

UNIVERSITY OF LJUBLJANA
INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
DEPARTMENT OF MATHEMATICS
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PROPERTY

Josip Globevnik Edgar Lee Stout

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DISCS AND THE MORERA PROPERTY

Josip Globevnik¹ and Edgar Lee Stout²

1. Introduction and the main result.

In previous work [G1S], [G12] we have considered the problem of showing that a continuous function on a real hypersurface Γ in \mathbb{C}^N satisfies the tangential Cauchy-Riemann equations provided that its slices satisfy conditions of Morera type. For instance, these results imply that if $\Omega \subset \mathbb{C}^N$ is a bounded convex domain with smooth boundary, strictly convex at $z_0 \in b\Omega$, if L_0 is a complex line tangent to $b\Omega$ at z_0 and if f is a continuous function on $b\Omega$ such that $\int_{L \cap b\Omega} f \omega = 0$ for all complex lines L close to L_0 which meet Ω and for all $(1,0)$ forms with constant coefficients, then f is a CR function in a neighbourhood of z_0 . This fails to hold if L_0 is a complex line that meets Ω even under much stronger assumption of holomorphic extendibility along complex lines. Indeed, let \mathbb{B} be the open unit ball in \mathbb{C}^2 , and define a function f on $b\mathbb{B} \setminus \{z = 0\}$ by $f(z, w) = 1/\bar{z}$. It is easy to verify that for each complex line L close to the z -axis, $f|_{L \cap b\mathbb{B}}$ has a continuous extension to $L \cap \mathbb{B}$ which is holomorphic on $L \cap \mathbb{B}$, yet there is no open set in $b\mathbb{B}$ on which f is a CR function. So to conclude that f is a CR function one has to assume the holomorphic extension property for a larger family of analytic discs.

Let Δ be the open unit disc in \mathbb{C} . An *analytic disc attached to a manifold* $M \subset \mathbb{C}^N$ is a continuous map $\varphi: \bar{\Delta} \rightarrow \mathbb{C}^N$, holomorphic on Δ and such that $\varphi(b\Delta) \subset M$. A. E. Tumanov proved that if f is a function of class \mathcal{C}^1 on a generic submanifold M of \mathbb{C}^N , and φ_0 is an analytic disc attached to M such that for all analytic discs φ , attached to M and close to φ_0 , the function $\zeta \rightarrow f(\varphi(\zeta))$ has a continuous extension from $b\Delta$ to $\bar{\Delta}$ which is holomorphic on Δ , then f is a CR function in a neighbourhood of $\varphi_0(b\Delta)$ [Tu1].

In the present paper we consider a similar problem with conditions of Morera type for continuous functions on real hypersurfaces. Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with \mathcal{C}^2 boundary. A subset $D \subset \Omega$ is called a *transversely embedded analytic disc* if $D = V \cap \Omega$ where V is a one dimensional complex submanifold of an open neighbourhood of $\bar{\Omega}$ which intersects $b\Omega$ transversely such that $V \cap \Omega$ is biholomorphically equivalent to Δ . Then $V \cap b\Omega$ is a simple closed curve which bounds D in V and which we denote by bD . Clearly $D = \varphi(\Delta)$ where $\varphi: \bar{\Delta} \rightarrow \mathbb{C}^N$ is a one to one continuous map which is holomorphic and regular on Δ . We say that φ *parametrizes* D . Let D_0 be a transversely embedded analytic disc and let φ_0 be a parametrization of D_0 . A family \mathcal{D} of such discs is called a *neighbourhood of D_0* if there is an $\varepsilon > 0$ such that \mathcal{D} contains each transversely embedded analytic disc D which can be parametrized by a map φ_D satisfying $|\varphi_D(\zeta) - \varphi_0(\zeta)| < \varepsilon$ ($\zeta \in \Delta$).

As we shall see, the formulation of our principal result depends in an essential way on the linear structure of \mathbb{C}^N so that it has no obvious analogue in the setting of domains in complex manifolds. However, in the final part of the paper we deduce from the principal result certain consequences that do hold on domains in Stein manifolds.

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In formulating our main results it is convenient to use the notation that $\mathbb{C}^{1;0}[dz]$ denotes the space of all $(1, 0)$ -forms on \mathbb{C}^N with constant coefficients and $\mathbb{C}^{1;1}[dz]$ the space of all $(1, 0)$ -forms with coefficients that are polynomials of degree not more than one. These spaces are invariant under the action of the group of affine automorphisms of \mathbb{C}^N but under the action of no larger subgroup of the group $\mathbf{Aut}(\mathbb{C}^N)$.

The following is the principal result of the paper:

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with boundary of class \mathcal{C}^2 . Let $D_0 \subset \Omega$ be a transversely embedded analytic disc and let f be a continuous function in a neighbourhood of bD_0 in $b\Omega$. Let w_0 be a point of D_0 , z_0 a point of bD_0 . Assume that*

$$(1.1) \quad \int_{bD} f\omega = 0$$

for every $\omega \in \mathbb{C}^{1;1}[dz]$ and for every transversely embedded analytic disc D belonging to a neighbourhood \mathcal{D} of D_0 such that $w_0 \in D$. If Ω is strictly pseudoconvex at z_0 then the function f is a CR function in a neighbourhood of z_0 . If Ω is strictly convex at z_0 , it suffices to assume the vanishing of the integrals (1.1) only when $\omega \in \mathbb{C}^{1;0}[dz]$.

In the following N is a fixed positive integer at least two. By \mathbb{B} we denote the open unit ball in \mathbb{C}^N or in \mathbb{R}^n , depending on the context. Similarly, $\mathbb{B}(z, r)$ is the ball of radius r centered at the point z .

We describe the idea of the proof in the special case when $\Omega \subset \mathbb{C}^2$ is a convex domain and $D_0 = \Lambda_0 \cap b\Omega$ where Λ_0 is a complex line which meets Ω . Let L_0 be the complex line tangent to $b\Omega$ at z_0 . To prove that f is a CR function in a neighbourhood of z_0 it is, by [GIS, Th. 3.2.1], enough to prove that

$$(1.2) \quad \int_{L \cap b\Omega} f\omega = 0$$

for all $\omega \in \mathbb{C}^{1;0}[dz]$ and for all complex lines L which meet Ω and belong to a neighbourhood \mathcal{L} of L_0 in the space of complex lines. Fix such a line L and let $z \in L \cap b\Omega$. By the strict convexity of Ω at z_0 , z is close to z_0 provided that \mathcal{L} is sufficiently small. Let Λ be the complex line passing through w_0 and z . If \mathcal{L} is sufficiently small then $\Lambda \cap \Omega$ belongs to \mathcal{D} . For convenience assume that $z = 0$. Let ℓ and h be linear functions on \mathbb{C}^2 such that $L = \{z \in \mathbb{C}^2 : \ell(z) = 0\}$, $\Lambda = \{z \in \mathbb{C}^2 : h(z) = 0\}$. We show that there are $\alpha \in \mathbb{R}$ and $\tau > 0$ such that if $V_t = \{z \in \mathbb{C}^2 : \ell(z)h(z) = t^2 e^{i\alpha}\}$, $0 < t < \tau$, and if E is a unitary map such that $|E - I| < \tau$ then $E(V_t) \cap \Omega$ is a transversely embedded analytic disc which belongs to \mathcal{D} provided that \mathcal{L} and τ are small enough. For each t , $0 < t < \tau$, we choose E_t in such a way that $w_0 \in E_t(V_t)$ and that $E_t \rightarrow I$ as $t \rightarrow 0$. We show that if $t \rightarrow 0$ then $E_t(V_t) \cap b\Omega$ converges to $[\Lambda \cap b\Omega] \cup [L \cap b\Omega]$ in such a way that

$$\int_{E_t(V_t) \cap b\Omega} f\omega \rightarrow \int_{\Lambda \cap b\Omega} f\omega + \int_{L \cap b\Omega} f\omega.$$

for each smooth 1-form ω on \mathbb{C}^2 . Now, for each $\omega \in \mathbb{C}^{1;0}[dz]$ the terms on the left and the first term on the right vanish by the assumption so (1.2) holds.

In Section 2 we analyze carefully the intersections of the varieties $V_t = \{(z, w) \in \mathbb{C}^2 : zw = t^2, t > 0\}$ with certain smooth perturbations of the real hyperplane $\{(z, w) \in \mathbb{C}^2 : \operatorname{Re}(z + w) = 0\}$. In Section 3 we use the results of this analysis to show the existence of α and τ with the properties above and then prove the theorem.

2. Intersections in \mathbb{C}^2 .

To begin with, we study how certain small smooth perturbations of the real hyperplane

$$H = \{(z, w) \in \mathbb{C}^2 : \operatorname{Re}(z + w) = 0\}$$

intersect the variety $V_1 = \{(z, w) \in \mathbb{C}^2 : zw = 1\}$.

The first step in this direction is to compute explicitly the intersection $V_1 \cap H$ and to show that this intersection is transverse.

The transversality assertion goes as follows. Denote by Λ the (unique) complex line in H that passes through the origin. Thus, $\Lambda = \{(z, w) \in \mathbb{C}^2 : w = -z\}$. At a point $(z, \frac{1}{z})$ of V_1 , the tangent space to V_1 is the complex line $\{\zeta(1, -1/z^2) : \zeta \in \mathbb{C}\}$. This coincides with Λ if and only if $z^2 = 1$, *i.e.*, if and only if $z = \pm 1$. It follows that V_1 meets the real hyperplane H transversely.

