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THE CARDINAL NUMBER OF ALGEBRAS

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Abstract. If the cardinal number of a field $\mathbb{F}$ is c, the cardinal number of the set of nonisomorphic types of d–dimensional algebras over $\mathbb{F}$ is less or equal to c^{d^3}; the equality holds if at least one of c and d is infinite.

In this note we shall assume the validity of the Axiom of choice. We shall understand the designation $x = \infty$ as x is an infinite cardinal number. $\mathbb{F}$ will always mean a commutative field with characteristic $\operatorname{chr}\mathbb{F} = p$ and cardinality $\operatorname{crd}\mathbb{F} = c$.

Let V be a vector space over $\mathbb{F}$ with dimension $\dim V = d$ and $\mathcal{A}$ a family of all (nonassociative) algebras over V. $\mathcal{A}$ can be interpreted also as the set of all bilinear maps from V^2 into V. To be isomorphic is an equivalence relation in $\mathcal{A}$, for which we shall use the symbol $\cong$. The elements of the quotient set $\mathcal{A}/\cong$ will be called *algebraic types* and

$$\operatorname{nat}(c, d) := \operatorname{crd}(\mathcal{A}/\cong)$$

will be the cardinality of this set. Our intention is to determine $\operatorname{nat}(c, d)$, at least in the infinite case.

The definition of the symbol $\operatorname{nat}(c, d)$ at the first sight is not correct, since it is not obvious that to two different fields $\mathbb{F}_1$ and $\mathbb{F}_2$ with the same cardinal number c there belongs the same $\operatorname{nat}(c, d)$. For the finite fields it is trivialy true since any two fields with the same number of elements are necessarily

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isomorphic. But for infinite fields it is not so and for that reason we will temporarily write $\mathrm{nat}(\mathbb{F},d)$ instead of $\mathrm{nat}(c,d)$, wherever this uncertainty would be actual.

The first proposition is rather trivial:

PROPOSITION 1. $\mathrm{nat}(c,0) = 1$,
$\mathrm{nat}(c,1) = 2$.

For further work we need the following lemma.

LEMMA 2. *The cardinal number of the set of multiplication tables of algebras over a vector space V of dimension $d \neq 0$ is c^{d^3}.*

Proof. A multiplication table is a map $B^3 \to \mathbb{F}$, where B is a (well–ordered) base of the space V. The multiplication in V is namely precisely determined when we know $b_i b_j = \sum \lambda_{ijk} b_k$ for all pairs $b_i, b_j \in B$, hence when we entirely know the map $(b_i, b_j, b_k) \mapsto \lambda_{ijk}$. Of course, at fix (i,j) only finitely many λ_{ijk} are different from 0; therefore, the cardinal number of the set of multiplication tables $\leq \mathrm{crd}(\mathbb{F}^{B^3}) = c^{d^3}$.

On the other hand, the set of multiplication tables can be represented with the set of maps $B^2 \to V$. The consequence is that the cardinal number of the set of multiplication tables is equal to $(\mathrm{crd}\, V)^{d^2}$. If $d < \infty$ then $\mathrm{crd}\, V = c^d$ and $(\mathrm{crd}\, V)^{d^2} = c^{d^3}$. If $d = \infty$ then $\mathrm{crd}\, V = dc$ ([2], p. 245) and $(\mathrm{crd}\, V)^{d^2} = (dc)^{d^2} = d^{d^2} \cdot c^{d^2} \geq c^{d^2} = c^{d^3}$. □

COROLLARY 3. $d \neq 0 \;\Rightarrow\; \mathrm{nat}(\mathbb{F},d) \leq c^{d^3}$.

PROPOSITION 4. $c = \infty \;\wedge\; 2 \leq d < \infty \;\Rightarrow\; \mathrm{nat}(c,d) = c$.

Proof. Since $d^3 < \infty$ then $c^{d^3} = c$ and from Corollary 3 it follows: $\mathrm{nat}(\mathbb{F},d) \leq c$.

