.*

The proposition below may easily be deduced from [4, Lemma 2.1].

Proposition 2.3 *Let G be a group and Q a generating set of G such that $1 \notin Q = Q^{-1}$. Let $X = \text{Cay}(G, Q)$ and $H = \text{Aut } X$. Then $N_H(G) \cap H_1 \cong \text{Aut}(G, Q)$.*

We end this section with the following consequence of a classical result of Jordan [11, Theorem 13.3].

Theorem 2.4 *A primitive group which contains a 3-cycle is either alternating or symmetric.*

3 Proof of Theorem 1.1

We start by outlining the strategy of the proof of Theorem 1.1. For $n = 3k \geq 9$ let $a = a_n, b = b_n$ be as in the statement of Theorem 1.1, let $t = t_n$ be the permutation of \wp_n which fixes $3i$ and interchanges $3i - 1$ and $3i - 2$ for each $i \in \wp_k$, and let $G_n = \langle \lambda_a, \lambda_b, \alpha_t \rangle$. We will first show that the group $\langle a, b \rangle$ is indeed the alternating group A_n (Lemma 3.1), implying that $G_n = \langle L_{A_n}, \alpha_t \rangle$. Next, based on a detailed analysis of induced 24-cycles in Y_n (Lemma 3.2) we will prove a result about 2-paths in Y_n (Lemma 3.3) enabling us to take our next step which will consist in showing that the action of a vertex stabilizer in Y_n is faithful and semiregular on the set of neighbors and thus of order 2 or 4. Finally, we will show that the latter possibility cannot occur. More precisely, we will prove that $\text{Aut } Y_n = G_n$. This will then force the graphs X_n to be edge- but not vertex-transitive.

Lemma 3.1 *Let $n = 3k \geq 9$ and let a and b have the meaning defined in the statement of Theorem 1.1. Then $\langle a, b \rangle = A_n$.*

PROOF. Clearly, $\langle a, b \rangle \leq A_n$ for both a and b are products of disjoint 3-cycles. Since the action digraph $Act(\langle a, b \rangle, \wp_n, \{a, b\})$ (see Figure 1) is connected, we have that $\langle a, b \rangle$ is transitive on \wp_n . We now show that $\langle a, b \rangle$ is primitive on \wp_n . To see this let W be a block of $\langle a, b \rangle$ containing 3. Since b fixes 3, it follows that W is fixed by b . Thus W is a union of orbits of b . If W contains any orbit of b of length 3, say if $\{3i - 2, 3i - 1, 3i + 3\} \subseteq W$ for some $i \in \wp_{k-1}$, then a must also fix W , for W and the image of W under a intersect in $3i - 1$. Hence W is fixed by both a and b , and thus W is an orbit of $\langle a, b \rangle$, forcing $W = \wp_n$. Hence we may assume that $W \subseteq \{3, 3k - 2, 3k - 1\}$. Assume that $3k - 1 \in W$. Observe that

$$ab^{-1} = (3, 6, 9, \dots, 3k - 3, 3k, 3k - 2, 3k - 1, 3k - 4, \dots, 8, 5, 2)$$

is a $(2k + 1)$ -cycle fixing each $3i - 2$, $i \in \wp_{k-1}$. Since $(ab^{-1})^{k+1}$ takes 3 to $3k - 1$, it follows that $(ab^{-1})^{k+1}$ fixes W . But then since $(ab^{-1})^{k+1}$ takes $3k - 1$ to 6 we have that $6 \in W$, a contradiction. A similar contradiction is derived in the case when $3k - 2 \in W$. It follows that $W = \{3\}$ and thus $\langle a, b \rangle$ is primitive on \wp_n . This fact is of crucial importance in deducing that $\langle a, b \rangle = A_n$. Note that

$$ab = (2, 3)(1, 6, 5) \dots (3i - 2, 3i + 3, 3i + 2) \dots (3k - 8, 3k - 3, 3k - 4)(3k - 5, 3k, 3k - 2, 3k - 1)$$

and that

$$ba = (1, 3)(2, 4, 6) \dots (3i - 1, 3i + 1, 3i + 3) \dots (3k - 7, 3k - 5, 3k - 3)(3k - 4, 3k - 2, 3k - 1, 3k).$$

In particular

$$(ab)^{12} = 1 = (ba)^{12}. \tag{1}$$

Hence $(ab)^3 = (2, 3)(3k - 5, 3k, 3k - 2, 3k - 1)$ and $(ba)^3 = (1, 3)(3k - 4, 3k - 2, 3k - 1, 3k)$ and so $((ab)^3(ba)^3)^5 = (1, 2, 3)$ is a 3-cycle. Hence $\langle a, b \rangle$ is a primitive group containing a 3-cycle and so $\langle a, b \rangle = A_n$ in view of Theorem 2.4. ■