The determination of $V_1 \cap H$ uses the identification of \mathbb{C}^2 with \mathbb{R}^4 obtained by setting $z = x + iy$ and $w = u + iv$. The point (z, w) lies in $V_1 \cap H$ if and only if $x + u = 0$, $xu - yv = 1$, and $uy + xv = 0$. These equations imply that $u = 0$, $x = 0$, and $yv = -1$. Thus

$$V_1 \cap H = \{(x + iy, u + iv) : x = u = 0, yv = -1\},$$

a hyperbola contained in the (real) (y, v) -plane in \mathbb{C}^2 . When studying the intersection of V_1 with perturbations of H , it is useful to view this hyperbola $yv = -1$ as the union of two graphs in a new coordinate system with the lines $y = \pm v$ as the coordinate axes. For this purpose introduce new coordinates in \mathbb{R}^4 by means of the real orthogonal transformation

$$X = \frac{-x + u}{\sqrt{2}}, \quad Y = \frac{-y + v}{\sqrt{2}}, \quad U = \frac{x + u}{\sqrt{2}}, \quad V = \frac{y + v}{\sqrt{2}}.$$

The inverse of this is the orthogonal transformation given by

$$x = \frac{-X + U}{\sqrt{2}}, \quad y = \frac{-Y + V}{\sqrt{2}}, \quad u = \frac{X + U}{\sqrt{2}}, \quad v = \frac{Y + V}{\sqrt{2}}.$$

With respect to the (X, Y, U, V) -coordinate system, the equation of H is $U = 0$, and

$$V_1 \cap H = \{(0, \sqrt{V^2 + 2}, 0, V) : V \in \mathbb{R}\} \cup \{(0, -\sqrt{V^2 + 2}, 0, V) : V \in \mathbb{R}\}.$$

The perturbations of H that we need to consider are graphs $Gr(\varphi)$ of the form

$$Gr(\varphi) = \{(X, Y, U, V) : U = \varphi(X, Y, V) : X, Y, V \in \mathbb{R}\}$$

where φ is a function of class \mathcal{C}^1 on \mathbb{R}^3 such that

$$(2.1) \quad |\varphi(p)| < \eta \quad (p \in \mathbb{R}^3, |p| < \rho) \quad \text{and} \quad |(D\varphi)(p)| < \eta \quad (p \in \mathbb{R}^3)$$

with suitably chosen ρ and η . Note that in (2.1) the condition $|\varphi(p)| < \eta$ is assumed to hold only for $p \in \mathbb{R}^3$ in a fixed neighborhood of the origin; it need not hold on large subsets of \mathbb{R}^3 . Nevertheless, it turns out that if η is small enough, then $Gr(\varphi) \cap H$ is similar to $V_1 \cap H$ and, in particular, it is a union of two graphs:

Lemma 2.1. *There are $\eta > 0$, $\rho < \infty$, and $M < \infty$ such that whenever φ is a function of class \mathcal{C}^1 on \mathbb{R}^3 that satisfies (2.1), then*

- (a) $Gr(\varphi)$ is transverse to V_1 , whence $Gr(\varphi) \cap V_1$ is a closed submanifold of \mathbb{R}^4 of class \mathcal{C}^1 ,
(b) $Gr(\varphi) \cap V_1 = \Lambda_1 \cup \Lambda_2$ where Λ_1 and Λ_2 are disjoint curves of class \mathcal{C}^1 each of which is of the form

$$\Lambda_j = \{(X_j(V), Y_j(V), U_j(V), V) : V \in \mathbb{R}\}$$

where X_j, Y_j , and U_j are \mathcal{C}^1 functions on \mathbb{R} the first derivatives of which are bounded by M , and

- (c) $Y_1(V) > 0$ and $Y_2(V) < 0$ ($V \in \mathbb{R}$).

Remark 1. If $\varphi \equiv 0$, then $Gr(\varphi) = H$, $U_j = X_j = 0$, and Y_1, Y_2 are the functions $V \mapsto \pm\sqrt{V^2 + 2}$.

Remark 2. Having proved (a), to prove (b) it is enough to prove that there is a $\beta > 0$ such that for each $s \in \mathbb{R}$, $Gr(\varphi) \cap V_1$ intersects the hyperplane

$$E_s = \{(X, Y, U, s) : X, Y, U \in \mathbb{R}\}$$

at precisely two points and, moreover, that if $p \in Gr(\varphi) \cap V_1 \cap E_s$, then the angle between the hyperplane E_s and the (tangent line to) the curve $Gr(\varphi) \cap V_1$ at p is bounded below by β .

The proof of Lemma 2.1 requires some preliminaries.

Denote by L_1 and L_2 the z - and w -axes in \mathbb{C}^2 , respectively. Far from the origin, V_1 is a slight perturbation of $L_1 \cup L_2$, so to understand $Gr(\varphi) \cap V_1$ far from the origin, we first understand how $Gr(\varphi)$ intersects L , a small perturbation of L_1 or L_2 .

Lemma 2.2. *There are $\delta, \beta > 0$ such that if φ is a \mathcal{C}^1 function on \mathbb{R}^3 with*

$$(2.2) \quad |(D\varphi)(p)| < \delta \quad (p \in \mathbb{R}^3)$$

and if f, g are functions of class \mathcal{C}^1 on \mathbb{R}^2 that satisfy

$$(2.3) \quad \|f\|_{\mathcal{C}^1(\mathbb{R}^2)} < \delta, \quad \|g\|_{\mathcal{C}^1(\mathbb{R}^2)} < \delta$$

then with $L = \{(x, y, f(x, y), g(x, y)) : x, y \in \mathbb{R}\}$ we have that

- (a) $Gr(\varphi)$ is transverse to L ,
(b) for each s , $Gr(\varphi) \cap L \cap E_s$ consists of a single point, and
(c) if $p \in Gr(\varphi) \cap E_s$, then the angle between E_s and $Gr(\varphi) \cap L$ at p is bounded below by β .

Proof. The z -axis $L_1 = \{(x, y, 0, 0) : x, y \in \mathbb{R}\}$ intersects H transversely, and the line $L_1 \cap H$ intersects E_s transversely at an angle that does not depend on s . Further, (2.3) implies that at each point p of L , the tangent space $T_p L$ is arbitrarily close to L_1 , uniformly with respect to $p \in L$ provided that δ is small enough, and (2.2) implies that each point $p \in Gr(\varphi)$, $T_p Gr(\varphi)$ is arbitrarily close to H , uniformly with respect to p , provided that δ is small enough. This shows that $Gr(\varphi)$ is transverse to L if δ is small enough, which

proves (a). It also shows that the tangent line to $L \cap Gr(\varphi)$ at a point p is arbitrarily close to $L_1 \cap H$ when δ is small enough, which proves that there is a $\beta > 0$ satisfying (c) when δ is small enough.

If $s \in \mathbb{R}$, then $L \cap E_s = \{(x, y, f(x, y), g(x, y)) : y + g(x, y) = s\sqrt{2}\}$. If δ is small enough, then a simple one-variable argument shows that given $s, x \in \mathbb{R}$, there is a unique $y = y(x, s) = s\sqrt{2} + \psi(x, s)$ such that $y + g(x, y) = s\sqrt{2}$, and, moreover, that $\psi(x, s)$ is arbitrarily small, uniformly with respect to $x, s \in \mathbb{R}$ when δ is sufficiently small. By the implicit mapping theorem ψ is of class \mathcal{C}^1 . By differentiating the equality $\psi(x, s) + g(x, s\sqrt{2} + \psi(x, s)) = 0$, we see that $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial s}$ are arbitrarily small, uniformly in $x, s \in \mathbb{R}$ when δ is small. If we put $t = -x/\sqrt{2}$ and pass to X, Y, U, V -coordinates, we can write $L \cap E_s = F_s(\mathbb{R})$ where $F_s(t) = (t + A(t, s), -s + B(t, s), -t + A(t, s), s)$ with $\|A\|_{\mathcal{C}^1}, \|B\|_{\mathcal{C}^1}$ arbitrarily small provided that δ is small enough. The set $E_s \cap Gr(\varphi)$ has two (connected) components $\Omega_s^- = \{(X, Y, U, s) : U < \varphi(X, Y, s)\}$ and $\Omega_s^+ = \{(X, Y, U, s) : U > \varphi(X, Y, s)\}$. If $p \in Gr(\varphi) \cap E_s$, let n_p be the unit normal vector to $Gr(\varphi) \cap E_s$ at p that points in the direction of Ω_s^+ . By (2.2), n_p is arbitrarily close to $(0, 0, 1, 0)$, uniformly with respect to p if δ is small enough. Further, for sufficiently small δ , $\dot{F}_s(t)/|\dot{F}_s(t)|$ is arbitrarily close to $(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0)$, uniformly with respect to s and t . Thus, granted that δ is small enough, if $\langle \cdot, \cdot \rangle$ denotes the real inner product on \mathbb{R}^4 , then

$$\left\langle \frac{\dot{F}_s(t)}{|\dot{F}_s(t)|}, n_p \right\rangle < 0 \quad (t \in \mathbb{R}, p \in Gr(\varphi) \cap E_s),$$

which shows that only way that $F_s(t)$ can meet $Gr(\varphi)$ as t increases is when it passes from Ω_s^+ to Ω_s^- , which implies that $Gr(\varphi) \cap E_s \cap L$ contains at most one point.

To see that $Gr(\varphi) \cap E_s \cap L \neq \emptyset$ we must show that $\varphi(t + A(t, s), -s + B(t, s), s) = -t + A(t, s)$ for at least one t . For small δ this follows from the one-variable fact that if p, q are two functions of class \mathcal{C}^1 on \mathbb{R} such that $p'(t) < \mu < \nu < q'(t)$, $t \in \mathbb{R}$, for two constants μ and ν , then their graphs intersect.

Lemma 2.3. *If $R > 0$, then the δ of Lemma 2.2 can be chosen so small that if (2.2) and (2.3) are satisfied, and if, in addition, φ satisfies*

$$(2.4) \quad |\varphi(X, Y, V)| < \delta \quad (|X|, |Y|, |V| < 2R)$$

then $|s| \leq R$ and $(X, Y, U, s) \in Gr(\varphi) \cap L \cap E_s$ imply that $|X| < R$ and $|Y| < 2R$.

Proof. Choose $\omega \in (0, \min\{1, R\})$ such that $2\omega/(1 - \omega) < R - \omega$. Let A, B be as in the proof of Lemma 2.2. Choose $\delta < \omega$ so small that Lemma 2.2 holds, that $|A(t, s)| < \omega, |B(t, s)| < \omega$ ($t, s \in \mathbb{R}$), and that φ satisfies

$$(2.5) \quad \left| \frac{d}{dt} \varphi(t + A(t, s), -s + B(t, s), s) \right| < \omega \quad (t, s \in \mathbb{R})$$

Assume also that φ satisfies (2.4). Recall that $Gr(\varphi) \cap L \cap E_s = \{(\lambda + A(\lambda, s), -s + B(\lambda, s), -\lambda + A(\lambda, s), s)\}$ where $-\lambda + A(\lambda, s) = \varphi(\lambda + A(\lambda, s), -s + B(\lambda, s), s)$. It follows that

$$-\lambda + A(\lambda, s) = \int_0^\lambda \frac{d}{dt} \varphi(t + A(t, s), -s + B(t, s), s) dt + \varphi(A(0, s), -s + B(0, s), s).$$

Since $|A(0, s)|, |B(0, s)| < R$, it follows by (2.4) that $|s| \leq R$ implies that

$$|\varphi(A(0, s), -s + B(0, s), s)| < \delta < \omega$$

whence (2.5) implies that $|\lambda| < 2\omega/(1 - \omega) < R - \omega$ so $|-\lambda + A(\lambda, s)| < R$. Clearly $| -s + B(t, s)| < R + R = 2R$. This completes the proof.