Let $U(\omega)$ be an algebra with Table 1 as a multiplication table. It is easy to see that $U(\tau) \cong U(\omega)$ if and only if $\tau = \omega$. There are c such algebras, which gives: $\mathrm{nat}(\mathbb{F},d) \geq c$. □

$\cdot$	b_0	b_1	b_2	$\dots$
b_0	$(1+\omega)b_0$	ωb_1	ωb_2	$\dots$
b_1	b_1	0	0	$\dots$
b_2	b_2	0	0	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 1.

PROPOSITION 5. *If* $d = \infty$ *then* $\mathrm{nat}(\mathbb{F}, d) \geq c^d$.

Proof. We shall make the proof by constructing c^d nonisomorphic algebras. Let T and U be two algebras with bases $B_T = \{a, b_i | i \in I\}$ and $B_U = \{e, f_i | i \in I\}$, $\mathrm{crd}\, I = d - 1 = d$. Let us well–order the set I and denote the first element with 1 and the successor of an i with i'. The multiplication will be defined in the following way:

$$\begin{array}{rcl} a^2 &=& b_1 \ ; \\ ab_i &=& b_{i'} + \lambda_i a \ , \quad b_i a = a \ , \quad b_i b_j = 0 \quad (i, j \in I) \ ; \\ e^2 &=& f_1 \ ; \\ ef_i &=& f_{i'} + \mu_i e \ , \quad f_i e = e \ , \quad f_i f_j = 0 \quad (i, j \in I) \ . \end{array}$$

Here, $i \mapsto \lambda_i$ and $i \mapsto \mu_i$ are arbitrary maps from I to $\mathbb{F}$.

Suppose that $H : T \to U$ is an isomorphism:

$$\begin{array}{rcl} H(a) &=& \alpha e + \sum\limits_{i \in I} \beta_i f_i \ ; \\ H(b_j) &=& \gamma_j e + \sum\limits_{i \in I} \delta_{ji} f_i \quad (j \in I) \ . \end{array}$$

Both upper sums and all sums in the continuation are of course finite.

$$\begin{array}{rcl} H(b_j^2) &=& H(0) = 0 \ , \\ &=& H(b_j)^2 = \gamma_j \sum \delta_{ji}(1 + \mu_i)e + \gamma_j^2 f_1 + \gamma_j \sum \delta_{ji} f_{i'} \ . \end{array}$$

Hence, $\gamma_j = 0$ $(j \in I)$ and, because of the surjectivity of H, $\alpha \neq 0$.

$$\begin{array}{rcl} H(b_j a) &=& H(a) = \alpha e + \sum \beta_i f_i \ , \\ &=& H(b_j)H(a) = \alpha \sum \delta_{ji} e \ . \end{array}$$

Then: $\beta_j = 0$, $\sum\ \delta_{ji} = 1\ \ (j \in I)$.

$$\begin{aligned} H(a^2) &= H(b_1) = \sum\ \delta_{1i} f_i\ , \\ &= H(a)^2 = \alpha^2 f_1\ . \end{aligned}$$

This implies: $\delta_{1i} = 0\ \ (i \in I \setminus \{1\})$. Then: $\delta_{11} = \alpha^2 = 1$.

$$\begin{aligned} H(ab_j) &= H(b_{j'} + \lambda_j a) = \alpha\lambda_j e + \delta_{j'1} f_1 + \sum\ \delta_{j'i'} f_{i'}\ , \\ &= H(a)H(b_j) = \alpha \sum\ \delta_{ji}\mu_i e + \alpha \sum\ \delta_{ji} f_{i'}\ , \end{aligned}$$