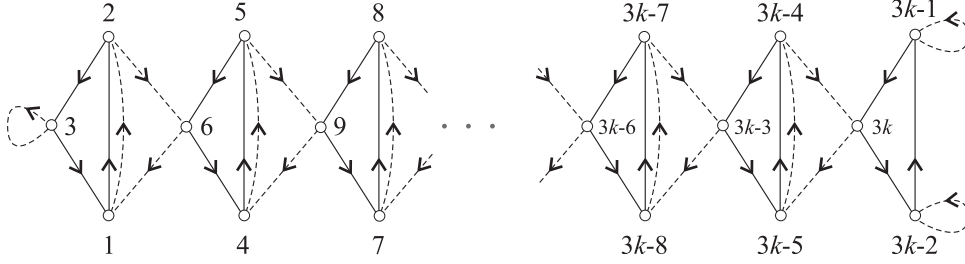


Figure 1: Action digraph $Act(\langle a, b \rangle, \varphi_n, \{a, b\}) = Act(A_n, \varphi_n, \{a, b\})$

Induced 24-cycles in Y_n are now going to be studied via the action of the group $G = G_n$. (We omit the index n for simplicity reasons.) Clearly, an induced cycle in Y_n of length greater than 3 must necessarily arise from a $Q_{a,b}$ -relation of the form $a^{\epsilon_1} b^{\omega_1} \dots a^{\epsilon_r} b^{\omega_r}$ for some positive integer r . Given a G -orbit \mathcal{C} of induced 24-cycles we have that $|\mathcal{C}| = [G : G_C]$, where $C \in \mathcal{C}$. But the restriction of G_C to C is isomorphic to a subgroup of D_{24} , the dihedral group of order 24, and moreover the action of G on induced 24-cycles is clearly faithful. The following result analyzes different possibilities for the setwise stabilizer G_C .

Lemma 3.2 *Let $n = 3k \geq 9$, let \mathcal{C} be a G -orbit of induced 24-cycles in Y_n and let $C \in \mathcal{C}$. If $G_C \leq D_{24}$ is the setwise stabilizer of C in G . Then either*

- (i) $|G_C|$ is not divisible by 8; or
- (ii) $G_C \cong D_{24}$ and C arises from a relation equivalent to $(ab)^{12}$; or
- (iii) $G_C \cong D_8$, and $k \in \{3, 4\}$ and C arises from a relation equivalent to $(ababa^{-1}b)^4$ for $k = 3$ and to $(aba^{-1}ba^{-1}b)^4$ for $k = 4$.

PROOF. Let R be a relation giving rise to the cycle C . If $|G_C|$ is not divisible by 8 then clearly G_C is isomorphic either to D_{24} or to D_8 . Let us first show that if $G_C \cong D_{24}$ then R is equivalent to $(ab)^{12}$. Namely in order for C to admit a rotational symmetry of order 12, R must be equivalent to $(x_1 x_2)^{12}$ for some $x_1 \in \{a, a^{-1}\}$ and $x_2 \in \{b, b^{-1}\}$. But without loss of generality $x_1 = a$, implying $x_2 = b$ as $(ab^{-1})^{12}$ is not a relation in G . Note that in view of (1) the sequence $(ab)^{12}$ is indeed a relation.

We may now assume that G_C is a proper subgroup of D_{24} . It remains to show that $|G_C| \neq 8$. Assume the contrary. Then clearly $G_C \cong D_8$ and the restriction of G_C to C contains a reflection of C . It may be inferred that $R = S^4 = (x_1x_2x_3x_4x_5x_6)^4$, where $x_{2i+1} \in \{a, a^{-1}\}$ and $x_{2i} \in \{b, b^{-1}\}$, and furthermore, because of reflection symmetry on C , one of the following holds: either $x_1 = x_3$ and $x_4 = x_6$ or $x_2 = x_4$ and $x_5 = x_1$ or $x_3 = x_5$ and $x_6 = x_2$. This gives us a total of 16 possibilities for R . Applying α_t on R (and if necessary a cyclic rotation), these 16 possibilities are reduced, up to equivalence, to the following 8 sequences.