To prove Lemma 2.1, we also need the following quantitative form of the inverse function theorem:

Lemma 2.4. *Let \mathcal{U} be an open subset of \mathbb{R}^n that contains the origin, and let $F : \mathcal{U} \rightarrow \mathbb{R}^n$ be a map of class \mathcal{C}^1 such that $F(0) = 0$ and $(DF)(0)$ is nonsingular. If $r > 0$ is so small that*

$$(2.6), \quad |(DF)(x) - (DF)(0)| < \frac{1}{8|(DF)(0)^{-1}|} \quad (|x| < r),$$

then given a \mathcal{C}^1 map $\varphi : \mathcal{U} \rightarrow \mathbb{R}^n$ such that

$$(2.7) \quad |(D\varphi)(x)| < \frac{1}{16|(DF)(0)^{-1}|} \quad (|x| < r)$$

and

$$(2.8) \quad |\varphi(0)| < \frac{r}{4|(DF)(0)^{-1}|},$$

there is a unique x with $|x| < r$ such that $F(x) + \varphi(x) = 0$.

Remark 3. The assumptions imply that if $|x| < r$, then $D(F + \varphi)(x)$ is nonsingular. Indeed, if $|x| < r$, then $DF(0)^{-1}D(F + \varphi)(x)$ is invertible:

$$\begin{aligned} |DF(0)^{-1}D(F + \varphi)(x) - I| &= |DF(0)^{-1}\{D(F + \varphi)(x) - DF(0)\}| \\ &\leq |DF(0)^{-1}\{|DF(x) - DF(0)| + |D\varphi(x)|\}| \\ &\leq |DF(0)^{-1}|\left\{\frac{1}{8} \frac{1}{|(DF)(0)^{-1}|} + \frac{1}{16} \frac{1}{|(DF)(0)^{-1}|}\right\} = 3/16 < 1. \end{aligned}$$

The proof of Lemma 2.4 is in the Appendix to this section.

Proof of Lemma 2.1. For each $t > 0$, set $D_t = \{(X, Y, U, V) : |V| < t\}$, and let L_1, L_2 be the coordinate axes in \mathbb{C}^2 . Sufficiently far away from the origin, V_1 is an arbitrarily small perturbation of $L_1 \cup L_2$, so by Lemmas 2.2 and 2.3, there are $R, 2 < R < \infty, \delta > 0$, and $\gamma > 0$ such that if φ satisfies (2.2) and (2.4), then
(i) $Gr(\varphi)$ is transverse to V_1 at each point of $\mathbb{C}^2 \setminus D_R$,

- (ii) for each s , $|s| \geq R$, the set $Gr(\varphi) \cap V_1 \cap E_s$ consists of precisely two points and at each of these points the angle between (the tangent line to) $Gr(\varphi) \cap V_1$ and E_s is at least γ , and
- (iii) if $|s| \leq R$ and if $(X, Y, U, s) \in Gr(\varphi) \cap V_1$, then $|X| < R$ and $|Y| < 2R$.

Let the ρ of condition (2.1) satisfy $\rho > 8R$. To complete the proof, it suffices to show that there are η , $0 < \eta < \delta$ and $\gamma' > 0$ such that if φ satisfies (2.1), then $G(\varphi)$ is transverse to V_1 at each point of \overline{D}_R and that for each s , $|s| < R$, the set $\{|X| < R, |Y| < 2R\} \cap V_1 \cap Gr(\varphi) \cap E_s$ consists of precisely two points at each of which the angle between (the tangent line to) $V_1 \cap Gr(\varphi)$ and E_s is at least γ' .

The equations of V_1 in the (X, Y, U, V) -coordinates are $-X^2 + Y^2 + U^2 - V^2 = 2$ and $XY - UV = 0$, so given $s \in \mathbb{R}$, we find $E_s \cap V_1 \cap Gr(\varphi)$ by solving $\tilde{F}(X, Y, U, V) = (0, 0, 0, s)$ where

$$\tilde{F}(X, Y, U, V) = (-X^2 + Y^2 + U^2 - V^2 - 2, XY - UV, U - \varphi(X, Y, V), V).$$

Let

$$F(X, Y, U, V) = (-X^2 + Y^2 + U^2 - V^2 - 2, XY - UV, U, V).$$

Then $F^{-1}(0, 0, 0, s) = V_1 \cap H \cap E_s = \{(0, \sqrt{2+s^2}, 0, s), (0, -\sqrt{2+s^2}, 0, s)\}$. We have $\det[DF(0, \pm\sqrt{2+s^2}, 0, s)] = -2(2+s^2)$ ($s \in \mathbb{R}$), so $(DF)(p)$ is invertible at each point of $V_1 \cap H$. Clearly $(DF)(p)$ and $(DF)(p)^{-1}$ depend continuously on $p = (0, \pm\sqrt{2+s^2}, 0, s)$ on each branch of $V_1 \cap H$, *i.e.*, they depend continuously on s . By compactness Lemma 2.4 now implies that

(A) there are r , $0 < r < \sqrt{R^2 + 2} - R$ and $\beta > 0$ such that whenever $p = (0, \pm\sqrt{s^2 + 2}, 0, s)$ with $|s| \leq R$ and φ satisfies $|\varphi| < \beta$ and $|D\varphi| < \beta$ on $(p + r\mathbb{B}) \cap H$, then $Gr(\varphi) \cap V_1 \cap E_s \cap (p + r\mathbb{B})$ consists of precisely one point, at which $D\tilde{F}$ is nonsingular.

The nonsingularity of $D\tilde{F}$ at p means that $Gr(\varphi)$ is transverse to V_1 at p and that the angle between $Gr(\varphi) \cap V_1$ and E_s at p is positive. Since this angle depends continuously on s , it follows that for $|s| \leq R$ it is bounded below by a positive constant.

The equation of $V_1 \cap H$ is $X = U = 0$, $Y = \pm\sqrt{2 + V^2}$. Since $R > 2$ and $r < \sqrt{R^2 + 2} - R$, it follows that $|X'| < R$ and $|Y'| < 2R$ whenever $|V| \leq R$, $p = (X, Y, 0, V) \in V_1 \cap H$, and $p' = (X', Y', U', V)$ satisfies $|p' - p| < r$. Let $\mathcal{U} = [\overline{D}_R \cap V_1 \cap H] + r\mathbb{B}$. By compactness there is some η , $0 < \eta < \min\{\delta, \beta\}$, such that $\overline{D}_R \cap V_1 \cap \{(X, Y, U, V) : |X| < R, |Y| < 2R, |U| < \eta\} \subset \mathcal{U}$. Since $\rho > 8R$, it follows that if φ satisfies (2.1) then $|\varphi| < \beta$ and $|D\varphi| < \beta$ on $\mathcal{U} \cap H$, which, by **(A)** implies that if $|s| \leq R$, then the only points (X, Y, U, V) in $E_s \cap V_1 \cap Gr(\varphi)$ that satisfy $|X| < R$, $|Y| < 2R$ are the two described in **(A)**.

It remains only to prove assertion (c). For this, note that $V_1 \setminus \mathbb{B}(0, r)$ is an arbitrarily small perturbation of $(L_1 \cup L_2) \setminus \mathbb{B}(0, r)$ where the L_i are the coordinate axes, provided that r is large enough. In particular, if ρ in Lemma 2.1 is large enough and $\eta > 0$ small enough then for, say, $|V| > \rho/4$, the tangent line to Λ_i is arbitrarily close to either $L_1 \cap H = \{X = U = V = 0\} = \{X = U = 0, V = -Y\}$ or to $L_2 \cap H = \{X = U = Y = 0\} = \{X = U = 0, V = Y\}$. In particular, there are disjoint neighborhoods \mathcal{W}_1 of $L_1 \cap H$ and \mathcal{W}_2 of $L_2 \cap H$ in the space of real lines passing through the origin such that

the tangent line to Λ_i for $|V| > \rho/4$ is either in \mathcal{W}_1 or in \mathcal{W}_2 . With no loss of generality, assume that if $\{X = p_1V, Y = p_2V, U = p_3V : V \in \mathbb{R}\}$ belongs to \mathcal{W}_1 , then $p_2 > \frac{1}{2}$ and if it belongs to \mathcal{W}_2 , then $p_2 < -\frac{1}{2}$. On the other hand, provided that η is small enough, $\{V_1 \cap Gr(\varphi) : |V| < \rho/2\}$ is an arbitrarily small perturbation of $\{V_1 \cap H : |V| < \rho/2\} = \{X = U = 0, Y = -\sqrt{2+V^2}\} \cup \{X = U = 0, Y = \sqrt{2+V^2}\}$. It follows that one of the functions Y_i , say Y_1 , is positive on $|V| \leq \rho/2$ and the other, Y_2 , is negative there. By the preceding discussion $Y_1'(\rho/2) > \frac{1}{2}$, and $Y_1'(\rho/2) < -\frac{1}{2}$, and $Y_2'(\rho/2) < -\frac{1}{2}$, and $Y_2'(\rho/2) > \frac{1}{2}$ provided that η is small enough. Since for $|V| \geq \rho/4$, either $Y_i'(V) > \frac{1}{2}$ or $Y_i'(V) < -\frac{1}{2}$, it follows, by the continuity of Y_i' that Y_1 decreases on $(-\infty, -\rho/4)$ and increases on $(\rho/4, \infty)$ and that Y_2 increases on $(-\infty, \rho/4)$ and decreases on $(\rho/4, \infty)$. This establishes (c) and completes the proof of Lemma 2.1.

We now want to understand how a surface S passing through the origin and almost tangent to $H = \{(z, w) : Re(z + w) = 0\}$ intersects the varieties $V_t = \{(z, w) : zw = t^2\}$ for small $t > 0$ in a neighborhood of the origin. To do this, we consider a \mathcal{C}^1 function ψ on \mathbb{R}^3 that satisfies $\psi(0) = 0$ and consider how $S = Gr(\psi)$ intersects V_t .

The main fact we use is the homogeneity condition that $V_t = tV_1$ when $t > 0$. This implies that $V_t \cap Gr(\psi) = t(V_1 \cap \frac{1}{t}Gr(\psi))$. Notice that $\frac{1}{t}Gr(\psi) = Gr(\varphi)$ if $\varphi(X, Y, V) = \frac{1}{t}\psi(tX, tY, tV)$. Also $(D\varphi)(X, Y, V) = (D\psi)(tX, tY, tV)$, so $|(D\varphi)(p)| < \tau$ for all $p \in \mathbb{R}^3$ if and only $|(D\psi)(p)| < \tau$ for all $p \in \mathbb{R}^3$.

