University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 36 (1998), 591

SPECIAL FORMAL SERIES SOLUTIONS OF LINEAR OPERATOR EQUATIONS

Sergei A. Abramov Marko Petkovšek
Anna Ryabenko

ISSN 1318-4865

February 12, 1998

Ljubljana, February 12, 1998

Special Formal Series Solutions of Linear Operator Equations

Sergei A. Abramov[*]
Computer Center of
the Russian Academy of Science
Vavilova 40, Moscow 117967, Russia
`sabramov@cs.msu.su`

Marko Petkovšek[†]
Department of Mathematics
University of Ljubljana
Jadranska 19, 1111 Ljubljana, Slovenia
`marko.petkovsek@fmf.uni-lj.si`

Anna Ryabenko[‡]
Department of Computational Mathematics & Cybernetics, Moscow State University
Moscow 119899, Russia
`ryabenko@cs.msu.su`

Abstract

The transformation which assigns to a linear operator L the recurrence satisfied by coefficient sequences of the polynomial series in its kernel, is shown to be an isomorphism of the corresponding operator algebras. We use this fact to help factoring q-difference and recurrence operators, and to find "nice" power series solutions of linear differential equations.

In particular, we characterize generalized hypergeometric series that solve a linear differential equation with polynomial coefficients at an ordinary point of the equation, and show that these solutions remain hypergeometric at any other ordinary point. Therefore to find all generalized hypergeometric series solutions, it suffices to look at a finite number of points: all the singular points, and a single, arbitrarily chosen ordinary point.

We also show that at a point $x = a$ we can have power series solutions with:

- polynomial coefficient sequence – only if the equation is singular at $a + 1$,
- non-polynomial rational coefficient sequence – only if the equation is singular at a.

1 Introduction and notation

The method of solving linear differential equations by means of power series has been known for centuries. Here we look at formal series that are based on other polynomial sequences besides the powers, and show how they can be used to reduce questions about operators of different types (e.g., differential, difference, q-difference) to questions about operators of a single type, namely recurrence operators.

We consider a transformation $\mathcal{R}_\mathcal{B}$ which assigns to a linear operator L acting on the polynomial algebra $K[x]$ its induced recurrence operator $\mathcal{R}_\mathcal{B}L$. The transformation is defined in Section 2. We show that $\mathcal{R}_\mathcal{B}$ is an isomorphism of the corresponding operator algebras. This result is applied in Sections 3, 4, and 5 to the cases of q-difference, recurrence, and differential operators. In particular, we show how transformation $\mathcal{R}_\mathcal{B}$ can help factor these operators. This is important because although general factorization algorithms are known [8], they are still highly impractical.

Subsections 5.1, 5.2, and 5.3 are devoted to the search for "nice" power series solutions in the differential case. We are interested in series with coefficients which are polynomial, rational, or hypergeometric in their subscript, respectively.

[*]Supported in part by grant 95-IN-RU-412 from the RFBR and INTASS.
[†]Supported in part by grant J2-6193-0101-94 of the Slovenian Ministry of Science and Technology.
[‡]Supported in part by grant 95-IN-RU-412 from the RFBR and INTASS.

Call a sequence $(c_n)_{n=0}^{\infty}$ *hypergeometric* if there is a rational function $R(x)$ such that $c_{n+1} = R(n)c_n$ for all *large enough* n. If c_n is hypergeometric and eventually nonzero then $R(x)$ is uniquely determined and we call it the *consecutive-term ratio* of c_n. Obviously, a rational sequence is hypergeometric, and the product of hypergeometric sequences is hypergeometric.

Two hypergeometric sequences a_n and b_n are *similar* if there is a rational function $r(x)$ such that $a_n = r(n)b_n$ for all large enough n. A linear combination of pairwise similar hypergeometric terms is obviously hypergeometric. Also, if a_n is hypergeometric and k a fixed integer, then a_{n+k} is similar to a_n.

A formal power series $y = \sum_{n=0}^{\infty} c_n x^n$ is called a *(generalized) hypergeometric series* if the sequence of coefficients $(c_n)_{n=0}^{\infty}$ is hypergeometric.

Lemma 1 *Let $y = \sum_{n=0}^{\infty} c_n x^n$ be a hypergeometric series, and $p(x)$ a polynomial. Then $p(x)y$ is a hypergeometric series with similar coefficients.*

Proof: Let $p(x) = \sum_{k=0}^{d} u_k x^k$ and $p(x)y = \sum_{n=0}^{\infty} b_n x^n$. Then

$$p(x)y = \sum_{k=0}^{d}\sum_{n=0}^{\infty} c_n u_k x^{n+k} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\min\{n,d\}} u_k c_{n-k},$$

so $b_n = \sum_{k=0}^{d} u_k c_{n-k}$ for $n \geq d$. This is a linear combination of hypergeometric terms which are all similar to c_n, hence b_n is hypergeometric and similar to c_n. □

Following [9], we denote the rising and falling factorial powers by

$$x^{\overline{n}} = \prod_{k=0}^{n-1}(x+k), \quad x^{\underline{n}} = \prod_{k=0}^{n-1}(x-k),$$

respectively.

We use $\mathbb{N}$ to denote the set of nonnegative integers. Throughout the paper, K denotes an arbitrary field of characteristic zero. We denote by E the shift operator on polynomials and rational functions over K, so that $Er(x) = r(x+1)$, for any $r \in K(x)$. Similarly, we denote by E_n the shift operator on sequences over K, so that $E_n a_n = a_{n+1}$ for any sequence $\langle a_n \rangle_{n=0}^{\infty}$ or $\langle a_n \rangle_{n\in\mathbb{Z}}$.

A preliminary version of this paper appeared as [5].

2 Compatible bases and transformation $\mathcal{R}_{\mathcal{B}}$

Let K be a field of characteristic zero. Denote by $K[x]$ the K-algebra of univariate polynomials over K, and by $\mathcal{L}_{K[x]}$ the K-algebra of linear operators $L : K[x] \to K[x]$. Further let $\mathcal{B} = \langle P_n(x) \rangle_{n=0}^{\infty}$ be a sequence of polynomials from $K[x]$ such that

P1. $\deg P_n = n$ for $n \geq 0$,

P2. $P_n \mid P_m$ for $0 \leq n < m$.

From **P1** it follows that $\{P_0, P_1, \ldots\}$ is a basis of $K[x]$.

Definition 1 A basis $\mathcal{B}$ of $K[x]$ satisfying **P1, P2**, and an operator $L \in \mathcal{L}_{K[x]}$ are *compatible* if there are $A, B \in \mathbb{N}$, and elements $\alpha_{i,n} \in K$ for $n \geq 0$ and $-A \leq i \leq B$, such that

$$LP_n = \sum_{i=-A}^{B} \alpha_{i,n}\, P_{n+i}, \tag{1}$$

with $P_k = 0$ when $k < 0$. □

In other words, L is compatible with $\mathcal{B}$ if the infinite matrix $[\alpha_{i-n,n}]_{i,n\in\mathbb{N}}$ corresponding to L in basis $\mathcal{B}$ is band-diagonal.

Example 1 Let $Dp(x) = p'(x)$ and $Ep(x) = p(x+1)$. Let $\mathcal{P} = \langle P_n(x)\rangle_{n=0}^\infty = \langle x^n\rangle_{n=0}^\infty$ be the *power basis*. Then $DP_n = nP_{n-1}$ and $EP_n = \sum_{k=0}^n \binom{n}{k} P_k$, so $\mathcal{P}$ is compatible with D (take $A = 1$, $B = 0$, $\alpha_{-1,n} = n$, $\alpha_{0,n} = 0$), but not with E.

On the other hand, let $\mathcal{C} = \langle P_n(x)\rangle_{n=0}^\infty = \langle \binom{x}{n}\rangle_{n=0}^\infty$ be the *binomial coefficient basis*. Then $EP_n = P_n + P_{n-1}$ and $DP_n = \sum_{k=0}^{n-1}(-1)^{n+k}/(k-n)P_k$, so $\mathcal{C}$ is compatible with E (take $A = 1$, $B = 0$, $\alpha_{-1,n} = \alpha_{0,n} = 1$), but not with D.