1. $(ab)^{12}$
2. $(ab^{-1})^{12}$
3. $(ababab^{-1})^4$
4. $(ababa^{-1}b)^4$
5. $(abab^{-1}ab^{-1})^4$
6. $(abab^{-1}a^{-1}b^{-1})^4$
7. $(aba^{-1}bab^{-1})^4$
8. $(aba^{-1}ba^{-1}b)^4$

Of course, sequence no. 1 is a relation but C cannot arise from it as $G_C = 8$. Sequence no. 2 is clearly not a relation for we know from the proof of Lemma 3.1 that ab^{-1} has order $2k+1$. We check the remaining sequences on the action digraph $Act(A_n, \varphi_n, \{a, b\})$ (see Figure 1). Firstly, sequence no. 3 is not a relation for we have that R takes 4 to 8 if $k=3$ and it takes 4 to 5 if $k \geq 4$. Secondly, it may be checked that sequence no. 4 is a relation if $k=3$. For $k \geq 4$ we have that R takes 1 to 5 and so it is not a relation. Thirdly, sequence no. 5 is not a relation as R takes 1 to 9 if $k=3$, it takes 1 to 10 if $k=4$ and it takes 1 to 12 if $k \geq 5$. Next, sequence no. 6 is not a relation for R takes 1 to 4, 2, 8, 14, 20, 26 and 25 if $k=3$, $k=4$, $k=5$, $k=6$, $k=7$, $k=8$, $k=9$ and $k \geq 10$. Similarly, sequence no. 7 is not a relation as R takes 1 to 7 if $k=3$, it takes 4 to 10 if $k=4$ and it takes 1 to 12 if $k \geq 5$. Finally, it may be checked that sequence no. 8 is a relation if $k=4$. On the other hand for $k=3$ we have that R takes 1 to 2 and for $k \geq 5$ we have that R takes 1 to 12. This proves Lemma 3.2. ■

We observe that there are four G -orbits on 2-paths in Y_n , their representatives being the 2-paths (id, a, a^{-1}) , (id, b, b^{-1}) , (id, a, ab) and (id, a, ab^{-1}) . Any 2-path belonging to the same G -orbits as the four 2-paths above will be said to be of types aa , bb , ab and ab^{-1} , respectively. Note that 2-paths of types aa and bb are contained on triangles, whereas 2-paths of types ab and ab^{-1} are not. In the lemma below we show that Y_n possesses a feature which distinguishes also between 2-paths of types ab and ab^{-1} .

Lemma 3.3 *Let $n = 3k \geq 15$. The numbers of induced 24-cycles containing a given 2-path of type ab and a given 2-path of type ab^{-1} in Y_n are of different parities.*

PROOF. Let \mathcal{P}_1 and \mathcal{P}_2 denote the G -orbits of 2-paths of types ab and ab^{-1} , respectively, and let ν_1 and ν_2 be the numbers of induced 24-cycles containing a given 2-path of type ab and a given 2-path of type ab^{-1} , respectively. We therefore have $\nu_1 = \sum_{\mathcal{C}} \nu_{\mathcal{C}}(\mathcal{P}_1)$, where \mathcal{C} runs over all orbits of induced 24-cycles. Similarly, $\nu_2 = \sum_{\mathcal{C}} \nu_{\mathcal{C}}(\mathcal{P}_2)$. Since the action of G is regular on both \mathcal{P}_1 and \mathcal{P}_2 , Lemma 2.1 implies that

$$\nu_{\mathcal{C}}(\mathcal{P}_1) + \nu_{\mathcal{C}}(\mathcal{P}_2) = \frac{|\mathcal{C}|(\mu_{\mathcal{P}_1}(\mathcal{C}) + \mu_{\mathcal{P}_2}(\mathcal{C}))}{|G|}$$

But $|\mathcal{C}| = [G : G_{\mathcal{C}}] = \frac{|G|}{|G_{\mathcal{C}}|}$ and for every cycle in \mathcal{C} each of the 24 2-paths is either of type ab or of type ab^{-1} . We must therefore have $\nu_{\mathcal{C}}(\mathcal{P}_1) + \nu_{\mathcal{C}}(\mathcal{P}_2) = 24/|G_{\mathcal{C}}|$, where $\mathcal{C} \in \mathcal{C}$ is arbitrary. Now using Lemma 3.2 we have that this sum is always even except for the special orbit of 24-cycles arising from the relation $(ab)^{12}$. Therefore $\nu_{\mathcal{C}}(\mathcal{P}_1)$ and $\nu_{\mathcal{C}}(\mathcal{P}_2)$ are of same parity, unless \mathcal{C} is the orbit of cycles arising from the relation $(ab)^{12}$. Hence ν_1 and ν_2 are of different parities and the result follows. ■

We have now gathered all the relevant information on the cycle and path structure of the graphs Y_n . To prove Theorem 1.1 one additional concept is needed. Given a graph X and a 2-path $P = [u, v, w]$ in X we let $\omega_m(P) = \omega_m(u, v, w)$ denote the number of induced k -cycles containing the 2-path P . The proof of the following simple observation is straightforward and is omitted.