If η, ρ , and M are as in Lemma 2.1, then the conditions (2.1) for $\varphi(p) = \frac{1}{t}\psi(tp)$ are

$$(2.9) \quad \left| \frac{1}{t}\psi(tp) \right| < \eta \quad (|p| < \rho) \quad \text{and} \quad |(D\psi)(p)| < \eta \quad (p \in \mathbb{R}^3).$$

Lemma 2.5. *There are $\tau > 0$ and $M < \infty$ such that if the smooth real function ψ on \mathbb{R}^3 satisfies*

$$(2.10) \quad \psi(0) = 0 \quad \text{and} \quad |(D\psi)(p)| < \tau \quad (p \in \mathbb{R}^3),$$

then for every $t > 0$

(a) $Gr(\psi)$ is transverse to V_t , and

(b) $Gr(\psi) \cap V_t = \Lambda_1 \cup \Lambda_2$ where Λ_1 and Λ_2 are \mathcal{C}^1 curves of the form

$$\Lambda_j = \{(X_j(V), Y_j(V), U_j(V), V) : V \in \mathbb{R}\}$$

where the functions X_j, Y_j , and U_j are of class \mathcal{C}^1 on \mathbb{R} and where the first derivative of each is bounded uniformly by M , and

(c) $Y_1(V) > 0$ and $Y_2(V) < 0$ ($V \in \mathbb{R}$).

Moreover, given $\epsilon > 0$, there is $\delta > 0$ such that if

$$(2.11) \quad W = \{(X, Y, U, V) : |V| < \delta\},$$

then for all $t > 0$

$$\text{length}(W \cap V_t \cap Gr(\psi)) < \epsilon.$$

Proof. Let η, ρ and M be as in Lemma 2.1. This lemma and the preceding discussion imply that to prove (a)-(c) it suffices to show that $\tau > 0$ in (2.10) can be chosen so small that (2.10) implies (2.9) for every $t > 0$. Let $\tau = \min\{\eta, \eta/\rho\}$, and assume that ψ satisfies (2.10). The second inequality in (2.9) is obviously satisfied. To prove the first inequality, let $|p| < \rho$. Then (2.10) implies that

$$\left| \frac{1}{t} \psi(tp) \right| = \left| \frac{1}{t} \int_0^t (D\psi)(\lambda p) \cdot p d\lambda \right| \leq \frac{1}{t} \tau |p| t < \frac{\eta}{\rho} \rho = \eta.$$

Finally, if $\delta > 0$ is so small that $\delta \sqrt{3M^2 + 1} < \epsilon$, and if W is given by (2.11) then

$$\begin{aligned} \text{length}(W \cap V_t \cap Gr(\varphi)) &= \sum_{j=1}^2 \int_{-\delta}^{\delta} \sqrt{X_j'(t)^2 + Y_j'(t)^2 + U_j'(t)^2 + 1} dt \\ &\leq 4\delta \sqrt{3M^2 + 1} < \epsilon. \end{aligned}$$

This completes the proof.

APPENDIX TO SECTION 2

The proof of Lemma 2.4 depends on the following standard result:

Lemma 2.A.1. *Let \mathcal{U} be a neighborhood of $0 \in \mathbb{R}^n$, and let $F : \mathcal{U} \rightarrow \mathbb{R}^n$ be a \mathcal{C}^1 map with $F(0) = 0$ and $(DF)(0) = I$. If $r > 0$ is so small that $|(DF)(0)(x) - (DF)(0)| < \frac{1}{2}$ when $|x| < r$, then for each y , $|y| < \frac{r}{2}$, there is precisely one x , $|x| < r$, such that $F(x) = y$.*

Proof. Let $G(x) = F(x) - x$. Then $|(DG)(x)| < \frac{1}{2}$ when $|x| < r$. Thus, if $|x|, |y| < r$, then $|G(x) - G(y)| = \left| \int_0^1 (DG)(x + t(y-x))(y-x) dt \right| \leq \frac{1}{2} |x-y|$, so $\frac{1}{2} |x-y| \geq |G(x) - G(y)| = |F(x) - F(y) - (x-y)| \geq |x-y| - |F(x) - F(y)|$, and $|F(x) - F(y)| \geq \frac{1}{2} |x-y|$, whence F is one-to-one on $W = \{|x| < r\}$. Let $|y| < \frac{r}{2}$ and set $x_0 = 0$, $x_n = y - G(x_{n-1})$, $n = 1, 2, \dots$. Then $|x_{n+1} - x_n| = |G(x_n) - G(x_{n-1})| \leq \frac{1}{2} |x_n - x_{n-1}|$. Note that x_n, x_{n-1} do lie in W : $|x_1| = |y| < \frac{r}{2}$, so $|x_1 - x_0| < \frac{r}{2} \leq \frac{1}{2} |x_2 - x_1| \leq \frac{1}{2} |x_1 - x_0| \leq \frac{1}{2} \frac{r}{2}$, etc. Thus, $x_n \rightarrow x$, and $x = y - F(x) + x$, whence $y = F(x)$. This completes the proof of the Lemma.

Lemma 2.A.2. *Let \mathcal{U} be a neighborhood of the origin in \mathbb{R}^n , and let $F : \mathcal{U} \rightarrow \mathbb{R}^n$ be a \mathcal{C}^1 map with $F(0) = 0$ and $(DF)(0)$ nonsingular. If $r > 0$ is so small that*

$$|(DF)(x) - (DF)(0)| < \frac{1}{2|(DF)(0)^{-1}|} \quad \text{when } |x| < r$$

then for each z with $|z| < \frac{r}{2|(DF)(0)^{-1}|}$ there is a unique x , $|x| < r$, such that $F(x) = z$.

Proof. Consider the map $x \mapsto (DF)(0)^{-1} \circ F(x) = G(x)$. We have that $G(0) = 0$ and that $|(DG)(x) - (DG)(0)| = |DF(0)^{-1}[(DF)(x) - (DF)(0)]| \leq |(DF)(0)^{-1}| \cdot |(DF)(x) - (DF)(0)| < \frac{1}{2}$ when $|x| < r$. By Lemma 2.A.2, given y , $|y| < \frac{r}{2}$, there is a unique x , $|x| < r$, such that $G(x) = y$, i.e., $F(x) = (DF)(0)(y)$. Consequently, for each $z \in \frac{r}{2}(DF)(0)(\mathbb{B})$,

there is a unique x , $|x| < r$, such that $F(x) = z$. As $\frac{1}{|(DF)(0)^{-1}|}\mathbb{B} \subset (DF)(0)(\mathbb{B})$ it follows that for each z , $|z| < \frac{r}{2|(DF)(0)^{-1}|}$ there is precisely one x with $|x| < r$ such that $F(x) = z$. This completes the proof.

Proof of Lemma 2.4. Put $G(x) = F(x) + \varphi(x) - \varphi(0)$ so that $G(0) = 0$, and $(DG)(x) = (DF)(x) + (D\varphi)(x)$. We have

$$(DG)(0)^{-1} = [(DF)(0) + (D\varphi)(0)]^{-1} = (DF)(0)^{-1} \sum_{k=0}^{\infty} [-(DF)(0)^{-1}D\varphi(0)]^k.$$

By (2.7) $|(DF)(0)^{-1}D\varphi(0)| \leq |(DF)(0)^{-1}||D\varphi(0)| < \frac{1}{2}$, so

$$(2.12). \quad |(DG)(0)^{-1}| \leq 2|(DF)(0)^{-1}|$$

Further, if $|x| < r$, then by (2.7) and (2.8)

$$\begin{aligned} |(DG)(x) - (DG)(0)| &\leq |(DF)(x) - (DF)(0)| + |(D\varphi)(0)| + |(D\varphi)(x)| \\ &< \frac{1}{8} \frac{1}{|(DF)(0)^{-1}|} + \frac{1}{8} \frac{1}{|(DF)(0)^{-1}|} < \frac{1}{4} \frac{1}{|(DF)(0)^{-1}|} \leq \frac{1}{2|(DG)(0)^{-1}|}. \end{aligned}$$

so by Lemma 2.A.2, for each $|z| < \frac{r}{2|(DG)(0)^{-1}|}$ there is a unique x , $|x| < r$, such that $G(x) = z$. In particular, by (2.8) and (2.12), this holds for $z = -\varphi(0)$. Accordingly, there is a unique x , $|x| < r$, such that $F(x) + \varphi(x) = 0$. This completes the proof of Lemma 2.4.

3. Proof of Theorem 1.1.

We begin with two lemmas of a general character, which will be used in the main part of the proof of the theorem.

The proof of the first of these is quite short but is decidedly nontrivial, as it depends essentially on some complicated results in the theory of polynomial convexity

Lemma 3.1. *If $\varphi : \bar{\Delta} \rightarrow \mathbb{C}^N$ is a continuous map that is holomorphic on Δ , that is one-to-one on $\bar{\Delta}$ and that carries $b\Delta$ onto a rectifiable simple closed curve, then the set $X = \varphi(\bar{\Delta})$ is polynomially convex.*

Proof. Let Σ be the image of $b\Delta$ under φ , and denote by $\widehat{\Sigma}$ the polynomially convex hull of Σ . It is a result of Alexander [Al1] that $\widehat{\Sigma} \setminus \Sigma$ is a closed analytic subvariety of the domain $\mathbb{C}^N \setminus \Sigma$. Moreover, by [Al2], this variety is irreducible. As $\varphi(\Delta)$ is a closed analytic subvariety of $\mathbb{C}^N \setminus \Sigma$ that is contained in $\widehat{\Sigma}$, it follows that, as claimed, $X = \widehat{\Sigma}$.

Lemma 3.2. *Let $P \subset \mathbb{C}^2$ be a bounded domain. Let $V_t = \{(z, w) : zw = t^2\}$, $t > 0$, and suppose that $V_t \cap P \neq \emptyset$ and that $V_t \cap bP = \Gamma$ is a simple closed curve. Then $V_t \cap P$ is biholomorphically equivalent to a disc.*

Proof. The projection $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ given by $\pi(z, w) = z$ carries V_t biholomorphically onto $\mathbb{C} \setminus \{0\}$. Let Q be the bounded component of $\mathbb{C} \setminus \pi(\Gamma)$. Since P is bounded, it follows that

if $z \in \mathbb{C} \setminus \bar{Q}$, $z \neq 0$, then $(z, \frac{t^2}{z}) \in \mathbb{C}^2 \setminus \bar{P}$. We show that $0 \notin Q$. Otherwise, there are points $z_n \in Q$ with $z_n \rightarrow 0$ such that $(z_n, \frac{t^2}{z_n}) \in P$ for all n , contradicting the boundedness of P . Thus, $0 \notin Q$, and $\{(z, \frac{t^2}{z}) : z \in Q\} = P \cap V_t$. As Q is biholomorphically equivalent to a disc, the same is true of $P \cap V_t$. This completes the proof.

We now begin the proof of Theorem 1.1 itself. Initially we work in \mathbb{C}^2 . We shall deal first with the case that Ω is strictly convex at z_0 .