Now let $Lp(x) = xp(x)$, and take any basis $\mathcal{B} = \langle P_n(x)\rangle_{n=0}^\infty$ which satisfies **P1** and **P2**. Then $LP_n = \sum_{k=0}^{n+1} a_k(n)P_k$ for some constants $a_k(n) \in K$. Because of **P2**, $\sum_{k=0}^{n-1} a_k(n)P_k$ is divisible by P_n. Being a polynomial of degree at most $n-1$, it must vanish, therefore $LP_n = a_{n+1}(n)P_{n+1} + a_n(n)P_n$. So any basis $\mathcal{B}$ satisfying **P1**, **P2** is compatible with multiplication by the independent variable (take $A = 0$, $B = 1$, $\alpha_{0,n} = a_n(n), \alpha_{1,n} = a_{n+1}(n)$). □

Let $l_n : K[x] \to K$ be linear functionals such that $l_n(P_m) = \delta_{mn}$. Property **P2** implies that $l_n(P_kP_m) = 0$ when $n < \max\{k,m\}$. Therefore $K[x]$ naturally embeds into the algebra $K[[\mathcal{B}]]$ of formal series of the form

$$y = \sum_{n=0}^\infty c_nP_n(x) \qquad (c_n \in K), \tag{2}$$

with multiplication defined by

$$\left(\sum_{n=0}^\infty c_nP_n(x)\right)\left(\sum_{n=0}^\infty d_nP_n(x)\right) = \left(\sum_{n=0}^\infty e_nP_n(x)\right)$$

where

$$e_n = \sum_{\max\{j,k\}\le n\le j+k} c_jd_k\, l_n(P_jP_k).$$

If $\mathcal{B}$ and L are compatible then L can be extended to $K[[\mathcal{B}]]$ by setting

$$L\sum_{n=0}^\infty c_nP_n(x) = \sum_{n=0}^\infty \sum_{i=-A}^{B} \alpha_{i,n-i}c_{n-i}P_n(x) = \sum_{n=0}^\infty \sum_{i=-B}^{A} \alpha_{-i,n+i}c_{n+i}P_n(x) \tag{3}$$

with A, B and $\alpha_{i,n}$ as in (1), and $c_n = 0$ when $n < 0$. Clearly, a formal series $y \in K[[\mathcal{B}]]$ satisfies $Ly = 0$ if and only if its coefficient sequence $c = \langle c_n\rangle_{n\in\mathbb{Z}}$ satisfies the recurrence

$$\sum_{i=-B}^{A} \alpha_{-i,n+i}c_{n+i} = 0 \qquad (n \ge 0) \tag{4}$$

where, again, $c_n = 0$ when $n < 0$. Thus relative to the basis $\mathcal{B}$, any operator L compatible with $\mathcal{B}$ induces a recurrence operator

$$\mathcal{R}_\mathcal{B}L = \sum_{i=-B}^{A} \alpha_{-i,n+i}E_n^i \tag{5}$$

where E_n is the shift operator w.r.t. n ($E_n^kc_n = c_{n+k}$ for $k \in \mathbb{Z}$).

Example 2 Using (5) and Example 1 we find that

$$\begin{aligned}\mathcal{R}_\mathcal{P}D &= (n+1)E_n,\\ \mathcal{R}_\mathcal{C}E &= E_n + 1.\end{aligned}$$

Also, since $x \cdot x^n = x^{n+1}$ and $x \cdot \binom{x}{n} = (n+1)\binom{x}{n+1} + n\binom{x}{n}$, we find that

$$\begin{aligned}\mathcal{R}_\mathcal{P}x &= E_n^{-1},\\ \mathcal{R}_\mathcal{C}x &= n(E_n^{-1}+1).\end{aligned}$$ □

Now fix a basis $\mathcal{B} = \langle P_n(x) \rangle_{n=0}^{\infty}$ of $K[x]$ having properties **P1**, **P2**, and denote by $\mathcal{L}_{\mathcal{B}}$ the set of operators $L \in \mathcal{L}_{K[x]}$ compatible with $\mathcal{B}$.

Proposition 1 *Let $\sigma : K[[\mathcal{B}]] \to K^{\mathbb{Z}}$ be the mapping assigning to the formal series $y = \sum_{n=0}^{\infty} c_n P_n$ its coefficient sequence $c = \langle c_n \rangle_{n \in \mathbb{Z}}$ extended by taking $c_n = 0$ whenever $n < 0$. Then for any $L \in \mathcal{L}_{\mathcal{B}}$,*

$$\sigma\, Ly = (\mathcal{R}_{\mathcal{B}} L)\, \sigma y.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} K[[\mathcal{B}]] & \xrightarrow{L} & K[[\mathcal{B}]] \\ \sigma \downarrow & & \downarrow \sigma \\ K^{\mathbb{Z}} & \xrightarrow{\mathcal{R}_{\mathcal{B}} L} & K^{\mathbb{Z}}. \end{array}$$

Proof: Let $y = \sum_{n=0}^{\infty} c_n P_n(x)$. Then $\sigma\, Ly = \left\langle \sum_{i=-B}^{A} \alpha_{-i,n+i} c_{n+i} \right\rangle_{n \in \mathbb{Z}} = \left\langle \sum_{i=-A}^{B} \alpha_{i,n-i} c_{n-i} \right\rangle_{n \in \mathbb{Z}} = (\mathcal{R}_{\mathcal{B}} L)\, \sigma y$. Here the first and last equalities follow from (5) and (3), respectively. □

Corollary 1 *For $L \in \mathcal{L}_{\mathcal{B}}$ and $y = \sum_{n=0}^{\infty} c_n P_n \in K[[\mathcal{B}]]$, we have $L \sum_{n=0}^{\infty} c_n P_n = \sum_{n=0}^{\infty} ((\mathcal{R}_{\mathcal{B}} L) c)_n P_n$.*

Proof: $L \sum_{n=0}^{\infty} c_n P_n = Ly = \sum_{n=0}^{\infty} \sigma(Ly)_n P_n = \sum_{n=0}^{\infty} ((\mathcal{R}_{\mathcal{B}} L) \sigma y)_n P_n = \sum_{n=0}^{\infty} ((\mathcal{R}_{\mathcal{B}} L) c)_n P_n$. □

Proposition 2 *$\mathcal{L}_{\mathcal{B}}$ is a K-algebra.*

Proof: Let $\lambda_1, \lambda_2 \in K$, $L_1, L_2 \in \mathcal{L}_{\mathcal{B}}$, and

$$L_1 P_n = \sum_{i=-A_1}^{B_1} \alpha_{i,n} P_{n+i}, \quad L_2 P_n = \sum_{j=-A_2}^{B_2} \beta_{j,n} P_{n+j}. \tag{6}$$

Then $\lambda_1 L_1 + \lambda_2 L_2$ is clearly compatible with $\mathcal{B}$ (take $A = \max\{A_1, A_2\}$, $B = \max\{B_1, B_2\}$), hence it belongs to $\mathcal{L}_{\mathcal{B}}$. Also,

$$(L_2 L_1) P_n = \sum_{i=-A_1}^{B_1} \alpha_{i,n} L_2 P_{n+i} = \sum_{i=-A_1}^{B_1} \sum_{j=-A_2}^{B_2} \alpha_{i,n} \beta_{j,n+i} P_{n+i+j} = \sum_{k=-A_1-A_2}^{B_1+B_2} \gamma_{k,n} P_{n+k} \tag{7}$$

where

$$\gamma_{k,n} = \sum_i \alpha_{i,n} \beta_{k-i,n+i}. \tag{8}$$

Here $\alpha_{i,n}$ and $\beta_{j,n}$ are considered zero, unless $-A_1 \le i \le B_1$ and $-A_2 \le j \le B_2$. So $L_2 L_1$ is compatible with $\mathcal{B}$ (take $A = A_1 + A_2$ and $B = B_1 + B_2$). Hence $L_2 L_1 \in \mathcal{L}_{\mathcal{B}}$. □

Definition 2 $\mathcal{E}$ denotes the K-algebra of recurrence operators of the form

$$M = \sum_{i=-s}^{r} a_i(n) E_n^i \tag{9}$$

with $r, s \in \mathbb{N}$ and $a_i : \mathbb{Z} \to K$ for $-s \le i \le r$. We regard these operators as acting on the K-algebra of two-way infinite sequences $K^{\mathbb{Z}}$. □

Theorem 1 *The transformation*

$$\mathcal{R}_{\mathcal{B}} : \mathcal{L}_{\mathcal{B}} \to \mathcal{E}$$

defined in (5), is an isomorphism of K-algebras.

Proof: First we show that $\mathcal{R}_\mathcal{B}$ is a K-algebra homomorphism. Clearly $\mathcal{R}_\mathcal{B}(\lambda_1 L_1 + \lambda_2 L_2) = \lambda_1 \mathcal{R}_\mathcal{B} L_1 + \lambda_2 \mathcal{R}_\mathcal{B} L_2$. Using (5), (7) and (8) we find that

$$\mathcal{R}_\mathcal{B}(L_2 L_1) = \sum_{k=-B_1-B_2}^{A_1+A_2} \gamma_{-k,n+k} E_n^k = \sum_{i,k} \alpha_{i,n+k} \beta_{-k-i,n+k+i} E_n^k. \tag{10}$$

On the other hand, using (5) and (6),

$$\begin{aligned}(\mathcal{R}_\mathcal{B} L_2)(\mathcal{R}_\mathcal{B} L_1) &= \left(\sum_{j=-A_2}^{B_2} \beta_{-j,n+j} E_n^j\right)\left(\sum_{i=-A_1}^{B_1} \alpha_{-i,n+i} E_n^i\right) \\ &= \sum_{i,j} \beta_{-j,n+j} \alpha_{-i,n+i+j} E_n^{i+j} = \sum_{i,k} \alpha_{-i,n+k} \beta_{i-k,n+k-i} E_n^k\end{aligned}$$

which turns into (10) after replacing i by $-i$. Hence $\mathcal{R}_\mathcal{B}(L_2 L_1) = (\mathcal{R}_\mathcal{B} L_2)(\mathcal{R}_\mathcal{B} L_1)$.

Consider the mapping $\mathcal{S}_\mathcal{B} : \mathcal{E} \to \mathcal{L}_\mathcal{B}$ defined as follows. For $M \in \mathcal{E}$ as given in (9), let $\mathcal{S}_\mathcal{B} M = L \in \mathcal{L}_\mathcal{B}$ where

$$L P_n = \sum_{i=-r}^{s} a_{-i}(n+i) P_{n+i} \qquad (n \geq 0), \tag{11}$$

with $P_k = 0$ for $k < 0$. It is easy to see that $\mathcal{S}_\mathcal{B}$ is the inverse of $\mathcal{R}_\mathcal{B}$. This proves that $\mathcal{R}_\mathcal{B}$ is one-to-one and onto, and hence a K-algebra isomorphism. □

In the next three sections, we apply these results to the cases of q-difference, recurrence, and differential operators, respectively.