Proposition 3.4 *Let X be a connected graph such that for any two adjacent vertices $u, v \in V(X)$, and for any for any neighbors $x, y \in N(v) \setminus \{u\}$ there exists an integer $m \geq 3$ such that $\omega_m(u, v, x) \neq \omega_m(u, v, y)$. Then no nonidentity automorphism of X fixes two adjacent vertices.*

Proof of Theorem 1.1. Let $H_n = \text{Aut } X_n$. Recall that the permutation t fixes $3i$ and interchanges $3i - 1$ and $3i - 2$ for each $i \in \wp_k$. Hence α_t interchanges a and a^{-1} and b and b^{-1} , and so $\alpha_t \in \text{Aut}(A_n, Q_{a,b}) \leq (G_n)_1$. Thus $G_n = \langle L_{A_n}, \alpha_t \rangle \leq H_n$.

Clearly, L_{A_n} is a normal subgroup of index 2 in G_n . Suppose first that n is even. A short calculation shows that L_{A_n} has trivial intersection with $Z(H_n) = \langle \lambda_t \alpha_t \rangle \cong \mathbb{Z}_2$. This shows that $H_n \cong A_n \times \mathbb{Z}_2$ in this case. On the other hand for n odd, we clearly have that $H_n \cong S_n$.

It remains to show that $H_n = G_n$. We first prove that $[H_n : L_{A_n}]$ is either 2 or 4. For a 2-path P in Y_n we let $\omega_m(P)$ be denoted by $\omega_m(aa)$, $\omega_m(bb)$, $\omega_m(ab)$ and $\omega_m(ab^{-1})$ if P is of types aa , bb , ab and ab^{-1} , respectively. Observe that $\omega_3(ab) = 0 = \omega_3(ab^{-1})$, whereas $\omega_3(aa) > 0$ and $\omega_3(bb) > 0$. Moreover, for $k \geq 5$, we have by Lemma 3.3 that $\omega_{24}(ab)$ and $\omega_{24}(ab^{-1})$ are of different parities. On the other hand, for $k = 3$, it may be checked that that $\omega_{14}(ab) = 0$ whereas $\omega_{14}(ab^{-1}) \neq 0$. Similarly, for $k = 4$, it may be checked that $\omega_{18}(ab) = 0$ whereas $\omega_{18}(ab^{-1}) \neq 0$. Thus applying Proposition 3.4 we conclude that no nonidentity automorphism of Y_n fixes two adjacent vertices. It follows that $[H_n : L_{A_n}] \in \{2, 4\}$. In particular, it follows that $(H_n)_1$ is isomorphic either to \mathbb{Z}_2 or to $\mathbb{Z}_2 \times \mathbb{Z}_2$ with each of the sets $\{a, a^{-1}\}$, $\{a, b\}$, and $\{a, b^{-1}\}$ being a block of imprimitivity. Assume that $(H_n)_1$ has order 4. Then G_n is of index 2 in H_n and so normal in H_n . It is easily seen that L_{A_n} is a characteristic subgroup of G_n and therefore a normal subgroup of H_n . Hence $(H_n)_1 = N_{H_n}(L_{A_n}) \cap (H_n)_1$. In view of Proposition 2.3 we have $(H_n)_1 = \text{Aut}(A_n, Q_{a,b})$. In particular, $\text{Aut}(A_n, Q_{a,b}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The same holds for its restriction to $Q_{a,b}$. Hence there exists $\gamma \in \text{Aut}(A_n, Q_{a,b})$ whose restriction to $Q_{a,b}$ is $(a, a^{-1})(b, b^{-1})$. Now recall that, since $n \neq 6$, the automorphisms of A_n are of the form α_x , $x \in S_n$, and so the existence of such a γ contradicts the fact that a and b are not conjugate in S_n . We conclude that $(H_n)_1 \cong \mathbb{Z}_2$ has order 2. Therefore H_n coincides with G_n and so it is isomorphic to $\cong A_n \times \mathbb{Z}_2$ for n even and to S_n for n odd. In particular, the graph $X_n = L^{-1}(Y_n)$ is edge- but not vertex-transitive and the edge-stabilizer is isomorphic to \mathbb{Z}_2 . ■

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