By the assumption of convexity, there are a ball $\mathbb{B}(z_0, r)$ centered at z_0 and a function ϱ on $\mathbb{B}(z_0, r)$ with nonvanishing gradient and positive definite Hessian such that $\mathbb{B}(z_0, r) \cap b\Omega = \{w \in \mathbb{B}(z_0, r) : \varrho(w) = 0\}$ and $\mathbb{B}(z_0, r) \cap \Omega = \{w \in \mathbb{B}(z_0, r) : \varrho(w) < 0\}$. Denote by $\Lambda(z_0, b\Omega)$ the complex line passing through z_0 and tangent to $b\Omega$ at z_0 . There is a neighborhood \mathcal{L} of $\Lambda(z_0, b\Omega)$ in the space of all complex lines in \mathbb{C}^2 such that if $\Lambda \in \mathcal{L}$ meets $\mathbb{B}(z_0, r) \cap \Omega$, then Λ meets $\mathbb{B}(z_0, r) \cap b\Omega$ transversely in a closed curve that bounds a convex domain in Λ . We shall show, after passing to a smaller \mathcal{L} if necessary, that

$$(3.1) \quad \int_{\Lambda \cap \mathbb{B}(z_0, r) \cap b\Omega} f\omega = 0 \quad (\Lambda \in \mathcal{L})$$

for all $\omega \in \mathbb{C}^{1;0}[dz]$. By [GIS, Th.3.2.1] this implies that f is a CR-function in a neighborhood of z_0 .

Given $\Lambda \in \mathcal{L}$ which meets $\mathbb{B}(0, r) \cap \Omega$ we shall show that the neighbourhood \mathcal{D} contains transversely embedded analytic discs D_n , $n \in \mathbb{N}$, and D , such that as $n \rightarrow \infty$, bD_n converges to $bD \cup [\Lambda \cap \mathbb{B}(z_0, r) \cap b\Omega]$ in the sense that for every continuous 1-form γ on \mathbb{C}^2 ,

$$(3.2) \quad \lim_{n \rightarrow \infty} \int_{bD_n} \gamma = \int_{bD} \gamma + \int_{\Lambda \cap \mathbb{B}(z_0, r) \cap b\Omega} \gamma.$$

If $\omega \in \mathbb{C}^{1;0}[dz]$, then by hypothesis $\int_{bD_n} f\omega = 0$, ($n \in \mathbb{N}$), and $\int_{bD} f\omega = 0$. Applying (3.2) with $\gamma = f\omega$ gives (3.1).

To construct such sequences, first consider the following special case. We again use the notation that $V_t = \{(z, w) \in \mathbb{C}^2 : zw = t^2\}$ ($t > 0$).

Lemma 3.3. *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain with boundary of class \mathcal{C}^2 . Let $0 \in b\Omega$, and let $Re(z + w) = 0$ be the tangent plane to $b\Omega$ at 0 . Denote by L_1 and L_2 the coordinate axes of \mathbb{C}^2 , which we assume to intersect $b\Omega$ transversely and in such a way that $L_j \cap b\Omega = \Lambda_j$ is a simple closed curve that bounds a domain $D_j \subset L_j$, $j = 1, 2$. There is then $\tau > 0$ such that if $0 < t \leq \tau$ and if E is a unitary map of \mathbb{C}^2 to itself with $|E - I| < \tau$, then $E(V_t)$ is transverse to $b\Omega$ and $D_{tE} = E(V_t) \cap \Omega$ is a transversely embedded analytic disc. Moreover, if $\delta > 0$ is sufficiently small, then $\overline{D_{tE}} \setminus \delta\mathbb{B}$ is an arbitrarily small perturbation of $[\overline{D_1} \setminus \delta\mathbb{B}] \cup [\overline{D_2} \setminus \delta\mathbb{B}]$ provided that $t > 0$ and $|E - I|$ are sufficiently small.*

Proof of Lemma 3.3. Step 1. If $r > 0$ is sufficiently small, then $b(r\mathbb{B})$ is transverse to Λ_1 and to Λ_2 , and each Λ_i intersects $b(r\mathbb{B})$ at exactly two points. In addition, $b(r\mathbb{B})$ intersects $b\Omega$ transversely in a slight perturbation of $H \cap b(r\mathbb{B})$ where $H = \{Re(z + w) = 0\}$. Both $b(r\mathbb{B})$ and $b\Omega$ are transverse to L_1 and L_2 . For each i , $D_i \setminus r\bar{\mathbb{B}}$ is bounded by the union of an arc $\lambda_i = \Lambda_i \setminus r\bar{\mathbb{B}}$ and a circular arc γ_i contained in $b(r\mathbb{B})$.

On any compact set missing the origin $E(V_t)$ is an arbitrarily small perturbation of $L_1 \cup L_2$ provided that $t > 0$ and $|E - I|$ are small enough. In this case, $b\Omega$ and $b(r\mathbb{B})$ cut out of V_t two domains $D_i(t, E, r)$, and by the transversality mentioned above, each $D_i(t, E, r)$ is a slight perturbation of $D_i \setminus r\overline{\mathbb{B}}$. It is bounded by two smooth arcs, one lying near λ_i , the other near γ_i .

Suppose for the moment that we have proved that

(B) there is $\tau > 0$ such that for each t , $0 < t \leq \tau$, $E(V_t)$ is transverse to $b\Omega$ and $E(V_t) \cap b\Omega$ is a simple closed curve Λ_{Et} .

For $0 < t \leq \tau$ define D_{tE} to be $\Omega \cap E(V_t)$. By Lemma 3.2 each D_{tE} is a transversely embedded analytic disc.

Step 2. We first describe what we are going to do to prove (B). If $V_0 = \{zw = 0\} = L_1 \cup L_2$, then $V_0 \cap b\Omega$ is a figure eight, a union of two simple closed curves, Λ_1 and Λ_2 , in \mathbb{C}^2 which meet only at the origin. (Note that they are not tangent to each other there). Fix a small ball \mathcal{B} centered at the origin. Outside \mathcal{B} the variety V_t is a small perturbation of V_0 when $t > 0$ is small enough. It follows that $b\Omega \cap E(V_t) \setminus \mathcal{B}$ is, for small $t > 0$ a small perturbation of $(\Lambda_1 \setminus \mathcal{B}) \cup (\Lambda_2 \setminus \mathcal{B})$, which is a union to two disjoint arcs. Now we use Lemma 2.5 to analyze what happens in \mathcal{B} as we pass from $t = 0$ to a small positive t . A careful analysis will show that $\mathcal{B} \cap V_0$, the cross in the figure eight, gets replaced with two arcs that connect the endpoints of the two arcs whose union is $[b\Omega \cap E(V_t)] \setminus \mathcal{B}$ in such a way that the union of all four arcs is a simple closed curve.

Now let us give the details of the proof of (B). Let $\delta > 0$ be smaller than τ in Lemma 2.2. Since $Re(z + w) = 0$ is tangent to $b\Omega$ at 0, it follows that there are $r > 0$ and a smooth function ψ on \mathbb{R}^3 such that $\psi(0) = 0$, $|(D\psi)(p)| < \frac{\delta}{2}$ ($p \in \mathbb{R}^3$) and such that $\mathbb{B}(0, r) \cap Gr(\psi) = \mathbb{B}(0, r) \cap b\Omega$. There is a $\nu > 0$ such that if E is a unitary map with $|E - I| < \nu$, then $E(b\Omega)$ is transverse to both L_1 and L_2 , and there is a unique function ψ_E , smooth on \mathbb{R}^3 with $\psi_E(0) = 0$, $|(D\psi_E)(p)| < \delta$ ($p \in \mathbb{R}^3$) and such that $E(Gr(\psi)) = Gr(\psi_E)$.

Provided that δ has been chosen small enough Lemma 2.2 shows that $Gr(\psi_E) \cap L_i = \{(\Phi_{iE}(V), V) : V \in \mathbb{R}\}$ where $\Phi_{iE} = (X_{iE}, Y_{iE}, U_{iE})$ is smooth on \mathbb{R} and $\Phi'_{1E}(V)$ is arbitrarily close to $(0, -1, 0)$, and $\Phi'_{2E}(V)$ is arbitrarily close to $(0, 1, 0)$ uniformly in $V \in \mathbb{R}$ provided that δ and γ are small enough. In particular, we may assume that for $V \in \mathbb{R}$

$$(3.3) \quad |X'_{iE}(V)| < \frac{1}{2}, |U'_{iE}(V)| < \frac{1}{2}, -\frac{3}{2} < Y'_{1E}(V) < -\frac{1}{2}, \frac{1}{2} < Y'_{2E}(V) < \frac{3}{2}$$

Thus, Y_{1E} is strictly decreasing, Y_{2E} is strictly increasing, and since $Y_{iE}(0) = 0$ it follows that $Y_{1E}(V) > 0$ and $Y_{2E}(V) < 0$ when $V < 0$ while $Y_{1E}(V) < 0$ and $Y_{2E}(V) > 0$ when $V > 0$.

Step 3. We have to take into account that $Gr(\psi_E)$ coincides with $E(\Omega)$ only within $\mathbb{B}(0, r)$. With M as in Lemma 2.5, choose $\omega > 0$ so small that if

$$\mathcal{A} = \{|X|, |Y|, |U| < (M + 2)\omega \text{ and } |V| < \omega\}$$

then $\mathcal{A} \subset B(0, R)$. The bounds (3.3) imply that

$$(3.4) \quad |X_{iE}(\pm\omega)| < \frac{1}{2}\omega, |U_{iE}(\pm\omega)| < \frac{1}{2}\omega, \text{ and } |Y_{iE}(\pm\omega)| < \frac{3}{2}\omega.$$

As δ is smaller than the τ of Lemma 2.5, this lemma implies that for each $t > 0$,

$$Gr(\psi_E) \cap V_t = \{(\Phi_{1Et}(V), V) : V \in \mathbb{R}\} \cup \{(\Phi_{2Et}(V), V) : V \in \mathbb{R}\}$$

where $\Phi_{iEt} = (X_{iEt}, Y_{iEt}, U_{iEt})$ is smooth on \mathbb{R} and $Y_{1Et}(V) > 0$, and $Y_{2Et}(V) < 0$ for all $V \in \mathbb{R}$.