which gives: $\lambda_j = \sum\ \delta_{ji}\mu_i$, $\delta_{j'1} = 0\ \ (j \in I)$, $\delta_{j'i'} = \alpha\delta_{ji}\ \ (i, j \in I)$. Then: $\delta_{1'1} = 0$, $\delta_{1'1'} = \alpha$, $\delta_{1'i''} = 0\ \ (i \in I)$. But since $\sum\ \delta_{1'i} = 1$, we find $\alpha = 1$. Using the transfinite induction we prove now: $\delta_{jj} = 1$, $\delta_{ij} = 0\ \ (i, j \in I, i \neq j)$. Then: $\lambda_j = \mu_j\ \ (j \in I)$. Algebras T and U are therefore isomorphic only if the maps $i \mapsto \lambda_i$ and $i \mapsto \mu_i$ coincide. Since there are $\mathrm{crd}(\mathbb{F}^I) = c^d$ such maps, there are also c^d such algebras. $\square$

Hence, we found out that if $\mathrm{crd}\, V = \infty$ then $\mathrm{nat}(\mathbb{F}, d) = \mathrm{nat}(c, d) = c^{d^3}$. In the following theorem we divide this result into three special cases.

Theorem 6.
$$\begin{aligned} 2 \le d < \infty = c \quad &\Rightarrow \quad \mathrm{nat}(c, d) = c \\ \infty = d < c \quad &\Rightarrow \quad \mathrm{nat}(c, d) = c^d \\ c \le d = \infty \quad &\Rightarrow \quad \mathrm{nat}(c, d) = 2^d \end{aligned}$$

The problem remains how to determine $\mathrm{nat}(c, d)$ for $c, d < \infty$. It is not really difficult to write a computer program for determination of these numbers for low c, d. With a FORTRAN program we computed:

nat(2, 2)=52	nat(5, 2)=877
nat(3, 2)=162	nat(7, 2)=2975
nat(4, 2)=402	nat(11, 2)=16507

nat(2, 3) = 801168

Table 2.

We have found the determination of finite $\mathrm{nat}(c, d)$ a very difficult combinatorial problem. Here we shall only derive usefull boundaries for these numbers.

PROPOSITION 7. *For $c < \infty$ and $0 < d < \infty$ it holds:*

$$\mathrm{nat}(c,d) \geq 1 + \frac{c^{d^3} - 1}{c^{\frac{d^2-d}{2}}(c-1)(c^2-1)\dots(c^d-1)} \ .$$

P r o o f. The number of different bases in an algebra is equal to the order of $GL(d, \mathbb{F})$, which is

$$\mathrm{crd}\, GL(d, \mathbb{F}) = c^{\frac{d^2-d}{2}}(c-1)(c^2-1)\dots(c^d-1)$$

([4], p. 114). Each base has its own multiplication table. But since some of multiplication tables happen to be equal, the number of different multiplication tables is at most equal to $\mathrm{crd}\, GL(d, \mathbb{F})$. If we treat the trivial algebra, which has only one multiplication table, separately, we have the estimate: $(\mathrm{nat}(c,d) - 1) \cdot \mathrm{crd}\, GL(d, \mathbb{F}) \geq c^{d^3} - 1$. □

If we compare this estimate with the numbers from Table 2, it seems to be surprisingly good.

With the intention to get the upper bound of $\mathrm{nat}(c,d)$ $(d \geq 2)$, we shall count out firstly the algebras with the following property:
(P) $\forall x, y \in V \;\; \exists \alpha, \beta \in \mathbb{F} : xy = \alpha x + \beta y$.
Except in the case $c = d = 2$, there always exist two unique linear forms $\varphi, \psi : V \to \mathbb{F}$ such that

$$\forall x, y \in V : xy = \varphi(y)x + \psi(x)y$$

([1], p. 234). If $\varphi = \psi = 0$, we get the trivial algebra. If $\varphi \neq 0$ and $\psi = \omega\varphi$ for certain $\omega \in \mathbb{F}$, we get the algebra from Table 1, if we choose a base so that $\varphi(b_0) = 1$, $\varphi(b_i) = 0 \;\; (i \neq 0)$. If $\varphi = 0$ and $\psi \neq 0$, the algebra is opposite to the algebra from Table 1 for $\omega = 0$. In case $\varphi \neq 0$ and $\psi \neq \omega\varphi$ for any $\omega \in \mathbb{F}$ we can find a base $\{c_1, c_2, \dots\}$ with Table 3 as a multiplication table, in which we choose: $\varphi(c_1) = \psi(c_2) = 1$, $\varphi(c_{i\neq 1}) = \psi(c_{i\neq 2}) = 0$.

