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AUTOMATIC CONTINUITY OF HOMOMORPHISMS INTO NORMED QUADRATIC ALGEBRAS

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Automatic continuity of homomorphisms into normed quadratic algebras

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Abstract. Let B be a real or complex finite dimensional quadratic algebra. Each homomorphism from any (nonassociative) Banach algebra into B is continuous iff B has no isotropic elements.
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According to [2], a normed algebra $(B, \|.\|_B)$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is *ACHR-algebra* if it has the following property: for any complete normed (nonassociative) algebra $(A, \|.\|_A)$, all homomorphisms $\varphi : A \to B$ are continuous. In [2] it is proven that smooth normed algebras are ACHR. In this note we will prove the extension of this theorem, that quadratic algebras without isotropic elements, which are equipped with a kind of "natural" norms, are still ACHR. By the definition, x is *isotropic* if $x \neq 0 = x^2$.

An $\mathbb{F}$-algebra B is *quadratic* if it has a nonzero unit e and every element $x \in B$ generates a subalgebra of dimension ≤ 2. x then satisfies a quadratic equation

$$x^2 - 2\tau(x)x + \nu(x)e = 0, \tag{1}$$

where τ is linear and ν quadratic form, $\tau(e) = \nu(e) = 1$. If $B_0 := \operatorname{Ker}\tau$, then $B = \mathbb{F}e \oplus B_0$. According to this direct sum we define a map called *conjugation*:

$$\alpha e + a \quad \mapsto \quad \overline{\alpha e + a} := \alpha e - a,$$

which is a linear involution. There are some formulae which are almost obvious:

$$\tau(x) = \tau(\overline{x}),\ \nu(x) = \nu(\overline{x}) \qquad (x \in B);$$

$$\tau(xy) = \tau(\overline{x}\,\overline{y}) \qquad (x, y \in B);$$

$$\tau(x\,\overline{x}) = \nu(x) \qquad (x \in B);$$

$$\overline{(\alpha e + a)(\beta e + b)} - \overline{\beta e + b}\;\overline{\alpha e + a} = \tau(ab - ba)e \qquad (\alpha, \beta \in \mathbb{F};\ a, b \in B_0);$$

$$\nu(x) = 2\tau(x)^2 - \tau(x^2) \qquad (x \in B);$$

$$a^2 = -\nu(a)e \qquad (a \in B_0); \tag{2}$$

$$x + \overline{x} = 2\tau(x)e, \quad x\,\overline{x} = \overline{x}\,x = \nu(x)e \qquad (x \in B); \tag{3}$$

$$\nu(\alpha x) = \alpha^2 \nu(x) \qquad (\alpha \in \mathbb{F};\ x \in B);$$

$$\nu(\alpha e + x) = \alpha^2 + 2\alpha\tau(x) + \nu(x) \qquad (\alpha \in \mathbb{F};\ x \in B);$$

$$\nu(\alpha e + a) = \alpha^2 + \nu(a) \qquad (\alpha \in \mathbb{F};\ a \in B_0); \tag{4}$$

$$\nu(x+y) + \nu(x-y) = 2[\nu(x) + \nu(y)] \qquad (x, y \in B);$$

$$\nu\left((\alpha e + \beta x)(\gamma e + \delta x)\right) = \nu(\alpha e + \beta x)\nu(\gamma e + \delta x) \qquad (\alpha, \beta, \gamma, \delta \in \mathbb{F};\ x \in B);$$

$$\nu(x^n) = \nu(x)^n \qquad (x \in B).$$

Quadratic algebra is power-associative.

For some $a \in B_0$, if $\nu(a) = 0$ or 1 or -1, we put $\varepsilon := 0$ or i or 1, respectively; hence $\varepsilon^2 = -\nu(a)$. Then for any $n = 1, 2, 3, \ldots$ we have $(\alpha e + \beta a)^n = \lambda_n e + \mu_n a$, where

$$\begin{aligned}
\lambda_n &= \frac{1}{2}\left[(\alpha + \varepsilon\beta)^n + (\alpha - \varepsilon\beta)^n\right], \\
\mu_n &= \frac{1}{2\varepsilon}\left[(\alpha + \varepsilon\beta)^n - (\alpha - \varepsilon\beta)^n\right] \qquad (\varepsilon \neq 0), \\
&= n\beta\alpha^{n-1} \qquad (\varepsilon = 0,\ n \neq 1), \\
&= \beta \qquad (n = 1).
\end{aligned}$$

These formulae are correct in both real and complex case.