If $t_0 > 0$ and $\gamma > 0$ are sufficiently small, then by transversality, the set $\{\Phi_{1Et}(\omega), \Phi_{2Et}(\omega)\}$ is a small perturbation of the set $\{\Phi_{1I}(\omega), \Phi_{2I}(\omega)\}$ and the set $\{\Phi_{1Et}(-\omega), \Phi_{2Et}(-\omega)\}$ is a small perturbation of the set $\{\Phi_{1I}(-\omega), \Phi_{2I}(-\omega)\}$ whenever $0 < t \leq t_0$ and $|E - I| < \gamma$. In particular by (3.4) we may suppose that

$$(3.5) \quad |X_{iEt}(\pm\omega)| < \omega, |U_{iEt}(\pm\omega)| < \omega \text{ and } |Y_{iEt}(\pm\omega)| < 2\omega.$$

Since by Lemma 2.5 $|X'_{iEt}|, |Y'_{iEt}|, |U'_{iEt}| < M$ on \mathbb{R} , (3.5) implies that for $|V| < \omega$,

$$|X_{iEt}(V)| < (M + 1)\omega, |U_{iEt}(V)| < (M + 1)\omega, \text{ and } |Y_{iEt}(V)| < (M + 2)\omega$$

so, provided that t_0 and γ are small enough $(X_{iEt}(V), Y_{iEt}, U_{iEt}(V), V) \in \mathcal{A}$ when $|V| < \omega$. Since $\mathcal{A} \subset \mathbb{B}(0, r)$ where $Gr(\psi_E)$ coincides with $E(b\Omega)$, it follows that $\mathbb{B}(0, r) \cap Gr(\psi_E) \cap V_t \cap \{|V| < \omega\} = \mathbb{B}(0, r) \cap E(b\Omega) \cap V_t \cap \{|V| < \omega\}$.

To see that $E(V_t) \cap b\Omega$ is a simple closed curve if $t > 0$ and $|E - I|$ are small is equivalent to seeing that $V_t \cap E(b\Omega)$ is a simple closed curve provided that $t > 0$ and $|E - I|$ are small.

To see that $V_t \cap E(b\Omega)$ is a simple closed curve, observe first that $\Lambda_i = \Lambda_{iI}$ consists of a short arc $\Lambda_i^s = \{(\Phi_{iI}(V), V) : |V| < \omega\}$ and a long arc Λ_i^ℓ , which joins the points $(\Phi_{iI}(\omega), \omega)$ and $(\Phi_{iI}(-\omega), -\omega)$. If $\omega > 0$ is small enough then the long arc Λ_E^ℓ meets $\bar{\mathcal{A}}$ only at its endpoints at which it is transverse to the hyperplanes $V = \omega$ and $V = -\omega$, respectively. This transversality, together with the transversality of L_1 and L_2 to $b\Omega$ implies that when $t = 0$ changes to t , $0 < t \leq t_0$ and I changes to E , $|E - I| < \gamma$, then, provided that t_0 and γ are small enough, the long arc Λ_i^ℓ will change arbitrarily little to an arc Λ_{iEt}^ℓ with endpoints (T_{iEt}^+, ω) close to $(\Phi_{iI}(\omega), \omega)$ and $(T_{iEt}^-, -\omega)$ close to $(\Phi_{iI}(-\omega), -\omega)$, which will still meet $\bar{\mathcal{A}}$ only at its endpoints. In particular,

(C) the Y -coordinates of T_{1Et}^- and T_{2Et}^+ will be positive, and the Y -coordinates of T_{1Et}^+ and T_{2Et}^- will be negative.

We have $\Lambda_{1Et}^\ell \cup \Lambda_{2Et}^\ell = [V_t \cap E(b\Omega)] \setminus \mathcal{A}$.

On the other hand, provided that $t_0 > 0$ and γ are small enough, for every E , $|E - I| < \gamma$, and for every t , $0 < t \leq t_0$, $\mathcal{A} \cap E(b\Omega) \cap V_t = \mathcal{A} \cap Gr(\psi_E) \cap V_t = \{(\Phi_{1Et}(V), V) : |V| < \omega\} \cup \{(\Phi_{2Et}(V), V) : |V| < \omega\}$ where $Y_{1Et}(V) > 0$, and $Y_{2Et}(V) < 0$ for all $V \in \mathbb{R}$. Now V_t intersects $E(b\Omega) \setminus \mathcal{A}$ transversely provided that γ and t_0 are small enough, and by Lemma 2.5, V_t meets $E(b\Omega) \cap \mathcal{A}$ transversely. Thus $E(V_t)$ intersects $b\Omega$ transversely, which implies that $E(V_t) \cap b\Omega$ is a closed, one-dimensional submanifold of \mathbb{C}^2 . As $V_t \cap E(b\Omega) \setminus \mathcal{A} =$

$\Lambda_{1Et}^\ell \cup \Lambda_{2Et}^\ell$ and $V_t \cap (b\Omega) \cap \mathcal{A} = \{(\Phi_{1Et}(V), V) : |V| < \omega\} \cup \{(\Phi_{2Et}(V), V) : |V| < \omega\}$, it follows that

$$(3.6) \quad E(b\Omega) \cap V_t = \Lambda_{1Et}^\ell \cup \Lambda_{2Et}^\ell \cup \{(\Phi_{1Et}(V), V) : |V| < \omega\} \\ \cup \{(\Phi_{2Et}(V), V) : |V| < \omega\}.$$

The last two arcs are contained in \mathcal{A} , and the first two miss \mathcal{A} . As $\Lambda_1^\ell \cap \Lambda_2^\ell = \emptyset$, it follows that $\Lambda_{1Et}^\ell \cap \Lambda_{2Et}^\ell = \emptyset$ if t_0 and γ are small enough. The last two arcs in (3.6) are also disjoint since $Y_{1Et}(V) > 0$ and $Y_{2Et}(V) < 0$ for all $V \in \mathbb{R}$. This, together with **(C)**, implies that the only way for the right side of (3.6) to be a closed submanifold of \mathbb{C}^2 is for

$$\Phi_{1Et}(-\omega) = T_{1Et}^- \quad \Phi_{1Et} = T_{2Et}^+$$

and

$$\Phi_{2Et}(-\omega) = T_{2Et}^- \quad \Phi_{2Et}(\omega) = T_{1Et}^+.$$

Thus, $E(b\Omega) \cap V_t$ consists of the long arc Λ_{1Et}^ℓ joining (T_1^+, ω) with $(T_1^-, -\omega)$ followed by the arc $\{(\Phi_{1Et}(V), V) : |V| < \omega\}$ joining $(T_1^-, -\omega)$ with (T_2^+, ω) followed by the long arc Λ_{2Et}^ℓ joining (T_2^+, ω) with $(T_2^-, -\omega)$ followed finally by the arc $\{(\Phi_{2Et}(V), V) : |V| < \omega\}$ joining $(T_2^-, -\omega)$ with (T_1^+, ω) . This proves that $E(b\Omega) \cap V_t$ is a simple closed curve provided that $t > 0$ and $|E - I|$ are small enough.

This completes the proof of Lemma 3.3.

Discussion 1. Given $\epsilon > 0$, the ω in the proof of Lemma 3.3 can be chosen so small that $4\omega\sqrt{3M^2 + 1} < \epsilon$. This implies that the length of $E(b\Omega) \cap \mathcal{A} \cap V_t$ does not exceed ϵ . Indeed, $\text{length}(E(b\Omega) \cap \mathcal{A} \cap V_t) = \text{length}(V_t \cap Gr(\psi_E) \cap \mathcal{A}) = \int_{-\omega}^{\omega} [\sum_{j=1}^2 [X'_{jEt}(v)^2 + Y'_{jEt}(v)^2 + V'_{jEt}(v)^2 + 1]]^{\frac{1}{2}} dv$, and since $|X'_{jEt}(v)|$, $|Y'_{jEt}(v)|$, $|V'_{jEt}(v)|$ are all bounded by M , it follows that $\text{length}(E(b\Omega) \cap \mathcal{A} \cap V_t) \leq 4\omega\sqrt{3M^2 + 1} < \epsilon$.

Discussion 2. The implicit mapping theorem implies that given an interior point w of the long arc Λ_i^ℓ , there are a neighborhood $W(w)$ of w , a $\nu(w) > 0$, and a $t_0(w) > 0$ such that $|E - I| < \nu(w)$ and $0 < t \leq t_0(w)$ imply that $\Lambda_{iEt}^\ell \cap W(w)$ is a smooth graph over $\Lambda_i^\ell \cap W(w)$. These graphs depend smoothly on E and t . The same is true at the endpoints of Λ_i^ℓ . Since we want the endpoints of Λ_{iEt}^ℓ to belong to $|V| = \omega$, we have to write, *e.g.*, in a sufficiently small neighborhood $W(w)$ of $w = (\Phi_{iI}(\omega), \omega)$, $\Lambda_{iI} \cap W(w) = \{(\Phi_{iI}(t), t) : \omega \leq t < \omega + \gamma\}$ for some small $\gamma > 0$, and then $\Lambda_{iEt}^\ell \cap W(w) = \{(\Phi_{iEt}(t), t) : \omega \leq t < \omega + \gamma\}$. A partition of unity argument together with the compactness of Λ_i^ℓ shows that given a smooth 1-form α on \mathbb{C}^2 , we have

$$(3.7) \quad \lim_{E \rightarrow I, t \rightarrow 0} \int_{\Lambda_{iEt}^\ell} \alpha = \int_{\Lambda_i^\ell} \alpha.$$

Similar reasoning applies to show that if t_0 and ν are small enough, then the lengths of Λ_{iEt}^ℓ , $|E - I| < \nu$, $0 < t \leq t_0$, are uniformly bounded. We already know that the lengths of $E(b\Omega) \cap V_t \cap \mathcal{A}$ are uniformly bounded. Thus, the lengths of $E(b\Omega) \cap V_t$ are uniformly bounded provided that $|E - I| < \nu$, $0 < t \leq t_0$.

Discussion 3. Let $\pi_1(z, w) = z, \pi_2(z, w) = w$ be the coordinate projections in \mathbb{C}^2 . For each $t, 0 < t < \tau$, let $D_{tE}^i = \pi_i(E^{-1}(D_{tE}))$, $i = 1, 2$. Since the $E^{-1}(D_{tE})$ are discs in V_t , $0 < t < \tau$, the D_{tE}^i are Jordan domains in \mathbb{C} , bounded by simple closed curves $\pi_i(E(V_t) \cap b\Omega)$, which, when $t \rightarrow 0$, $E \rightarrow I$, by the last statement of Lemma 3.3, converge to D_i in the following sense: Given $\varepsilon > 0$ there is a $\delta, 0 < \delta < \tau$, such that if $0 < t < \delta$ and $|E - I| < \delta$ then there is a homeomorphism $\psi_{tE}^i: bD_i \rightarrow bD_{tE}^i$ such that $|\psi_{tE}^i(w) - w| < \varepsilon$ ($w \in bD_i$), $i = 1, 2$. This implies [Po, p.26] that given a conformal map $\varphi_1: \Delta \rightarrow D_1$ and $\varepsilon > 0$ there is a $\delta, 0 < \delta < \tau$, such that whenever $0 < t < \delta$ and $|E - I| < \delta$, there is a conformal map $\varphi_{tE}^1: \Delta \rightarrow D_{tE}^1$ such that $|\varphi_{tE}^1 - \varphi_1| < \varepsilon$ on Δ which further implies that there is a parametrization $\zeta \mapsto (\varphi_{tE}^1(\zeta), t^2/\varphi_{tE}^1(\zeta)) = \Phi_{tE}(\zeta)$ of $E^{-1}(D_{tE})$ such that $|\Phi_{tE}(\zeta) - \Phi_1(\zeta)| < 2\varepsilon + \text{diam}(D_2)$ ($\zeta \in \Delta$) whenever $0 < t < \delta$, $|E - I| < \delta$ where $\Phi_1(\zeta) = (\varphi_1(\zeta), 0)$ is a parametrization of D_1 . In particular, given a neighbourhood \mathcal{D} of D_1 in the space of transversely embedded analytic discs, $D_{tE} \in \mathcal{D}$ ($0 < t < \delta, |E - I| < \delta$) provided that $\text{diam}(D_2)$ and δ are small enough.