3 q-Difference operators

Let $q \in K \setminus \{0\}$ be such that $q^n \neq 1$ for all $n \in \mathbb{N} \setminus \{0\}$. Define $Q \in \mathcal{L}_{K[x]}$ by $Qp(x) = p(qx)$, and consider operators of the form

$$L = \sum_{k=0}^{r} p_k(x) Q^k \tag{12}$$

where $r \in \mathbb{N}$, $p_k \in K[x]$, and $p_r \neq 0$. They form the skew polynomial algebra $K[x, Q]$ with commutation rule $Qx = qxQ$. As $Qx^n = q^n x^n$, operator Q is compatible with the power basis $\mathcal{P} = \langle x^n \rangle_{n=0}^\infty$ (take $A = B = 0$, $\alpha_{0,n} = q^n$). To describe transformation $\mathcal{R}_\mathcal{P}$ on $K[x, Q]$, it suffices to give it on the two generators Q and x. Using (5) we have

$$\begin{aligned}\mathcal{R}_\mathcal{P} : \quad Q &\mapsto q^n \quad \text{(termwise multiplication by } q^n\text{)}, \\ x &\mapsto E_n^{-1} \quad \text{(back-one shift)}.\end{aligned}$$

Thus $\mathcal{R}_\mathcal{P}$ maps $K[x, Q]$ into $K[q^n, E_n^{-1}]$. For symmetry, write $x = q^n$. Then $E_n q^n = q^{n+1} = qx = Qx$. As the coefficients of $\mathcal{R}_\mathcal{P} L$ do not depend on n directly but only on q^n, the transformation $q^n \mapsto x$, $E_n \mapsto Q$ embeds $\mathcal{R}_\mathcal{P} L$ into $K[x, Q, Q^{-1}]$. Now extend $\mathcal{R}_\mathcal{P}$ to a mapping of the skew Laurent-polynomial algebra $K[x, x^{-1}, Q, Q^{-1}]$ into itself by

$$\begin{aligned}\mathcal{R}_\mathcal{P} : \quad & Q \mapsto x, \qquad Q^{-1} \mapsto x^{-1}, \\ & x \mapsto Q^{-1}, \qquad x^{-1} \mapsto Q.\end{aligned}$$

In four steps

$$\begin{aligned}\mathcal{R}_\mathcal{P} : \quad & Q \mapsto x \mapsto Q^{-1} \mapsto x^{-1} \mapsto Q, \\ & x \mapsto Q^{-1} \mapsto x^{-1} \mapsto Q \mapsto x,\end{aligned}$$

we are back to where we started, so $\mathcal{R}_\mathcal{P}$ is an automorphism of $K[x, x^{-1}, Q, Q^{-1}]$ of order 4.

Write $L \in K[x, x^{-1}, Q, Q^{-1}] \setminus \{0\}$ as

$$L = \sum_{i,k} c_{i,k}\, x^i Q^k. \tag{13}$$

Then

$$\mathcal{R}_\mathcal{P} L = \sum_{i,k} c_{i,k} Q^{-i} x^k = \sum_{i,k} c_{i,k} q^{-ik} x^k Q^{-i} = \sum_{i,k} \tilde{c}_{i,k} x^i Q^k \tag{14}$$

where $\tilde{c}_{i,k} = c_{-k,i} q^{ik}$. From (14) we see that for q-difference operators of the form (13) transformation $\mathcal{R}_\mathcal{P}$ has a simple geometric description. Apart from multiplication by certain powers of q, it corresponds to counter-clockwise rotation of the coefficient array $c_{i,k}$ around $c_{0,0}$ by 90°:

$c_{i,k}$	$\dots$	Q^{-1}	Q^0	Q^1	$\dots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	
x^{-1}	$\dots$	$c_{-1,-1}$	$c_{-1,0}$	$c_{-1,1}$	$\dots$
x^0	$\dots$	$c_{0,-1}$	$c_{0,0}$	$c_{0,1}$	$\dots$
x^1	$\dots$	$c_{1,-1}$	$c_{1,0}$	$c_{1,1}$	$\dots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	

$\xrightarrow{\mathcal{R}_\mathcal{P}}$

$\tilde{c}_{i,k}$	$\dots$	Q^{-1}	Q^0	Q^1	$\dots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	
x^{-1}	$\dots$	$c_{-1,1}q$	$c_{0,1}$	$c_{1,1}q^{-1}$	$\dots$
x^0	$\dots$	$c_{-1,0}$	$c_{0,0}$	$c_{1,0}$	$\dots$
x^1	$\dots$	$c_{-1,-1}q^{-1}$	$c_{0,-1}$	$c_{1,-1}q$	$\dots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	

Definition 3 The *effective order* of L is

$$\rho(L) = \max\{k \in \mathbb{Z};\ c_{i,k} \neq 0 \text{ for some } i\} - \min\{k \in \mathbb{Z};\ c_{i,k} \neq 0 \text{ for some } i\},$$

and the *effective degree* of L is

$$\delta(L) = \max\{i \in \mathbb{Z};\ c_{i,k} \neq 0 \text{ for some } k\} - \min\{i \in \mathbb{Z};\ c_{i,k} \neq 0 \text{ for some } k\}.$$

□

Obviously, $\rho(\mathcal{R}_\mathcal{P} L) = \delta(L)$ and $\delta(\mathcal{R}_\mathcal{P} L) = \rho(L)$.

The fact that $\mathcal{R}_\mathcal{P}$ is an automorphism of $K[x, x^{-1}, Q, Q^{-1}]$ can be exploited to find factors of degree 0 and 1 (and any order) for operators in $K[x, x^{-1}, Q, Q^{-1}]$.

Proposition 3 *A q-difference operator $L \in K[x, x^{-1}, Q, Q^{-1}]$ has a non-trivial left (resp. right) factor $L_1 \in K[x, x^{-1}, Q, Q^{-1}]$ of effective degree d, if and only if the induced operator $\mathcal{R}_\mathcal{P} L$ has a non-trivial left (resp. right) factor $M_1 \in K[x, x^{-1}, Q, Q^{-1}]$ of effective order d.*

Proof: If $L = L_1 L_2$ with $\delta(L_1) = d$ then $\mathcal{R}_\mathcal{P} L = (\mathcal{R}_\mathcal{P} L_1)(\mathcal{R}_\mathcal{P} L_2)$ and $\rho(\mathcal{R}_\mathcal{P} L_1) = \delta(L_1) = d$. Conversely, if $\mathcal{R}_\mathcal{P} L = M_1 M_2$ with $\rho(M_1) = d$ then $L = (\mathcal{R}_\mathcal{P}^{-1} M_1)(\mathcal{R}_\mathcal{P}^{-1} M_2)$ and $\delta(\mathcal{R}_\mathcal{P}^{-1} M_1) = \rho(\mathcal{R}_\mathcal{P}(\mathcal{R}_\mathcal{P}^{-1} M_1)) = \rho(M_1) = d$. For right factors the proof is analogous. □

To find factors of L of effective degree 0, find factors of $\mathcal{R}_\mathcal{P} L$ of effective order 0. Write $\mathcal{R}_\mathcal{P} L = x^{-a} M Q^{-b}$ where $M = \sum_{k=0}^{r} p_k(x) Q^k$ and $p_k(x)$ are polynomials. For left factors of effective order 0, compute $\gcd_{0 \le k \le r} p_k(x)$. For right factors of effective order 0, compute $\gcd_{0 \le k \le r} p_k(q^{-k} x)$.

Example 3 Let

$$L_1 = Q^2 - (qx^2 + 1)Q + qx^2, \quad L_2 = Q^2 - (q^3 x^2 + 1)Q + qx^2.$$

Then

$$\mathcal{R}_\mathcal{P} L_1 = x^2 - (qQ^{-2} + 1)x + qQ^{-2} = (x^2 - x) - qQ^{-2}(x - 1) = (x - qQ^{-2})(x - 1),$$
$$\mathcal{R}_\mathcal{P} L_2 = x^2 - (q^3 Q^{-2} + 1)x + qQ^{-2} = (x^2 - x) - q(x - 1)Q^{-2} = (x - 1)(x - qQ^{-2}),$$

giving factorizations

$$L_1 = (Q - qx^2)(Q - 1), \quad L_2 = (Q - 1)(Q - qx^2).$$

□

To find factors of L of effective degree 1, find factors of $\mathcal{R}_\mathcal{P} L$ of effective order 1 using algorithm `qHyper` of [4].