Further, we can define an interesting bilinear form:

$$\begin{aligned}
\pi(x, y) &:= \frac{1}{2}\left[\nu(x+y) - \nu(x) - \nu(y)\right] && (5) \\
&= \frac{1}{2}\left[\nu(x) + \nu(y) - \nu(x-y)\right] \\
&= \frac{1}{4}\left[\nu(x+y) - \nu(x-y)\right] && (6) \\
&= \frac{1}{2}\tau(x\overline{y} + y\overline{x}) \quad = \quad \frac{1}{2}\tau(x\overline{y} + \overline{y}x) \\
&= 2\tau(x)\tau(y) - \frac{1}{2}\tau(xy + yx) \qquad (x, y \in B).
\end{aligned}$$

A few formulae about the form π:

$$\pi(x, y) = \pi(y, x) = \pi(\overline{x}, \overline{y}) \qquad (x, y \in B);$$

$$\pi(\alpha e + a, \beta e + b) = \alpha\beta + \pi(a, b) \qquad (\alpha, \beta \in \mathbb{F};\ a, b \in B_0);$$

$$\pi(x, \overline{y}) = \pi(\overline{x}, y) = \frac{1}{2}\tau(xy + yx) \qquad (x, y \in B);$$

$$\pi(x,x) = \nu(x), \quad \pi(e,x) = \tau(x) \qquad (x \in B);$$

(7) $$ab + ba = -2\pi(a,b)e \qquad (a,b \in B_0).$$

B_0 itself is an algebra as well for a multiplication

$$a,b \quad \mapsto \quad a \times b := ab - \tau(ab)e.$$

This product is anticommutative:

$$a \times a = 0 \qquad (a \in B_0);$$

$$\overline{a \times b} = \overline{b} \times \overline{a} \qquad (a,b \in B_0).$$

Suppose that $(B, \|.\|)$ is a normed quadratic algebra:

$$\|xy\| \leq \|x\| \cdot \|y\| \qquad (x,y \in B).$$

In case the previously defined hyperplan B_0 were not closed, there would exist a sequence $\{a_n\} \subset B_0$ such that $\lim_{n\to\infty} a_n = e + b \quad (b \in B_0 \setminus \{0\})$. Squaring both sides of this equation and considering (2) and the fact that the squaring is a continuous function, we get: $-\lim_{n\to\infty} \nu(a_n)e = [1-\nu(b)]e + 2b$, which is a contradiction. Therefore, B_0 is closed and τ is continuous. Because of (1), ν is also continuous.

If one wants to compute the spectral radius $\rho(x) := \lim_{n\to\infty} \|x^n\|^{1/n}$, the norm $\|.\|$ may be replaced with any other norm because of the fact that the subalgebra generated by x is twodimensional. Therefore, one can suppose that $\|\alpha e + a\| = |\alpha| + \|a\|$ for $\alpha \in \mathbb{F}, a \in B_0$. Straightforward computation, in which the limit

$$\lim_{n\to\infty} \left(|\lambda^n + \mu^n| + \kappa\,|\lambda^n - \mu^n|\right)^{1/n} = \max\{|\lambda|, |\mu|\}$$

is used ($\kappa \geq 0$, λ and μ arbitrary), shows that

$$\rho(\alpha e + a) = \max\left|\alpha \pm \sqrt{-\nu(a)}\right| \qquad (\alpha \in \mathbb{F};\ a \in B_0).$$

Since the estimate $\max|\lambda \pm \mu| \geq |\lambda|$ holds for any numbers λ, μ, we get