Let α be a continuous 1-form on C^2 with compact support. Given $\epsilon > 0$, there is a $\delta > 0$ such that for every smooth compact curve Σ in \mathbb{C}^2 , we have

$$\left| \int_{E(\Sigma)} \alpha - \int_{\Sigma} \alpha \right| = \left| \int_{\Sigma} E^* \alpha - \alpha \right| < \epsilon \cdot \text{length}(\Sigma)$$

whenever E is a unitary map with $|E - I| < \delta$.

Lemma 3.4. Let $\Omega, \Lambda_1, \Lambda_2$ and V_t , $t > 0$, be as in Lemma 3.3. If α is a continuous 1-form on \mathbb{C}^2 , then

$$(3.8) \quad \lim_{E \rightarrow I, t \rightarrow 0} \int_{b\Omega \cap E(V_t)} \alpha = \int_{\Lambda_1 \cup \Lambda_2} \alpha = \int_{b\Omega \cap V_0} \alpha.$$

Proof. With no loss of generality we suppose that α has compact support. Suppose that we have proved that

$$(3.9) \quad \lim_{E \rightarrow I, t \rightarrow 0} \int_{E(b\Omega) \cap V_t} \alpha = \int_{\Lambda_1 \cup \Lambda_2} \alpha.$$

Since the lengths of $E(b\Omega) \cap V_t$, $|E - I| < \delta, 0 < t < t_0$, are uniformly bounded, the preceding discussion implies that given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left| \int_{\tilde{E}(E(b\Omega) \cap V_t)} \alpha - \int_{E(b\Omega) \cap V_t} \alpha \right| < \epsilon$$

whenever $|\tilde{E} - I| < \delta$, $|E - I| < \gamma$, and $0 < t < t_0$. In particular, taking $\tilde{E} = E^{-1}$ yields

$$\left| \int_{b\Omega \cap E^{-1}(V_t)} \alpha - \int_{E(b\Omega) \cap V_t} \alpha \right| < \epsilon$$

whenever $|E^{-1} - I| < \delta$, $|E - I| < \gamma$, $0 < t < t_0$. Now $E \rightarrow I$ is equivalent to $E^{-1} \rightarrow I$, so

$$\lim_{E^{-1} \rightarrow I, t \rightarrow 0} \int_{b\Omega \cap E^{-1}(V_t)} \alpha = \lim_{E \rightarrow I, t \rightarrow 0} \int_{E(b\Omega) \cap V_t} \alpha$$

provided that the limit on the right exists. Thus (3.9) implies (3.8). It remains to prove (3.9).

Let $\epsilon > 0$, and let L be a uniform bound for the coefficients of α . By the Discussion 1 above, one can choose $\omega > 0$, $t_0 > 0$, and $\nu > 0$ such that

$$\text{length}(E(b\Omega) \cap \mathcal{A} \cap (\Lambda_1 \cup \Lambda_2)) < \frac{\epsilon}{16L}$$

and

$$\text{length}(E(b\Omega) \cap \mathcal{A} \cap V_t) < \frac{\epsilon}{16L}$$

whenever $|E - I| < \nu$ and $0 < t < t_0$. It follows that when $0 < t < t_0$ and $|E - I| < \nu$,

$$(3.10) \quad \left| \int_{E(b\Omega) \cap \mathcal{A} \cap V_t} \alpha \right| < \frac{\epsilon}{4} \quad \text{and} \quad \left| \int_{E(b\Omega) \cap \mathcal{A} \cap (\Lambda_1 \cup \Lambda_2)} \alpha \right| < \frac{\epsilon}{4}.$$

Further, by (3.7), we can pass to smaller $\nu > 0$ and $t_0 > 0$ if necessarily to get

$$(3.11) \quad \left| \int_{\Lambda_{iEt}^\ell} \alpha - \int_{\Lambda_i^\ell} \alpha \right| < \frac{\epsilon}{4} \quad (0 < t < t_0, |E - I| < \nu, i = 1, 2.)$$

Thus, if $0 < t < t_0$ and $|E - I| < \nu$, then (3.10) and (3.11) imply that

$$\begin{aligned} \left| \int_{E(b\Omega) \cap V_t} \alpha - \int_{\Lambda_1 \cup \Lambda_2} \alpha \right| &\leq \sum_{i=1}^2 \left| \int_{\Lambda_{iEt}^\ell} \alpha - \int_{\Lambda_i^\ell} \alpha \right| + \left| \int_{E(b\Omega) \cap \mathcal{A} \cap V_t} \alpha \right| \\ &\quad + \left| \int_{E(b\Omega) \cap \mathcal{A} \cap (\Lambda_1 \cup \Lambda_2)} \alpha \right| \leq \epsilon. \end{aligned}$$

This proves (3.9). The proof of Lemma 3.4 is complete.

Lemma 3.5. *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain with $0 \in b\Omega$. Assume that $b\Omega$ is of class \mathcal{C}^2 in a neighborhood U of $(L_1 \cup L_2) \cap b\Omega$, L_1, L_2 the coordinate axes, and that L_1, L_2 intersect $b\Omega$ transversely so that $D_j = L_j \cap \Omega$ are transversely embedded analytic discs, $j = 1, 2$. Let $w_1 \in D_1$. There is then a sequence $\{A_n\}_{n=1,2,\dots} \subset \Omega$ of transversely embedded analytic discs such that $w_1 \in A_n$, $bA_n \subset U$ ($n \in \mathbb{N}$) and such that for each continuous 1-form α on \mathbb{C}^2*

$$(3.12) \quad \lim_{n \rightarrow \infty} \int_{bA_n} \alpha = \int_{bD_1} \alpha + \int_{bD_2} \alpha.$$

Proof. Suppose to begin with that $b\Omega$ is of class \mathcal{C}^2 .

Assume first that $Re(z + w) = 0$ is the tangent space to $b\Omega$ at 0. By Lemma 3.3 there is $\tau > 0$ such that $b\Omega \cap E(V_t)$ is a simple closed curve contained in U that bounds the analytic disc $E(V_t) \cap \Omega$ transverse to $b\Omega$ provided that $|E - I|$ and t are small enough. Since on a compact set missing 0, V_t is an arbitrarily small perturbation of $L_1 \cup L_2$ provided that t is small enough, it follows that given τ_n ($\tau_n < \tau$), there are t_n , $0 < t_n < \tau_n$ and E_n , $|E_n - I| < \tau_n$, such that $w_1 \in E_n(V_{t_n})$. Put $A_n = E_n(V_{t_n}) \cap \Omega$. Then the A_n are transversely embedded analytic discs and by Lemma 3.4, (3.12) holds for every continuous 1-form on \mathbb{C}^2 .

In the general case, let $Re(pz + qw) = 0$ be the equation of the tangent space to $b\Omega$ at 0. Since L_1, L_2 are transverse to $b\Omega$ it follows that $p \neq 0, q \neq 0$, so $F(z, w) = (\frac{z}{p}, \frac{w}{q})$ is an isomorphism of \mathbb{C}^2 with $F(0) = 0, F(L_i) = L_i, i = 1, 2$, and $Re(z + w) = 0$ is the tangent space to $b\tilde{\Omega}$ if $\tilde{\Omega} = F(\Omega)$. We are now in the situation above with Ω replaced by $\tilde{\Omega}$ and with D_j replaced by $\tilde{D}_j = L_j \cap \tilde{\Omega}$. Thus, there is a sequence $\tilde{A}_n \subset \tilde{\Omega}$ of transversely embedded analytic discs such that $b\tilde{A}_n \subset F(U)$, $F(w_1) \in \tilde{A}_n$ and such that

$$\lim_{n \rightarrow \infty} \int_{b\tilde{A}_n} \beta = \int_{b\tilde{D}_1} \beta + \int_{b\tilde{D}_2} \beta$$

for every continuous 1-form β on \mathbb{C}^2 . In particular given a continuous 1-form α on \mathbb{C}^2 it follows that if $\beta = (F^{-1*})\alpha$ then

$$\lim_{n \rightarrow \infty} \int_{bA_n} F^*(F^{-1*})\alpha = \int_{bD_1} F^*(F^{-1*})\alpha + \int_{bD_2} F^*(F^{-1*})\alpha$$

where $A_n = F^{-1}(\tilde{A}_n)$, which implies (3.12).

If the boundary $b\Omega$ is of class \mathcal{C}^2 only in a neighborhood of the intersection $(L_1 \cup L_2) \cap b\Omega$, then a small modification of the argument just given is required. The set U can be taken to lie in the subset of $b\Omega$ that is a manifold of class \mathcal{C}^2 . Then, in the proof of Lemma 3.3 it is enough to assume that $b\Omega$ is of class \mathcal{C}^2 only in a neighborhood of $(L_1 \cup L_2) \cap b\Omega$, since we are there intersecting $b\Omega$ with varieties that are small perturbations of $L_1 \cup L_2$.

Lemma 3.5 is proved.

Discussion. Again, as in Discussion 3 after Lemma 3.3, given a neighbourhood \mathcal{D} of D_1 in the space of transversely embedded analytic discs, A_n can be chosen to belong to \mathcal{D} provided that $\text{diam}(D_2)$ is small enough.

We now continue the proof of Theorem 1.1, but no longer under the restriction to domains in \mathbb{C}^2 . We are dealing with the case that Ω is strictly convex at z_0 .