4 Recurrence operators

Let $E \in \mathcal{L}_{K[x]}$ be defined by $Ep(x) = p(x+1)$, and consider operators of the form

$$L = \sum_{k=0}^{r} p_k(x) E^k \tag{15}$$

where $r \in \mathbb{N}$, $p_k \in K[x]$, and $p_r \neq 0$. They form the skew polynomial algebra $K[x, E]$ with commutation rule $Ex = (x+1)E$. As noted in Example 1, the operator E, and hence every operator from $K[x, E]$, is compatible with the binomial coefficient basis $\mathcal{C} = \langle \binom{x}{n} \rangle_{n=0}^{\infty}$. To describe $\mathcal{R}_\mathcal{C}$ on $K[x, E]$, it suffices to give it on the two generators E and x. Using (5) we have

$$\begin{array}{rrcl} \mathcal{R}_\mathcal{C}: & E & \mapsto & E_n + 1, \\ & x & \mapsto & n(1 + E_n^{-1}). \end{array}$$

Thus $\mathcal{R}_\mathcal{C}$ maps $K[x, E]$ into $K[n, E_n, E_n^{-1}]$. To compute $\mathcal{R}_\mathcal{C}^{-1}$ on $\mathcal{R}_\mathcal{C}(K[x, E])$, write $L \in K[x, E]$ as

$$L = \sum_{i,k} c_{i,k} \binom{x}{i} E^k. \tag{16}$$

Note that $\binom{x}{i} E^{-i} P_n(x) = \binom{x}{i}\binom{x-i}{n} = \binom{n+i}{i}\binom{x}{n+i} = \binom{n+i}{i} P_{n+i}(x)$, which together with (5) gives

$$\mathcal{R}_\mathcal{C}: \quad \binom{x}{i} E^{-i} \mapsto \binom{n}{i} E_n^{-i}.$$

As $\mathcal{R}_\mathcal{C} E^i = (E_n + 1)^i$, we have $\mathcal{R}_\mathcal{C}: \binom{x}{i} \mapsto \binom{n}{i}(1 + E_n^{-1})^i$, therefore

$$\mathcal{R}_\mathcal{C} L = \sum_{i,k} c_{i,k} \binom{n}{i} (1 + E_n^{-1})^i (1 + E_n)^k = \sum_i \binom{n}{i} (1 + E_n^{-1})^i r_i(E_n)$$

where $r_i(E_n) = \sum_k c_{i,k}(1 + E_n)^k$. So, if $M \in \mathcal{E}$ and $M = \sum_i \binom{n}{i}(1 + E_n^{-1})^i r_i(E_n)$, then $\mathcal{R}_\mathcal{C}^{-1} M = \sum_i \binom{x}{i} r_i(E - 1)$.

For symmetry, we could identify x with n and E with E_n. However, we cannot extend $\mathcal{R}_\mathcal{C}$ to $K[x, E, E^{-1}]$ because E^{-1} is not compatible with $\mathcal{C}$: $E^{-1}\binom{x}{n} = \binom{x-1}{n} = \sum_{k=0}^{n} (-1)^{n-k} \binom{x}{k}$.

It may happen that factoring $\mathcal{R}_\mathcal{C} L \in K[n, E_n, E_n^{-1}] \subset \mathcal{E}$ is easier than factoring $L \in K[x, E]$. If $\mathcal{R}_\mathcal{C} L = M_2 M_1$ then $L = L_2 L_1$ (where $L_i = \mathcal{R}_\mathcal{C}^{-1} M_i$, for $i = 1, 2$) is a factorization of L in $\mathcal{L}_\mathcal{C}$. But $K[n, E_n, E_n^{-1}]$ is larger than $\mathcal{R}_\mathcal{C}(K[x, E])$, so L_1, L_2 will not necessarily belong to $K[x, E]$. In fact, they need not even belong to $K[x, E, E^{-1}]$.

Example 4 Let $L = \mathcal{R}_\mathcal{C}^{-1} E_n^{-1}$. Then, using (11), we have $LP_n = P_{n+1}$. As $\sum_{k=0}^{x-1} P_n(k) = P_{n+1}(x)$, we see that L acts as the summation operator $Lp(x) = \sum_{k=0}^{x-1} p(k)$. Also, $(E-1)LP_n = (E-1)P_{n+1} = P_n$, so $(E-1)L - 1$ acts as the zero operator on $K[x]$. If $L \in K[x, E, E^{-1}]$ then it is not hard to see that $(E-1)L - 1 \in K[x, E, E^{-1}] \setminus \{0\}$, and, consequently, that its kernel is finite-dimensional. Therefore $L \notin K[x, E, E^{-1}]$. □

When L_1, L_2 do belong to $K[x, E, E^{-1}]$, we can factor L for the cost of factoring $\mathcal{R}_\mathcal{C} L$. In this case the following formulæ are useful to compute the inverse transformation:

$$\begin{array}{rrcl} \mathcal{R}_\mathcal{C}^{-1}: & E_n & \mapsto & E - 1, \\ & n & \mapsto & x(1 - E^{-1}), \\ & \binom{n}{i} E_n^{-i} & \mapsto & \binom{x}{i} E^{-i}. \end{array}$$

Example 5 Let $L = (x+4)E^4 - (7x+24)E^3 - (x^2-8x-28)E^2 + (6x^2+10x-1)E - 5(x+1)^2$. Algorithm `Hyper` of [10] shows that L has no right or left first-order factors in $K(x)[E]$ where K is any field of characteristic 0, so the full factorization algorithm of [8] needs to be used to check for existence of second-order factors. Instead, we compute here the induced recurrence operator

$$\mathcal{R}_{\mathcal{C}} L = (n+4)E_n^4 - (2n+8)E_n^3 - (n^2+10n+20)E_n^2 + (2n^2+3n-1)E_n + (7n^2+8n+2) + 2n(2n-1)E_n^{-1},$$

for which `Hyper` finds the factorization

$$E_n(\mathcal{R}_{\mathcal{C}} L) = M_2 M_1$$

where $M_2 = E_n^4 + 2E_n^3 - (n+1)E_n^2 - (2n+3)E_n - (n+1)$ and $M_1 = (n+1)E_n - 2(2n+1)$. Thus

$$L = \mathcal{R}_{\mathcal{C}}^{-1}(E_n^{-1} M_2) \mathcal{R}_{\mathcal{C}}^{-1} M_1.$$

Luckily, both $\mathcal{R}_{\mathcal{C}}^{-1}(E_n^{-1}M_2)$ and $\mathcal{R}_{\mathcal{C}}^{-1} M_1$ belong to $K[x, E, E^{-1}]$, namely

$$\begin{aligned} \mathcal{R}_{\mathcal{C}}^{-1}(E_n^{-1}M_2) &= E^3 - E^2 - (x+1)E = (E^2 - E - (x+1))E, \\ \mathcal{R}_{\mathcal{C}}^{-1} M_1 &= (x+1)E - 3(2x+1) + 5xE^{-1}, \end{aligned}$$

so we have found a factorization $L = L_2 L_1$ where

$$L_2 = E^2 - E - (x+1), \quad L_1 = (x+2)E^2 - 3(2x+3)E + 5(x+1). \qquad \square$$

5 Differential operators

Let $D \in \mathcal{L}_{K[x]}$ be defined by $Dp(x) = \frac{d}{dx}p(x)$, and consider operators of the form

$$L = \sum_{k=0}^{r} p_k(x) D^k \tag{17}$$

where $r \in \mathbb{N}$, $p_k \in K[x]$, and $p_r \neq 0$. They form the Weyl algebra $K[x, D]$ with commutation rule $Dx = 1 + xD$. As noted in Example 1, the operator D, and hence every operator from $K[x, D]$, is compatible with the power basis $\mathcal{P} = \langle x^n \rangle_{n=0}^{\infty}$. To describe $\mathcal{R}_{\mathcal{P}}$ on $K[x, D]$, it suffices to give it on the two generators D and x. Using (5) we have

$$\begin{aligned} \mathcal{R}_{\mathcal{P}} : \quad D &\mapsto (n+1)E_n, \\ x &\mapsto E_n^{-1}. \end{aligned} \tag{18}$$

For symmetry, we extend $\mathcal{R}_{\mathcal{P}}$ to the skew Laurent-polynomial ring $K[x, x^{-1}, D]$ by letting $\mathcal{R}_{\mathcal{P}} x^k = E_n^{-k}$, for all $k \in \mathbb{Z}$. Then $\mathcal{R}_{\mathcal{P}}$ becomes an isomorphism of $K[x, x^{-1}, D]$ onto $K[n, E_n, E_n^{-1}]$, the inverse being given by

$$\begin{aligned} \mathcal{R}_{\mathcal{P}}^{-1} : \quad n &\mapsto xD, \\ E_n^k &\mapsto x^{-k}, \quad \text{for } k \in \mathbb{Z}. \end{aligned}$$

Example 6 Let $\vartheta = xD$. Then $\mathcal{R}_{\mathcal{P}} \vartheta = E_n^{-1}(n+1)E_n = n$, hence for any polynomial $p \in K[x]$ we have $\mathcal{R}_{\mathcal{P}} : p(\vartheta) \mapsto p(n)$. Therefore

$$p(\vartheta) \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} p(n) c_n x^n, \tag{19}$$

by Corollary 1. $\square$

In the rest of the section we consider the problem of finding "nice" power series solutions of linear differential equations. Note that for any $a \in K$, the *shifted power basis* $\mathcal{P}_a = \langle (x-a)^n \rangle_{n=0}^{\infty}$ is also compatible with operators from $K[x, D]$. If

$$y = \sum_{n=0}^{\infty} c_n x^n \in K[[\mathcal{P}]] \tag{20}$$

is a formal series in basis $\mathcal{P}$, then for any $a \in K$ we denote by y_a the corresponding formal series having identical coefficients as y, but in basis $\mathcal{P}_a$:

$$y_a = \sum_{n=0}^{\infty} c_n (x-a)^n \;\in\; K[[\mathcal{P}_a]]. \tag{21}$$

Our goal is to find all $a \in K$ and all formal power series which satisfy $Ly_a = 0$, and whose coefficients c_n have a "nice" explicit representation in terms of n. Let

$$L_a = \sum_{k=0}^{r} p_k(x+a) D^k. \tag{22}$$

The following lemma allows us to consider only the basis $\mathcal{P}_0 = \mathcal{P}$.