(8) $$|\tau(x)| \leq \rho(x) \leq \|x\| \qquad (x \in B),$$

$$\frac{1}{\|e\|} \leq \|\tau\| \leq 1.$$

From (4) and the inequality $\rho(x)^2 \geq \left|\alpha + \sqrt{-\nu(a)}\right| \cdot \left|\alpha - \sqrt{-\nu(a)}\right|$ for $x = \alpha e + a \quad (\alpha \in \mathbb{F},\ a \in B_0)$ we find also

$$|\nu(x)| \leq \rho(x)^2 \leq \|x\|^2 \qquad (x \in B) \tag{9}$$

and, using (2):

$$|\nu(a)| = \left\|a^2\right\| / \|e\| \leq \|a\|^2 / \|e\| \qquad (a \in B_0).$$

From $\|x\| = \|y\| = 1$ and (6) and (9) it follows then: $|\pi(x,y)| \leq \frac{1}{4}\left(|\nu(x+y)| + |\nu(x-y)|\right) \leq \frac{1}{4}\left(\|x+y\|^2 + \|x-y\|^2\right) \leq 2$. Since $|\pi(x,y)| = \left|\pi\left(\frac{1}{\|x\|}x, \frac{1}{\|y\|}y\right)\right| \cdot \|x\| \cdot \|y\|$ for $x, y \neq 0$, we get:

$$|\pi(x,y)| \leq 2\,\|x\| \cdot \|y\| \qquad (x, y \in B).$$

(7) gives us a better estimate:

$$|\pi(a,b)| \leq \|a\| \cdot \|b\| / \|e\| \qquad (a, b \in B_0).$$

From (3) follows the continuity of the conjugation:

$$\|\overline{x}\| \leq (2\|e\| + 1)\|x\| \qquad (x \in B),$$

and from $a \times b = ab - \tau(ab)e$ the continuity of the anticommutative multiplication:

$$\|a \times b\| \leq (\|e\| + 1)\|a\| \cdot \|b\| \qquad (a, b \in B_0).$$

In [2] it is proven that if $\mathbb{F} = \mathbb{C}$ then a normed quadratic algebra is ACHR-algebra iff B has no isotropic elements, which is exactly when $B \cong \mathbb{C}$ or $\mathbb{C} \oplus \mathbb{C}$ (a direct sum of two onedimensional ideals). Also, if $B \cong \mathbb{R}$ then B is real ACHR-algebra. According to our main goal we will suppose from now on that $(B, \|.\|)$ *is at least twodimensional real normed quadratic algebra without isotropic elements* (which is, by [3], necessary for an algebra to be ACHR).

Since $(\alpha e + a)^2 = \left[\alpha^2 - \nu(a)\right] e + 2\alpha a = 0$ iff $\alpha = \nu(a) = 0$ (for $\alpha \in \mathbb{R}, a \in B_0$), it must hold: $\nu(a) \neq 0$ for any $a \in B_0 \setminus \{0\}$. If $a, b \in B_0$ were such that $\nu(a) > 0$ and $\nu(b) < 0$, the continuity of ν would imply the existence of a point $c = \lambda a + (1-\lambda) b$ on the line through a and b $(0 < \lambda < 1)$, for which $\nu(c) = 0$; but then $c^2 = 0$ and $c = 0$ and $b = \frac{\lambda}{\lambda - 1} a$, which would give a contradiction $\nu(b) = \left(\frac{\lambda}{\lambda-1}\right)^2 \nu(a) > 0$.

We shall join a constant ω to the algebra B:
$\omega = 1$, if $\nu(a) > 0$ for $a \in B_0 \setminus \{0\}$,
$\omega = -1$, if $\nu(a) < 0$ for $a \in B_0 \setminus \{0\}$.
Correcting the from π with the factor ω, we get a new bilinear form:

$$\langle \alpha e + a, \beta e + b \rangle := \alpha\beta + \omega\pi(a,b)$$

for $\alpha, \beta \in \mathbb{R}$ and $a, b \in B_0$. This form is an inner product and implies a new norm:

$$|\alpha e + a| := \sqrt{\langle \alpha e + a, \alpha e + a \rangle} = \sqrt{\alpha^2 + |\nu(a)|}$$

for $\alpha \in \mathbb{R}$ and $a \in B_0$. It is a routine to find the following relations:

$$|e| = 1; \quad |\overline{x}| = |x| \qquad (x \in B);$$

$$\langle x, y \rangle = \langle \overline{x}, \overline{y} \rangle \qquad (x, y \in B);$$

$$\langle e, x \rangle = \tau(x), \quad |\tau(x)| \le |x| \qquad (x \in B);$$

$$|\langle x, y \rangle| \le |x|\,|y| \qquad (x, y \in B);$$

$$|\pi(x, y)| \le |x|\,|y| \qquad (x, y \in B);$$

for $\omega = 1$ only:

$$\sqrt{\nu(x)} = |x| = \rho(x) \qquad (x \in B);$$

$$\langle x, y \rangle = \pi(x, y) \qquad (x, y \in B);$$

$$|xy + yx| \le 2|x|\,|y| \qquad (x, y \in B);$$

$$|(\alpha e + \beta a)(\gamma e + \delta a)| = |\alpha e + \beta a|\,|\gamma e + \delta a| \qquad (\alpha, \beta, \gamma, \delta \in \mathbb{R};\ a \in B_0);$$

for $\omega = -1$ only:

$$\sqrt{|\nu(x)|} \le |x| \le \rho(x) \le \sqrt{2}|x| \qquad (x \in B);$$

$$\sqrt{-\nu(a)} = |a| = \rho(a) \qquad (a \in B_0);$$

$$\langle x, y \rangle = \pi(x, \overline{y}) \qquad (x, y \in B);$$

$$|xy + yx| \le 2\sqrt{2}|x|\,|y| \qquad (x, y \in B);$$

$$|ab + ba| \le 2|a|\,|b| \qquad (a, b \in B_0);$$

$$|(\alpha e + \beta a)(\gamma e + \delta a)| \le \sqrt{2}|\alpha e + \beta a|\,|\gamma e + \delta a| \qquad (\alpha, \beta, \gamma, \delta \in \mathbb{R};\ a \in B_0).$$

In fact, all these relations have nothing to do with the norm $\|.\|$.

The spectral radius $\rho(.)$ is a norm, equivalent with the norm $|.|$! In the case $\omega = 1$ this statement is trivial. But also in the case $\omega = -1$ we need to prove only the subadditivity. From $-\pi(a,b) = \langle a,b\rangle \le |a|\,|b| = \sqrt{\nu(a)\nu(b)}$ it follows: $0 \le 2\pi(a,b) + 2\sqrt{\nu(a)\nu(b)}$; further from (5): $0 \le \nu(a+b) - \nu(a) - \nu(b) + 2\sqrt{\nu(a)\nu(b)}$, which gives: $|\nu(a+b)| \le |\nu(a)| + |\nu(b)| + 2\sqrt{|\nu(a)|\,|\nu(b)|}$, and: $\sqrt{|\nu(a+b)|} \le \sqrt{|\nu(a)|} + \sqrt{|\nu(b)|}$; the rest of the proof of subadditivity is trivial.

Because of $\|(x^\kappa)^n\|^{1/n} = \left(\|x^{\kappa n}\|^{\frac{1}{\kappa n}}\right)^\kappa$ it holds:

$$\rho\left(x^\kappa\right) = \rho(x)^\kappa, \quad \rho(x) = \lim_{\kappa\to\infty} \rho\left(x^\kappa\right)^{1/\kappa} \qquad (x \in B).$$

This implies also

$$\lim_{n\to\infty} |x^n|^{1/n} = \rho(x) \qquad (x \in B).$$

Further,

$$\rho\left((\alpha e + \beta a)(\gamma e + \delta a)\right) \le \rho(\alpha e + \beta a)\rho(\gamma e + \delta a) \qquad (\alpha,\beta,\gamma,\delta \in \mathbb{R};\ a \in B_0),$$

which follows from the relation $\|(xy)^n\|^{1/n} = \|x^n y^n\|^{1/n} \le \|x^n\|^{1/n}\,\|y^n\|^{1/n}$ for $x = \alpha e + \beta a,\ y = \gamma e + \delta a$. Also:

$$\rho(xy + yx) \le 2\rho(x)\rho(y) \qquad (x, y \in B).$$

In general, the norms $|.|$ and $\|.\|$ are not equivalent; only

$$|x| \le \rho(x) \le \|x\| \qquad (x \in B).$$

According to Open mapping theorem these norms are equivalent if B is a complete normed space in the topologies of both norms (which is true if $\dim B < \infty$).