$\cdot$	c_1	c_2	c_3	
c_1	c_1	0	0	$\dots$
c_2	$c_1 + c_2$	c_2	c_3	$\dots$
c_3	c_3	0	0	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 3.

If $H : (V_1, \varphi_1, \psi_1) \to (V_2, \varphi_2, \psi_2)$ is an isomorphism of two such algebras, there must hold for any $x \in V_1$:

$$\varphi_1(x) = \varphi_2(H(x)) \ , \quad \psi_1(x) = \psi_2(H(x)) \ .$$

With this fact it is easy to prove that these $c+3$ algebras are non–isomorphic.

In the case $c = d = 2$ there are 5 exceptional types; they are given in Table 4.

$\cdot$	a	b
a	αa	$\beta a + \gamma b$
b	$\delta a + \varepsilon b$	ζb

α		β	γ	δ	ε		ζ
0	0	0	0	0	1	0	1
0	0	0	1	1	0	0	0
0	0	1	0	0	1	0	0
1	0	0	0	0	0	0	1
1	0	1	1	1	1	0	1

Table 4.

Proposition 8. *If* $c < \infty$ *and* $1 < d < \infty$ *then there are*

$$\begin{array}{ll} c+3 & (c \neq 2 \text{ or } d \neq 2) \\ 10 & (c = d = 2) \end{array}$$

algebras with the property (P).

Now let $d = 2$ and suppose that (P) fails. Since for any linearly independent x, y their product xy is their linear combination, there exists such z that z and z^2 are linerly independent. Therefore, there is always a base $\{a, b\}$ which has Table 5 for its multiplication table. Changing the base $\{a, b\} \to \{\lambda a, \lambda^2 b\}$ we transform $\beta \to \lambda\beta$, $\delta \to \lambda\delta$, and with suitable choice of λ we can have $\beta = 1$, or $\beta = 0, \delta = 1$, or $\beta = \delta = 0$. The numbers of these multiplication tables are c^5 or c^4 or c^4 respectively. Since there is still some freedom in the case $\beta = \delta = 0$, we have:

$$\mathrm{nat}(c, 2) \leq c^5 + 2c^4 \ .$$

If $d > 2$ and (P) fails and if there is for any x, y linearly independent: $xy = \alpha x + \beta y$, there exists such z that z and $w = z^2$ are linearly independent. Let us deal with a triple $\{z, w, t\}$ where t is such that this triple is linearly independent. If $zw = \alpha z + \beta w$ and $zt = \gamma z + \delta t$, then $z(w + t) = (\alpha+\gamma)z+\beta w+\delta t$, which gives $\beta = \delta$, and $z(z+w+t) = (\alpha+\gamma)z+(\beta+1)w+\delta t$,

	a	b
a	b	$\alpha a + \beta b$
b	$\gamma a + \delta b$	$\varepsilon a + \zeta b$

Table 5.

which gives $\beta + 1 = \delta$, a contradiction. Therefore, there is always a base $\{a, b, c, \ldots\}$ with $ab = c$. With a change of this base $\{a, b, c, \ldots\} \to \{\lambda a, \mu b, \lambda\mu c, \ldots\}$ and with some straightforward resolution we get

Proposition 9.

$$c < \infty \quad \Rightarrow \quad \mathrm{nat}(c, 2) \leq c^4(c + 2) \ ,$$
$$c < \infty, 2 < d < \infty \quad \Rightarrow \quad \mathrm{nat}(c, d) \leq c^{d^3-d-3}(c + 4) \ .$$

We believe that this estimation is much worse than that from Proposition 7.

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