Lemma 2 *Let L, y_a, L_a, and y be as in (17), (21), (22), and (20), respectively. Then $Ly_a = 0$ if and only if $L_a y = 0$.*

Proof: Write $q_i(x) = p_i(x+a)$. Then

$$L\,(x-a)^n = \sum_i n^{\underline{i}} p_i(x)(x-a)^{n-i} = \sum_i n^{\underline{i}} q_i(x-a)(x-a)^{n-i}$$

and

$$L_a x^n = \sum_i n^{\underline{i}} p_i(x+a) x^{n-i} = \sum_i n^{\underline{i}} q_i(x) x^{n-i}.$$

Comparing these two expressions we see that the infinite matrix representing L in basis $\mathcal{P}_a$ agrees with that representing L_a in basis $\mathcal{P}$. Therefore $\mathcal{R}_{\mathcal{P}_a} L = \mathcal{R}_{\mathcal{P}} L_a$, hence

$$L\,y_a = 0 \;\Leftrightarrow\; (\mathcal{R}_{\mathcal{P}_a} L)\,c = 0 \;\Leftrightarrow\; (\mathcal{R}_{\mathcal{P}} L_a)\,c = 0 \;\Leftrightarrow\; L_a y = 0. \qquad \square$$

Write

$$p_k(x+a) = \sum_{i=0}^{d} u_{i,k} x^i \quad (0 \le k \le r) \tag{23}$$

where d and r are chosen so that some $u_{d,k}$ and some $u_{i,r}$ are nonzero. Define $u_{i,k} = 0$ whenever $i < 0$, $i > d$, $k < 0$, or $k > r$. Then, using (18), we obtain the corresponding recurrence operator

$$\begin{aligned} R_a &= \mathcal{R}_{\mathcal{P}} L_a = \sum_{i,k} u_{i,k} \mathcal{R}_{\mathcal{P}} x^i D^k = \sum_{i,k} u_{i,k} E_n^{-i}((n+1)E_n)^k = \sum_{i,k} u_{i,k} E_n^{-i}(n+1)^{\overline{k}} E_n^k \\ &= \sum_{i,k} u_{i,k}(n-i+1)^{\overline{k}} E_n^{k-i} = \sum_{j,k} u_{k-j,k}(n+j)^{\underline{k}} E_n^j. \end{aligned} \tag{24}$$

Lemma 3 *Let L_a and y be as in (22) and (20), respectively. Then $L_a y = 0$ if and only if the recurrence*

$$\sum_{j,k} u_{k-j,k}(n+j)^{\underline{k}} c_{n+j} = 0 \tag{25}$$

holds for all $n \in \mathbb{Z}$.

Proof: By (4), (5), and (24), $L_a y = 0$ if and only if (25) holds for all $n \ge 0$. Now assume that $n < 0$, and consider the nonzero terms in the sum in (25). They must have $k \ge j$ and $n+j \ge 0$ lest $u_{k-j,k}$ or c_{n+j} should vanish. But then $n+j+1 \le j \le k$, so

$$n+j-k+1 \le 0 \le n+j,$$

implying that $(n+j)^{\underline{k}} = 0$. Thus (25) holds trivially when $n < 0$. $\square$

Example 7 Let $\langle F_n\rangle_{n=0}^\infty$ be the sequence of Fibonacci numbers defined by $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. To find a homogeneous linear differential equation satisfied by their generating function $f(x) = \sum_{n=0}^\infty F_n x^n$, apply $\mathcal{R}_{\mathcal{P}}^{-1}$ to a recurrence operator R which annihilates the sequence $\langle F_n\rangle_{n\in\mathbb{Z}}$ where $F_n = 0$ for $n < 0$. Note that the operator $1 - E_n^{-1} - E_n^{-2}$ won't do because $F_0 = 1$ while $F_{-1} + F_{-2} = 0$. However, $R = n(1 - E_n^{-1} - E_n^{-2})$ does annihilate $\langle F_n\rangle_{n\in\mathbb{Z}}$, and $\mathcal{R}_{\mathcal{P}}^{-1}R = xD(1-x-x^2) = x(1-x-x^2)D - x(1+2x)$ indeed annihilates $f(x) = 1/(1-x-x^2)$. □

To avoid negative powers of E_n, multiply R_a on the left by E_n^b where $-b$ is the least exponent of E_n appearing in R_a:

$$b \;=\; -\min_{u_{i,k}\neq 0}(k-i) \;=\; -\min_{0\leq k\leq r}(k - \deg p_k) \;=\; \max_{0\leq k\leq r}(\deg p_k - k)\,.$$

Then (25) is equivalent to

$$\sum_{j=0}^{r+b} q_j(n)c_{n+j} = 0 \quad (n \in \mathbb{Z}) \tag{26}$$

where $q_j(n) = \sum_k u_{k-j+b,k}(n+j)^{\underline{k}}$. Note that by the definition of b, the coefficient of c_n in (26) is nonzero.

Thus the problem of finding "nice" power series solutions of $Ly_a = 0$ splits into two steps:

S1 Find all candidate values of a for which $Ly_a = 0$ may have solutions of the form (21) with "nice" c_n.

S2 For each candidate value of a, find "nice" solutions $c = \langle c_n\rangle_{n=0}^\infty$ of the corresponding recurrence (26).

Once **S1** has been solved and the candidate expansion points a have been found, the algorithms of [2], [1], and [10], resp., can be used at each a (assuming there are finitely many of them) to find all polynomial, rational, resp. hypergeometric solutions of the corresponding recurrence (26). In particular, a detailed description of an algorithm to find all hypergeometric series solutions of $Ly_a = 0$ given the expansion point a is presented in [11]. This solves **S2**.

A short discussion of **S1** in the case of hypergeometric coefficients is given in [11, Sec. 3.2], but a completely satisfactory solution has not been provided yet. Here we show how to find all $a \in K$ and all solutions (21) of $Ly_a = 0$ for which there exists:

1. a polynomial $p \in K[x]$ such that $c_n = p(n)$ for all large enough n (subsection 5.1),
2. a rational function $r \in K(x)$ such that $c_n = r(n)$ for all $n \geq 0$ (subsection 5.2),
3. a rational function $R \in K(x)$ such that $c_{n+1} = R(n)c_n$ for all large enough n (subsection 5.3).

Of course, the first two problems are special cases of the last one, but they are sufficiently interesting to warrant individual treatment. We also show that existence of a power series solution with rational coefficients implies existence of a solution with rational logarithmic derivative.

Let L be as in (17), and $a \in K$. If $p_r(a) = 0$ then L is *singular* at $x = a$, and a is a *singular point* of L. Otherwise a is an *ordinary point* of L.

If $f(x)$ and $g(x)$ are two formal power series such that $f(x) - g(x)$ is a polynomial, we write $f(x) \sim g(x)$. In particular, $f(x) \sim 0$ iff $f(x)$ is a polynomial.

5.1 Solutions with polynomial coefficients

Let $c_n = p(n)$ for some polynomial $p \in K[x]$ and for all large enough n. Then, as it is well known, c_n satisfies a linear recurrence with constant coefficients, and its generating function (20) is a rational function of x, of the form

$$y \sim \sum_{n=0}^\infty p(n)x^n = p(\vartheta)\sum_{n=0}^\infty x^n = p(\vartheta)\frac{1}{1-x} = \frac{P(x)}{(1-x)^{s+1}} \tag{27}$$

where ϑ is as in Example 6, P is a polynomial, $P(1) \neq 0$, and $\deg P = s = \deg p$. By Lemma 4 given in Section 5.3 below, $L_a y = 0$ implies that L_a is singular at $x = 1$, so L is singular at $x = a + 1$. Thus we have

Theorem 2 *Let L be a linear differential operator with polynomial coefficients, and c_n a polynomial function of n. If a series $y_a \sim \sum_{n=0}^{\infty} c_n(x-a)^n$ satisfies $Ly_a = 0$, then L is singular at $x = a+1$.*

Therefore to find solutions (21) of $Ly_a = 0$ with polynomial coefficients c_n, it suffices to consider all the roots of $p_r(x+1) = 0$ as candidate expansion points a, and to use the algorithm of [2] at each of them to find polynomial solutions of the corresponding recurrence (26).

5.2 Solutions with rational coefficients

Next we look for rational solutions c_n of (26). We request here that there is a rational function $r \in K(x)$ such that $c_n = r(n)$ for *all* $n \geq 0$. In particular, $r(x)$ can have no nonnegative integer poles. Solutions which are eventually rational are covered in subsection 5.3.

For polynomials $f, g \in K[x]$, $f \notin K$, $g \neq 0$, denote the *order of g at f* by

$$\nu_f(g) = \max\{k \in \mathbb{N};\ f^k \mid g\}.$$

Theorem 3 *Let $y_n = p(n)/q(n)$, with $p, q \in K[x]$ relatively prime polynomials, be a rational solution of the recurrence*

$$\sum_{i=0}^{s} q_i(n) y_{n+i} = h(n) \quad (n \geq 0) \tag{28}$$

where $q_0, q_1, \ldots, q_s, h \in K[x]$, and $q_0, q_s \neq 0$. Then, for any irreducible polynomial $f \in K[x] \setminus K$,

$$\nu_f(q) \ \leq\ \min\left\{\sum_{i=0}^{\infty} \nu_{E^i f}(E^{-s} q_s), \sum_{j=0}^{\infty} \nu_{E^{-j} f}(q_0)\right\}.$$

Note that because of characteristic zero, the two sums on the right have only finitely many nonzero terms.