If the norms $|.|$ and $\|.\|$ are equivalent, there exists such an $M > 0$ that for any $x, y \in B$: $|xy| \le M|x|\,|y|$. The consequence is that $x \mapsto M|x|$ is an algebraic norm. $(B, \|.\|)$ is ACHR-algebra iff so is $(B, M|.|)$.

If the norms $|.|$ and $\|.\|$ are equivalent on B_0, they are equivalent on a whole B. To see this, suppose that $\|a\| \le N|a|$ $(a \in B_0)$ and $M = \sqrt{2}\max\{N, \|e\|\}$. Then $\|\alpha e + a\| \le |\alpha|\,\|e\| + \|a\| \le |\alpha|\,\|e\| + N|a| \le \frac{M}{\sqrt{2}}(|\alpha| + |a|) \le M\sqrt{\alpha^2 + |a|^2} = M|\alpha e + a|$ for $\alpha \in \mathbb{R}, a \in B_0$.

Proposition 1. *If the norms $|.|$ and $\|.\|$ are equivalent and $\omega = 1$, then $(B, \|.\|)$ is ACHR-algebra.*

Proof. Let $(A, |||.|||)$ be a Banach algebra and $\varphi : A \to B$ a homomorphism. Suppose that there exists such $p \in A$ that $|||p||| < \sqrt{2} - 1$ and, if $\varphi(p) = \lambda e + \mu a$ $(\nu(a) = 1)$, that $|\varphi(p)| = \sqrt{\lambda^2 + \mu^2} = 1$. If $q = 2\lambda p - p^2$, then $|||q||| \le 2|||p||| + |||p|||^2 < 1$. Then the series $r = q + q^2 + qq^2 + q(qq^2) + \dots$ converges and r satisfies the equation $qr = r - q$. Since $\varphi(q) = 2\lambda\varphi(p) - \varphi(p)^2 = e$, the contradiction $\varphi(q)\varphi(r) = \varphi(r) - \varphi(q)$ follows. Therefore, if $|||p||| < \sqrt{2} - 1$, then $|\varphi(p)| < 1$ and φ is continuous. □

From the proof there follows an estimate for the norm of the homomorphism φ:

$$\frac{|\varphi(p)|}{|||p|||} \le \sqrt{2} + 1.$$

Denote by $(\hat{B}, \|.\|\hat{})$ a completion of $(B, \|.\|)$. If we extend τ and ν continuously to $\hat{\tau}$ and $\hat{\nu}$ over $\hat{B}$, we find out that $\hat{B}$ is still a quadratic algebra: $\hat{x}^2 - 2\hat{\tau}(\hat{x})\hat{x} + \hat{\nu}(\hat{x})e = 0$ $(\hat{x} \in \hat{B})$. If $\hat{B}_0 := \operatorname{Ker}\hat{\tau}$ then $B_0 \subset \hat{B}_0$, and if $\hat{\omega}$ stands for the sign of $\hat{\nu}$ over $\hat{B}_0$ then of course $\hat{\omega} = \omega$. If there is such $\hat{a} \in \hat{B}_0 \setminus \{0\}$, $\hat{a} = \lim a_n$, $\{a_n\} \subset B_0$, that $\hat{\nu}(\hat{a}) = 0$, then $0 = \lim \nu(a_n) = \lim |a_n|$. Therefore, $\hat{B}$ is without isotropic elements iff the norms $|.|$ and $\|.\|$ are equivalent on B_0.

Hence, we will suppose in the sequel that $\omega = -1$ *and the norms* $|.|$ *and* $\|.\|$ *are equivalent.* Since the completion $(\hat{B}, \|.\|\hat{})$ has the same proposed properties, we may suppose that $(B, \|.\|)$ itself is complete.

If $(A, |||.|||)$ is a Banach algebra and $\varphi : A \to B$ a homomorphism, then $\varphi(A)$ is a subalgebra in B and the closure $\overline{\varphi(A)}$ is a complete subalgebra. If $\nu(x) \neq 0$ for some $x \in \overline{\varphi(A)}$ then $\frac{1}{\nu(x)}\left(x^2 - 2\tau(x)x\right) = e \in \overline{\varphi(A)}$. But if $e \neq \overline{\varphi(A)}$, we may add a unit, say u, to A and define $\varphi(u) := e$; such an extension of φ is continuous iff the original φ is continuous. Besides, if $\dim \overline{\varphi(a)} \le 2$, φ is by [2] automatically continuous. Hence, without loss of generality we may suppose: $\overline{\varphi(A)} = B$.

The *separating space* of φ is the following closed linear subspace of B:

$$S(\varphi) := \{x \in B \,|\, \exists \{p_n\} \subset A : p_n \to 0 \text{ and } \varphi(p_n) \to x\}.$$

By Closed graph theorem, φ would be continuous iff $S(\varphi) = \{0\}$. Since φ is a homomorphism with dense range, $S(\varphi)$ is a twosided ideal in B. Suppose that $S(\varphi) \neq B$, which is equivalent with $e \notin S(\varphi)$, and $x \in S(\varphi)$, $x = \alpha e + \beta a$ $(\nu(a) = -1)$. $x\overline{x} = \left(\alpha^2 - \beta^2\right) e \in S(\varphi)$, therefore $\alpha^2 = \beta^2$ and $\alpha = \beta$ (if necessary, one replaces a with $a' = -a$). If $b \in B_0$

is an element orthogonal to a: $\langle a,b\rangle = 0$, and with $\nu(b) = -1$, we get from (7): $\frac{1}{2}(e+a)b + b\frac{1}{2}(e+a) = b \in S(\varphi)$, which gives a contradiction $e = b^2 \in S(\varphi)$. Therefore, if φ is not continuous then $S(\varphi) = B$.

In this case there exists such a sequence $\{p_n\} \subset A$ that $\lim p_n = 0$ and $\lim \varphi(p_n) = e$. Then it holds for enough big n: $\|e - \varphi(p_n)\| < \frac{1}{5}$ and $|||p_n||| < \frac{1}{5}$. Suppose that $\varphi(p_n) = \alpha_n e + b_n$ $(\alpha_n \in \mathbb{R}, b_n \in B_0)$. $\frac{1}{5} > \|e-\varphi(p_n)\| \geq |e-\varphi(p_n)| = |(1-\alpha_n)e-b_n| = \sqrt{(1-\alpha_n)^2 + |b_n|^2}$, which gives: $\frac{4}{5} < \alpha_n < \frac{6}{5}$, $0 \leq |b_n| < \frac{1}{5}$. If $q = \left(\alpha_n^2 - |b_n|^2\right)^{-1}\left(2\alpha_n p_n - p_n^2\right)$, then $|||q||| \leq \frac{(2\alpha_n + |||p_n|||)|||p_n|||}{\alpha_n^2 - |b_n|^2} < \frac{13}{15}$ and $\varphi(q) = e$. The series $r = q + q^2 + qq^2 + q(qq^2) + \ldots$ then converges and q and r fulfill the equation $qr = r - q$. Then $\varphi(q)\varphi(r) = \varphi(r) - \varphi(q)$, which is a contradiction. So we proved

Proposition 2. *If the norms $|.|$ and $\|.\|$ are equivalent and $\omega = -1$, then $(B, \|.\|)$ is ACHR-algebra.*

Collecting all the facts together, we find

Theorem 3. *Let $(B, \|.\|)$ be a normed quadratic algebra with a unit e over $\mathbb{R}$ or $\mathbb{C}$, τ and ν be the forms defined in (1), and $B_0 = \operatorname{Ker}\tau$. Consider the following two statements:*
(a) $\exists M > 0\ \forall a \in B_0 : \|a\|^2 \leq M|\nu(a)|$.
(b) $(B, \|.\|)$ is ACHR-algebra.
Then (a) $\Rightarrow$ (b). If B is commutative or finite dimensional then also (b) $\Rightarrow$ (a).

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