Proof: Let $r(n) = y_{n+1}/y_n$. Write

$$r = \frac{A}{B}\frac{EC}{C} \tag{29}$$

where $A, B, C \in K[x]$ and $\gcd(A, E^k B) = \gcd(A, C) = \gcd(B, EC) = 1$ for all $k \in \mathbb{N}$. This is possible for any nonzero rational function r (cf. [10, Lemma 3.1]). Because $r = Ey/y$ and y is rational, [10, Lemma 5.1] implies that there is a polynomial $v \in K[x]$ such that

$$\frac{B}{A} = \frac{Ev}{v}. \tag{30}$$

It follows that

$$\frac{Ey}{y} = \frac{v}{Ev}\frac{EC}{C},$$

hence $y = \lambda C/v$ for some constant λ. Since by assumption $y = p/q$ with p and q relatively prime, q divides v. Thus $\nu_f(q) \leq \nu_f(v)$.

Rewrite (30) as

$$Bv = A(Ev). \tag{31}$$

It follows that v divides $A(Ev)$, and, using this repeatedly, that v divides $A(EA)\cdots(E^{n-1}A)(E^n v)$ for any positive integer n. Since we work in characteristic 0, v and $E^n v$ will be relatively prime for large enough n. Therefore

$$\nu_f(v) \ \leq\ \sum_{j=0}^{\infty} \nu_f(E^j A) \ =\ \sum_{j=0}^{\infty} \nu_{E^{-j} f}(A).$$

In an analogous way we obtain from (31) that

$$\nu_f(v) \ \leq\ \sum_{i=0}^{\infty} \nu_f(E^{-i-1} B) \ =\ \sum_{i=0}^{\infty} \nu_{E^i f}(E^{-1} B).$$

We claim that $A \,|\, q_0$ and $B \,|\, E^{-s+1} q_s$. Assuming this, the theorem follows.

To prove the claim, note that in the homogeneous case ($h = 0$) it follows from [10, Theorem 4.1]. In the general case, express all y_{n+i} in (28) as rational multiples of $y_n = \lambda C(n)/v(n)$, use (29), and clear denominators to find that

$$\lambda \sum_{i=0}^{s} q_i \, (E^i C) \left(\prod_{j=0}^{i-1} E^j A \right) \prod_{j=i}^{s-1} E^j B = vh \prod_{j=0}^{s-1} E^j B. \tag{32}$$

From (31) it follows that A divides v and hence the right side of (32). Note that all terms with $i > 0$ on the left of (32) contain A as a factor, therefore A divides the term with $i = 0$ as well:

$$A \mid \lambda q_0 C \prod_{j=0}^{s-1} E^j B.$$

Because A is relatively prime with C and with all $E^j B$, $0 \le j \le s-1$, we conclude that $A \,|\, q_0$.

Similarly, all terms with $i < s$ on the left of (32), as well as the right side of (32), contain $E^{s-1}B$ as a factor, therefore $E^{s-1}B$ divides the term with $i = s$ as well:

$$E^{s-1} B \mid \lambda \, q_s (E^s C) \prod_{j=0}^{s-1} E^j A.$$

As $E^{s-1}B$ is relatively prime with $E^s C$ and with all $E^j A$, $0 \le j \le s-1$, we conclude that $E^{s-1} B \,|\, q_s$. □

Theorem 4 *Let $L = \sum_{k=0}^{r} p_k(x) D^k$ be a linear differential operator with polynomial coefficients, and $r \in K(x) \setminus K[x]$ a non-polynomial rational function which has no poles in $\mathbb{N}$. If the series $y_a = \sum_{n=0}^{\infty} c_n (x-a)^n$ where $c_n = r(n)$ for all $n \in \mathbb{N}$ satisfies $L y_a = 0$, then L is singular at $x = a$, and the equation $Ly = 0$ has a solution with rational logarithmic derivative over $\bar{K}$, the algebraic closure of K.*

Proof: Assume that L is not singular at $x = a$, hence that $p_r(a) = u_{0,r} \neq 0$. Then the leading term of (26) is the one with $k = r + b$, and its leading coefficient is

$$q_{r+b}(n) = \sum_k u_{k-r,k} (n + r + b)^{\underline{k}} = u_{0,r} (n + r + b)^{\underline{r}}.$$

We are going to use Theorem 3 on recurrence (26). The order of (26) in this case is $s = r + b$, so $q_s(n - s) = u_{0,r} n^{\underline{r}}$, therefore $\nu_{E^i f}(E^{-s} q_s) > 0$ for an irreducible f only if $f(n) = n - \alpha$ for some $\alpha \in \mathbb{N}$. As c_n has no poles in $\mathbb{N}$, it follows from Theorem 3 that c_n is a polynomial in n. We conclude that (22) can have non-polynomial rational solutions only when L is singular at $x = a$.

To prove the second assertion, recall that a function f is called d'Alembertian over K if it is annihilated by an operator $L = L_1 L_2 \cdots L_k$ where each $L_i \in K(x)[D]$ is of order one [3]. A d'Alembertian function satisfies $f(x) \in f_1(x) \int f_2(x) \int \cdots \int f_k(x) dx \cdots dx \, dx$ where the f_i have rational logarithmic derivatives. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n - \alpha)^k}$$

where $\alpha \in \bar{K} \setminus \mathbb{N}$. It is easy to see that the operator

$$L = \left(D - \frac{1}{1-x} \right) (xD - \alpha)^k$$

annihilates $f(x)$, which is consequently d'Alembertian over $\bar{K}$. Now, if c_n is a rational function of n, its partial fraction decomposition

$$c_n = p(n) + \sum_{i=0}^{s} \sum_{j=1}^{d_i} \frac{\beta_{i,j}}{(n - \alpha_i)^j}$$

together with (27) and the fact that d'Alembertian functions form a ring [6], shows that (20) is d'Alembertian as well. But if $L_a y = 0$ has a d'Alembertian solution then it also has a solution with rational logarithmic derivative [3, Theorem 4], and so does $Ly = 0$. □

Therefore to find solutions (21) of $Ly_a = 0$ with non-polynomial rational coefficients c_n, it suffices to consider the singular points of (17) as candidate expansion points a, and to use the algorithm of [1] at each of them to find rational solutions of the corresponding recurrence (26).

Example 8 The equation

$$2x(x-1)y''(x) + (7x-3)y'(x) + 2y(x) = 0 \tag{33}$$

is singular at $x = 0$ and $x = 1$. Let us find power series solutions at $x = 0$. Recurrence (26) in this case is

$$(n+1)(2n+3)c_{n+1} - (n+2)(2n+1)c_n = 0 \tag{34}$$

and is satisfied by the rational sequence $c_n = 2(n+1)/(2n+1)$. Thus (33) has a power series solution with rational coefficients

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{2(n+1)}{2n+1} x^n = \frac{1}{1-x} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n+1/2} \\ &\in \frac{1}{1-x} + \frac{1}{2\sqrt{x}} \int \frac{dx}{\sqrt{x}(1-x)} = \frac{1}{1-x} + \frac{1}{2\sqrt{x}} \log \frac{1+\sqrt{x}}{1-\sqrt{x}} + \frac{K}{\sqrt{x}}, \end{aligned}$$

which is d'Alembertian. Since $f(0) = 2$ it follows that $f(x) = \frac{1}{1-x} + \frac{1}{2\sqrt{x}} \log \frac{1+\sqrt{x}}{1-\sqrt{x}}$. Note that (33) is also satisfied by $g(x) = 1/\sqrt{x}$ which has rational logarithmic derivative. □

Remark 1 If, in notation of Theorem 4, $c_n = r(n)$ for large enough n but not for all $n \in \mathbb{N}$, then L need not be singular at $x = a$. For instance, the equation $(x-1)y'' + y' = 0$ has solution $y(x) = -\log(1-x) = \sum_{n=1}^{\infty} x^n/n$ with non-polynomial rational coefficients, although it is not singular at $x = 0$. This is because $c_0 = 0$ while $r(n) = 1/n$ has a pole at $n = 0$. Such solutions are covered in the next subsection. □

5.3 Solutions with hypergeometric coefficients

To find power series solutions with hypergeometric coefficients, instead of (20) and (21) it is more convenient to write

$$y = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}, \tag{35}$$

and

$$y_a = \sum_{n=0}^{\infty} b_n \frac{(x-a)^n}{n!} \tag{36}$$

respectively, where $b_n = c_n n!$ is hypergeometric iff c_n is. Note that b_n is undefined for $n < 0$. Then, after replacing k with $k + j - b$ and multiplying both sides with $(n+b)!$, (26) turns into

$$\sum_{j=0}^{r+b} q_j(n) b_{n+j} = 0 \quad \text{for large enough } n, \tag{37}$$

where $q_j(n) = \sum_k u_{k,k+j-b}(n+b)^{\underline{k}}$. Since $k + j - b \le r$, it follows that $\deg q_j(n) \le r + b - j$. In particular, $q_{r+b}(n)$ is a constant polynomial.

Theorem 5 *Let $x = a$ be an ordinary point of $L = \sum_{k=0}^{r} p_k(x) D^k$, and $y_a = \sum_{n=0}^{\infty} b_n (x-a)^n/n!$ a hypergeometric series which satisfies $Ly_a = 0$. Then there are polynomials $A, C \in K[x]$ with $\deg A \le 1$, such that*

$$b_{n+1} = A(n) \frac{C(n+1)}{C(n)} b_n \tag{38}$$

for all large enough n.

Proof: If $Ly_a = 0$ then by Lemma 2, $L_a y = 0$ where L_a and y are as in (22) and (35), respectively. By the preceding discussion, b_n is a hypergeometric solution of (37).

If b_n is eventually zero then the theorem holds trivially. Otherwise b_n is eventually nonzero (because it satisfies a homogeneous first-order recurrence with rational coefficients). Let $R(n)$ be the rational function equal to b_{n+1}/b_n for all large enough n. As any nonzero rational function, R can be written in the form

$$R = \zeta \frac{A}{B} \frac{EC}{C} \tag{39}$$

where $\zeta \in K \setminus \{0\}$, $A, B, C \in K[x]$ are monic, and $\gcd(A, E^k B) = \gcd(A, C) = \gcd(B, EC) = 1$ for all $k \in \mathbb{N}$.

By [10, Theorem 5.1], B divides the leading coefficient of recurrence (37) which is

$$q_{r+b}(n) = \sum_k u_{k,k+r} (n+b)^{\underline{k}} = u_{0,r} = p_r(a),$$

a nonzero constant because $x = a$ is an ordinary point of L. So $B = 1$, and it remains to show that $\deg A \leq 1$.

Again by [10, Theorem 5.1], ζ is a nonzero root of the algebraic equation

$$\sum_{k=0}^{r+b} \alpha_k \zeta^k = 0, \tag{40}$$

where α_k is the coefficient of n^M in $P_k(n) = q_k(n) \prod_{j=0}^{k-1} E^j A$, and $M = \max_{0 \leq k \leq r+b} \deg P_k$. Write $\delta = \deg A$. Since $\deg q_{r+b} = 0$ and $\deg q_k \leq r + b - k$ for $k < r + b$, it follows that $\deg P_{r+b} = (r+b)\delta$ and $\deg P_k \leq r + b - k(1-\delta)$ for $k < r + b$. If $\delta > 1$ then for $k < r + b$,

$$\deg P_{r+b} - \deg P_k \geq (r+b)\delta - (r + b - k(1-\delta)) = (\delta - 1)(r + b - k) > 0,$$

so $\deg P_k < \deg P_{r+b}$. Therefore $M = (r+b)\delta$ and all the α's are zero except α_{r+b}. Hence (40) has no nonzero roots, and (37) has no hypergeometric solution with $\delta > 1$. It follows that $\deg A = \delta \leq 1$. Writing A for ζA in (39) we obtain (38). □

Corollary 2 *Let $x = 0$ be an ordinary point of L, and $y = \sum_{n=0}^{\infty} b_n x^n / n!$ a hypergeometric series solution of $Ly = 0$. Then y is of one of the forms*

a) $y \sim p(x) e^{\zeta x}$, *or*

b) $y \sim p(x)(1 - \zeta x)^{\alpha}$, *or*

c) $y \sim p(x)/(1 - \zeta x)^s + q(x) \log(1 - \zeta x)$,

where $p, q \in K[x]$ are polynomials, $\zeta \in K \setminus \{0\}$, $\alpha \in K$, and $s \in \mathbb{N}$.

Proof: If y is a polynomial, this is trivially true. Otherwise, Theorem 5 implies that for all large enough n, $b_{n+1}/b_n = \zeta A(n) C(n+1)/C(n)$ where $\zeta \in K \setminus \{0\}$ and either $A(n) = 1$, or $A(n) = n - \alpha$ for some $\alpha \in K$. We distinguish three cases according to the form of A and the nature of α.

Case a) $A(n) = 1$
In this case, $b(n+1)/b(n) = \zeta C(n+1)/C(n)$, so $b_n = \lambda C(n) \zeta^n$ where λ is a constant. Hence by (19),

$$y \sim \lambda \sum_{n=0}^{\infty} C(n) \frac{(\zeta x)^n}{n!} = \lambda C(\vartheta) e^{\zeta x} = p(x) e^{\zeta x} \tag{41}$$

where $p(x)$ is some polynomial of degree $s = \deg C(n)$.

Case b) $A(n) = n - \alpha$, where $\alpha \notin \mathbb{N}$
In this case, $b(n+1)/b(n) = \zeta(n-\alpha)C(n+1)/C(n)$, so $b_n = \lambda C(n)(-\alpha)^{\overline{n}}\zeta^n$ where λ is a constant. Hence by (19),

$$y \sim \lambda \sum_{n=0}^{\infty} C(n)\frac{(-\alpha)^{\overline{n}}}{n!}(\zeta x)^n = \lambda C(\vartheta)\sum_{n=0}^{\infty}\binom{\alpha}{n}(-\zeta x)^n = \lambda C(\vartheta)(1-\zeta x)^{\alpha} = p(x)(1-\zeta x)^{\alpha - s} \qquad (42)$$

where $p(x)$ is some polynomial and $\deg p = s = \deg C$.

Case c) $A(n) = n - \alpha$, with $\alpha \in \mathbb{N}$
Here we still have the solution

$$y \sim \lambda C(\vartheta)(1-\zeta x)^{\alpha}$$

which in this case is simply a polynomial in x, corresponding to $s = q = 0$. But now there is another hypergeometric solution of (37), namely

$$b_n = \lambda C(n)(n-\alpha-1)!\zeta^{n-\alpha-1}, \qquad \text{for } n \geq \alpha + 1,$$

which, using (19), yields

$$\begin{aligned} y &\sim \lambda \sum_{n=\alpha+1}^{\infty} C(n)\frac{(n-\alpha-1)!}{n!}\zeta^{n-\alpha-1}x^n \\ &= \lambda C(\vartheta)\sum_{n=0}^{\infty}\frac{\zeta^n x^{n+\alpha+1}}{(n+1)^{\overline{\alpha+1}}} \\ &\in \lambda C(\vartheta)\int\int\cdots\int\frac{1}{1-\zeta x}dx\ldots dx\,dx \end{aligned}$$

where there are $\alpha + 1$ integral signs. It is straightforward to verify by induction on k that for any $n, k \in \mathbb{N}$,

$$\frac{d^k}{dx^k}((1-\zeta x)^n \log(1-\zeta x)) = \zeta^k k!(1-\zeta x)^{n-k} f_{n,k}(x) \qquad (43)$$

where

$$f_{n,k}(x) = \begin{cases} (-1)^k\binom{n}{k}(H_n - H_{n-k} + \log(1-\zeta x)), & k \leq n \\ (-1)^{n+1}/\left(k\binom{k-1}{n}\right), & k > n \end{cases}$$

and $H_n = \sum_{k=1}^{n} 1/k$. Taking $n = \alpha$ and $k = \alpha + 1$ in (43), we see that the nested integral of $1/(1-\zeta x)$ has the form $P(x)\log(1-\zeta x) + Q(x)$ where P and Q are polynomials of degree $\leq \alpha$. Finally

$$y \sim \frac{p(x)}{(1-\zeta x)^s} + q(x)\log(1-\zeta x), \qquad (44)$$

where p, q are polynomials, $\deg p \leq \alpha + s$, $\deg q \leq \alpha$, and $s = \deg C$. In fact, a more careful analysis shows that $p(x)$ is divisible by $(1-\zeta x)^t$ where $t = \min\{\alpha, s\}$. □

Lemma 4 *Let $L = \sum_{k=0}^{r} a_k(x)D^k$ be a linear differential operator with polynomial coefficients. If $y(x) = p(x)(1-\zeta x)^{\alpha}$ satisfies $Ly = 0$ where $\alpha \in K \setminus \mathbb{N}$, $\zeta \in K \setminus \{0\}$, and $p \in K[x]$ is relatively prime with $1 - \zeta x$, then $1 - \zeta x$ divides the leading coefficient $a_r(x)$ of L.*

Proof: By Leibniz' rule,

$$Ly(x) = \sum_{k=0}^{r} a_k(x)\sum_{j=0}^{k}\binom{k}{j}\frac{d^{k-j}p(x)}{dx^{k-j}}(-\zeta)^j \alpha^{\underline{j}}(1-\zeta x)^{\alpha-j} = 0. \qquad (45)$$

As α is not a nonnegative integer, $\alpha^{\underline{j}} \neq 0$ for $0 \leq j \leq k$. Multiplying (45) by $(1-\zeta x)^{r-\alpha}$ we see that $1 - \zeta x$ divides $a_r(x)p(x)$ and hence $a_r(x)$. □

Lemma 5 *Let $L = \sum_{k=0}^{r} a_k(x)D^k$ be a linear differential operator with polynomial coefficients. If*

$$y(x) = \frac{p(x)}{(1-\zeta x)^s} + q(x)(1-\zeta x)^t \log(1-\zeta x) \tag{46}$$

satisfies $Ly = 0$ where $p, q \in K[x]$, $q \neq 0$, $s, t \in \mathbb{N}$, $\zeta \in K \setminus \{0\}$, and $q(x)$ is relatively prime with $1 - \zeta x$, then $1 - \zeta x$ divides the leading coefficient $a_r(x)$ of L.

Proof: If y is as in (46) then clearly

$$Ly(x) = A(x) + B(x)\log(1-\zeta x)$$

where $A, B \in K(x)$ are rational power series. As $\log(1-\zeta x)$ is not a rational power series, $Ly = 0$ implies $A = B = 0$. We distinguish three cases.

Case 1 ($t \geq r$): Using (43) and Leibniz' rule,

$$B(x) = \sum_{k=0}^{r} a_k(x) \sum_{j=0}^{k} \binom{k}{j} \frac{d^{k-j}q(x)}{dx^{k-j}} \zeta^j j!(-1)^j \binom{t}{j} (1-\zeta x)^{t-j}.$$

As $B(x) = 0$, and all terms with $j \leq r-1$ above contain $(1-\zeta x)^{t-r+1}$ as an explicit factor, it follows that the term with $j = k = r$ is also divisible by this factor. Thus $1 - \zeta x$ divides $a_r(x)q(x)$ and hence $a_r(x)$.

Case 2 ($t < r$, $s > 0$): In this case we can assume that $p(x)$ is relatively prime with $1 - \zeta x$, and use the fact that $A(x) = 0$. Consider the exponent of $1 - \zeta x$ in the denominators of various contributions to $A(x)$. In those terms arising from applying L to $q(x)(1-\zeta x)^t \log(1-\zeta x)$ this exponent is at most $r - t$, according to (43). On the other hand,

$$L\, \frac{p(x)}{(1-\zeta x)^s} = \sum_{k=0}^{r} a_k(x) \frac{p_k(x)}{(1-\zeta x)^{s+k}}$$

where $p_k(x)$ is a polynomial relatively prime with $1 - \zeta x$, and $p_r \neq 0$. As $s > 0$ we have $s + r > r - t$. It follows that $1 - \zeta x$ divides $a_r(x)p_r(x)$ and hence $a_r(x)$.

Case 3 ($t < r$, $s = 0$): As $Lp(x)$ is a polynomial, $A(x)$ contains a term which is a constant multiple of $a_r(x)q(x)/(1-\zeta x)^{r-t}$ while the exponent of $1 - \zeta x$ in the denominators of all other terms of $A(x)$ is at most $r - t - 1$, according to (43). It follows that $1 - \zeta x$ divides $a_r(x)q(x)$ and hence $a_r(x)$. □

Corollary 3 *Let $x = a$ be an ordinary point of L, and $y_a = \sum_{n=0}^{\infty} c_n(x-a)^n/n!$ a hypergeometric series satisfying $Ly_a = 0$. Then for any other ordinary point $x = b$ of L, there is a hypergeometric series $w_b = \sum_{n=0}^{\infty} d_n(x-b)^n/n!$ satisfying $Lw_b = 0$.*

Proof: By Lemma 2, $Ly_a = 0$ implies that $L_a y = 0$ where $y = \sum_{n=0}^{\infty} c_n x^n/n!$, and L_a is as in (22). Because $x = 0$ is an ordinary point of L_a, the series y has one of the three forms listed in Corollary 2. Note that all three are d'Alembertian. Let $L_{\min} \in K(x)[D]$ be the monic operator of minimal order annihilating y. Then $L_{\min}$ is a right factor of L_a in $K(x)[D]$. We claim that in each of the three cases, and for any ordinary point b of L, there exists a hypergeometric series of the form $w_c = \sum_{n=0}^{\infty} d_n(x-c)^n/n!$ where $c = b - a$, such that $L_{\min} w_c = 0$ and hence that $L_a w_c = 0$. By Lemma 2, it then follows that $Lw_{c+a} = Lw_b = 0$ as desired.

To prove the claim, we give $L_{\min}$ and w_c separately for the three cases of Corollary 2. In each of them, it is easy to check that indeed $L_{\min} w_c = 0$. We write z for $x - c$. In what follows, $p_0, p, q \in K[x]$ and $\zeta \in K \setminus \{0\}$.

Case a) $y(x) = p_0(x) + p(x)e^{\zeta x}$ with $p \neq 0$: As $Lp_0(x)$ is rational while $L\left(p(x)e^{\zeta x}\right)$ is not unless it vanishes, $Ly(x) = 0$ implies that also $L\left(p(x)e^{\zeta x}\right) = 0$. Thus we can take $p_0(x) = 0$ and $y(x) = p(x)e^{\zeta x}$. Then

$$\begin{aligned} L_{\min} &= D - \left(\frac{p'(x)}{p(x)} + \zeta\right), \\ w_c &= p(z+c)e^{\zeta z}. \end{aligned} \tag{47}$$

Case b) $y(x) = p_0(x) + p(x)(1-\zeta x)^\alpha$ with $\alpha \in K \setminus \mathbb{N}$ and $p \neq 0$: As in case a), we can take $p_0(x) = 0$ and $y(x) = p(x)(1-\zeta x)^\alpha$. Then

$$\begin{aligned} L_{\min} &= D - \left(\frac{p'(x)}{p(x)} - \frac{\alpha\zeta}{1-\zeta x}\right), \\ w_c &= p(z+c)(1-\xi z)^\alpha \end{aligned} \tag{48}$$

with $\xi = \zeta/(1-\zeta c)$.

Case c) $y(x) = p_0(x) + p(x)/(1-\zeta x)^s + q(x)\log(1-\zeta x)$ with $s \in \mathbb{N}$ and $q \neq 0$: Here

$$\begin{aligned} L_{\min} &= \left(D - \frac{g'(x)}{g(x)}\right)\left(D - \frac{q'(x)}{q(x)}\right), \\ w_c &= p_0(z+c) + \frac{p(z+c)}{(1-\zeta c)^s(1-\xi z)^s} + q(z+c)\log(1-\xi z) \end{aligned} \tag{49}$$

with $g = q(y/q)'$ and $\xi = \zeta/(1-\zeta c)$.

In cases b) and c), we need to show that $1 - \zeta c \neq 0$. According to Lemmas 4 and 5, L_a is singular at $x = 1/\zeta$. But then L is singular at $x = a + 1/\zeta$, so $a + 1/\zeta \neq b$ because b is an ordinary point of L. Thus $1/\zeta \neq b - a = c$ and $\zeta c \neq 1$.

In cases a) and b), w_c is a polynomial multiple of a hypergeometric series, which by Lemma 1 is again a hypergeometric series. In case c), w_c is the sum of two such series. But the coefficients of $1/(1-\xi z)^s = \sum_{n=0}^\infty \binom{n+s-1}{s-1}\xi^n z^n$ as well as those of $\log(1-\xi z) = -\sum_{n=1}^\infty (\xi^n/n) z^n$ are both similar to ξ^n, hence, by Lemma 1, so are the coefficients of w_c which are thus hypergeometric. □

Therefore the following **algorithm** will find all solutions (21) of $Ly_a = 0$ with hypergeometric c_n:

1. For each singular point a of L, find all solutions $y = \sum_{n=0}^\infty c_n x^n$ of $L_a y = 0$ with hypergeometric c_n, using the algorithm of [11]. Then the corresponding y_a give all the hypergeometric series solutions at $x = a$.

2. Pick any ordinary point a of L. Find all solutions $y = \sum_{n=0}^\infty c_n x^n$ of $L_a y = 0$ with hypergeometric c_n, using either the algorithm of [11], or, since these solutions are d'Alembertian, the algorithm of [7], or a custom-designed algorithm for finding solutions of the three types described in Corollary 2. Then the corresponding y_a give all the hypergeometric series solutions at $x = a$. For any other ordinary point b of L, the series w_c given in (47), (48), and (49), respectively (with z replaced by $x - b$ and c by $b - a$), give all the hypergeometric series solutions at $x = b$.

References

[1] S. A. Abramov, Rational solutions of linear difference and q-difference equations with polynomial coefficients, *Progr. and Comp. Software* 21 (1995) 273–278.

[2] S. A. Abramov, M. Bronstein, M. Petkovšek, On polynomial solutions of linear operator equations, *Proc. ISSAC '95*, T. Levelt, Ed., Montreal, Canada, July 10–12, 1995 (ACM Press, New York 1995) 290–296.

[3] S. A. Abramov, M. Petkovšek, D'Alembertian solutions of linear differential and difference equations, *Proc. ISSAC '94*, M. Giesbrecht, Ed., Oxford, England, United Kingdom, July 20–22, 1994 (ACM Press, New York 1994) 169–174.

[4] S. A. Abramov, P. Paule, M. Petkovšek, q-Hypergeometric solutions of q-difference equations, *Discrete Math.*, to appear.

[5] S. A. Abramov, M. Petkovšek, Special power series solutions of linear differential equations, *Proc. FPSAC '96*, D. Stanton, Ed., Minneapolis, Minnesota, June 25–29, 1996 (Univ. of Minnesota, 1996) 1–8.

[6] S. A. Abramov, E. V. Zima, Minimal completely factorable annihilators, *Proc. ISSAC '97*, W. W. Küchlin, Ed., Maui, Hawaii, USA, July 21–23, 1997 (ACM Press, New York 1997) 290–297.

[7] M. Bronstein, Linear differential equations: breaking through the order 2 barrier, *Proc. ISSAC '92*, P. S. Wang, Ed., Berkeley CA, USA, July 27–29, 1992 (ACM Press, New York 1992) 42–48.

[8] M. Bronstein, M. Petkovšek, An introduction to pseudo-linear algebra, *Theor. Comp. Sci.* 157 (1996) 3–33.

[9] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics* (Addison-Wesley, Reading, Mass., 1989).

[10] M. Petkovšek, Hypergeometric solutions of linear recurrences with polynomial coefficients, *J. Symb. Comput.* 14 (1992) 243–264.

[11] M. Petkovšek, B. Salvy, Finding all hypergeometric solutions of linear differential equations, *Proc. ISSAC '93*, M. Bronstein, Ed., Kiev, Ukraine, July 6–8, 1993 (ACM Press, New York 1993) 27–33.