Lemma 3.6. *Let Ω be a bounded domain in \mathbb{C}^N with $b\Omega$ of class \mathcal{C}^2 . Suppose that $D_0 \subset \Omega$ is a transversely embedded analytic disc. Let $w_0 \in D_0$ and $z_0 \in bD_0$, and suppose that Ω is strictly convex at z_0 . Let \mathcal{D} be a neighbourhood of D_0 in the space of transversely embedded analytic discs. Denote by $\Lambda(z_0)$ a complex line that is tangent to $b\Omega$ at z_0 . There are an open ball B centered at z_0 and a neighborhood \mathcal{L} of $\Lambda(z_0)$ in the space of all complex lines in \mathbb{C}^N such that for each $L \in \mathcal{L}$ that meets $\Omega \cap B$*

(a) $L \cap B \cap b\Omega$ is a compact convex curve, and

(b) there are $A \in \mathcal{D}$ and a sequence $\{A_n\}_{n=1,2,\dots} \subset \mathcal{D}$ such that $w_0 \in A, w_0 \in A_n$ ($n \in \mathbb{N}$) and such that for each smooth 1-form α on \mathbb{C}^2

$$(3.13) \quad \lim_{n \rightarrow \infty} \int_{bA_n} \alpha = \int_{bA} \alpha + \int_{L \cap Q \cap b\Omega} \alpha.$$

As in the two-dimensional analysis, this implies that $\int_{L \cap Q \cap b\Omega} f\omega = 0$. Granted this lemma, Theorem 1.1, in the case of convexity, now follows from the following lemma.

Lemma 3.7. *Let D be a bounded convex domain with bD of class \mathcal{C}^2 and strictly convex at $z_0 \in bD$. Let \mathcal{L} be an open set of complex lines in \mathbb{C}^N that contains a line tangent to bD at z_0 . If f is a continuous function on bD that with the property that $\int_{L \cap bD} f\alpha = 0$ whenever $L \in \mathcal{L}$ meets D and whenever $\alpha \in \mathbb{C}^{1,0}[dz]$, then f is a CR-function on a neighborhood of z_0 in bD .*

Proof. As \mathcal{L} contains L_0 , a complex line tangent to bD at z_0 and is open, the result follows, in the case that $N = 2$, from a result in [Gl2]. In the case of arbitrary N , a different analysis is necessary.

Thus, consider the case of general N . Let H_0 be the real hyperplane tangent to bD at z_0 , and let $T_0 \subset H_0$ be the complex hyperplane in \mathbb{C}^N that goes through z_0 . We shall show that if T is a complex hyperplane in \mathbb{C}^N that is near T_0 and that meets D , then $\int_{T \cap bD} f\vartheta = 0$ for all $(N, N-2)$ -forms ϑ on \mathbb{C}^N with constant coefficients. Granted this, the result we want is a consequence of Theorem 3.2.1 of [GlS].

The complex hyperplane T_0 is a disjoint union of complex lines parallel to L_0 . Continuity and the openness of \mathcal{L} imply the existence of an open set \mathcal{T} in the space of complex hyperplanes in \mathbb{C}^N such that $T_0 \in \mathcal{T}$ and such that each $T \in \mathcal{T}$ is a union of complex lines L each of which is parallel to an element of \mathcal{L} and each of which is either disjoint from \overline{D} , meets \overline{D} in a single point, or else meets bD in a small convex curve lying near z_0 . If \mathcal{T} is small enough, then as \mathcal{L} is open, each $T \in \mathcal{T}$ is a union of lines $\{L_\lambda\}_{\lambda \in \Lambda(T)}$ parallel to an element of \mathcal{L} with the additional property that if L_λ meets bD , then $L_\lambda \in \mathcal{L}$.

By hypothesis, for a given $T \in \mathcal{T}$, $\int_{L_\lambda \cap bD} f\omega = 0$ for every $\lambda \in \Lambda(T)$. Lemma 2.2.1 of [GlS] implies that for each $T \in \mathcal{T}$, $\int_{T \cap bD} f\vartheta = 0$ for each ϑ , an $(N, N-2)$ -form on \mathbb{C}^N with constant coefficients. The lemma is proved.

The proof of Lemma 3.6 depends on the following simple observation.

Lemma 3.8. *Let $\delta > 0$, and let V be a closed one-dimensional complex submanifold of an open set in \mathbb{C}^N such that for each $\zeta, |\zeta| < \delta$, V intersects the hyperplane $H_\zeta = \{z : z_1 = \zeta\}$ at one point and transversely. Assume that V meets H_0 at the origin. There is a biholomorphic map Φ of $\{z \in \mathbb{C}^N : |z_1| < \delta\}$ onto itself that fixes H_0 and has the property that $\Phi(V \cap \{|z_1| < \delta\}) = \{(\zeta, 0, \dots, 0) : |\zeta| < \delta\}$.*

Proof. Let $V \cap H_\zeta = \varphi(\zeta) = (\zeta, \varphi_2(\zeta), \dots, \varphi_N(\zeta))$, $|\zeta| < \delta$. The map $\varphi : \delta\Delta \rightarrow \mathbb{C}^N$ is holomorphic by the transversality. As it satisfies $\varphi(0) = 0$ the map Φ given by

$$\Phi(z_1, \dots, z_N) = (z_1, z_2 - \varphi_2(z_1), \dots, z_N - \varphi_N(z_1))$$

has the desired properties.

Proof of Lemma 3.6 The idea is very simple; we describe it in a special case. Since Ω is strictly convex at z_0 there are an open ball B centered at z_0 and a neighbourhood \mathcal{L} of $\Lambda(z_0)$ in the space of complex lines in \mathbb{C}^N such that if $L \in \mathcal{L}$ and $L \cap B$ meets Ω then $L \cap B$ meets $b\Omega$ transversely in a compact convex curve. Moreover, given a neighbourhood \mathcal{E} of z_0 in \mathbb{C}^N , $L \cap B \cap \overline{\Omega} \subset \mathcal{E}$ provided that \mathcal{L} is small enough. Let V be a one dimensional submanifold of an open neighbourhood of $\overline{\Omega}$ such that $V \cap \Omega = D_0$. We find a biholomorphic map G from a neighbourhood Q of $\overline{D_0}$ in \mathbb{C}^N to a domain $G(Q)$ in such a way that $G(z_0) = 0$ and that G maps $Q \cap V$ into the z_1 -axis. Shrink \mathcal{L} if necessary so that $L \cap B \cap \overline{\Omega} \subset Q$ whenever $L \in \mathcal{L}$ and assume that $L \in \mathcal{L}$ and that $z_0 \in L$. We modify G so that, in addition, it maps $L \cap B \cap Q$ into the z_2 -axis. Then we intersect $G(Q)$ with the two-dimensional subspace M spanned by z_1 - and z_2 -axes. This gives a domain in the copy M of \mathbb{C}^2 to which Lemma 3.5 applies to yield a sequence $\{\tilde{A}_n\}$ of transversely embedded analytic discs whose boundaries $b\tilde{A}_n$, in the sense of that lemma, converge to $G(bD_0) \cup G(L \cap B \cap \Omega)$. Then $A_n = G^{-1}(\tilde{A}_n)$, and $A_0 = D_0$ will do the job. Of course, we must be more careful when L does not pass through z_0 .

Step 1. Let V be a closed one-dimensional submanifold of an open neighborhood of $\overline{\Omega}$ that intersects $b\Omega$ transversely so that $D_0 = V \cap \Omega$, $bD_0 = V \cap b\Omega$. By Lemma 3.1 above, $\overline{D_0}$ is polynomially convex, so it has a Stein neighborhood basis. Thus, there are arbitrarily small neighborhoods Q of $\overline{D_0}$ in \mathbb{C}^N that are biholomorphically equivalent, say under F , to a domain P in \mathbb{C}^N and in such a way that $F(V \cap Q) = \{(z_1, \dots, z_N) \in P : z_2 = \dots = z_N = 0\}$, *i.e.*, that $F(V \cap Q)$ is the intersection of P with the z_1 -axis. This follows from a result of Docquier and Grauert on the existence of holomorphic tubular neighborhoods -see [GR, pp. 256-257]- and the holomorphic triviality of holomorphic vector bundles over discs. The strict convexity of Ω at z_0 implies that there are an open ball B centered at z_0 and a neighbourhood \mathcal{L} of $\Lambda(z_0)$ in the space of complex lines such that

(D) if $L \in \mathcal{L}$ and $L \cap B$ meets Ω then $L \cap B$ meets $b\Omega$ transversely in a small compact convex curve. Moreover, given a neighbourhood \mathcal{E} of z_0 , \mathcal{L} can be chosen so small that $B \cap L \cap \overline{\Omega} \subset \mathcal{E}$ for every $L \in \mathcal{L}$.

Passing to smaller P, Q if necessary we may assume that there are a neighbourhood \mathcal{T} of the z_1 -axis in the space of all complex lines passing through $\tilde{w}_0 = F(w_0)$ and a $\delta > 0$ such that whenever a complex line T' is parallel to a line $T \in \mathcal{T}$, $\text{dist}(T, T') < \delta$ then T' intersects $F(Q \cap b\Omega)$ transversely in a simple closed curve that bounds the domain $\mathcal{D}(T') = T' \cap F(Q \cap \Omega)$.

Step 2. Since V is transverse to $b\Omega$ at z_0 it follows that $\Lambda(z_0)$ is not tangent to V at z_0 so the complex tangent line to $F(\Lambda(z_0) \cap B \cap Q)$ at $\tilde{z}_0 = F(z_0)$ does not coincide with the z_1 -axis. Thus, after composing F with a unitary map that fixes the z_1 -axis we may, after passing to a smaller \mathcal{L} and δ , assume that there are a small open ball $\mathcal{E} \subset B \cap Q$ such that (D) holds, and a neighbourhood \mathcal{H} of $\{z_2 = 0\}$ in the space of complex hyperplanes passing through \tilde{w}_0 such that

(E) for each $L \in \mathcal{L}$, $F(\mathcal{E} \cap L)$ intersects each H' , a complex hyperplane parallel to an $H \in \mathcal{H}$, $\text{dist}(H, H') < 2\delta$, at precisely one point and transversely.

By passing to a smaller \mathcal{T} we can suppose that for each $T \in \mathcal{T}$ there is an $H \in \mathcal{H}$ such that $T \subset H$.

For each $T \in \mathcal{T}$ let

$$\mathcal{D}(T, \delta) = \cup\{\mathcal{D}(T') : T' \text{ parallel to } T, \text{dist}(T, T') < \delta\}$$

and

$$P(T, \delta) = \cup\{T' \cap P : T' \text{ parallel to } T, \text{dist}(T, T') < \delta\}.$$

Note that $\mathcal{D}(T, \delta) = P(T, \delta) \cap F(\Omega \cap Q)$ is a connected component of $P(T, \delta) \setminus S$ where $S = F(b\Omega \cap Q)$. Choose a neighbourhood \mathcal{P} of \tilde{z}_0 in Q so small that

(F) if $z \in \mathcal{P}$ and if T is the complex line passing through z and \tilde{w}_0 then $T \in \mathcal{T}$ and $\mathcal{P} \subset\subset P(T, \delta)$.

By (D) we can pass to a smaller \mathcal{L} such that

$$(3.14). \quad L \cap B \cap \bar{\Omega} \subset Q, \quad F(L \cap B \cap \bar{\Omega}) \subset \mathcal{P} \quad (L \in \mathcal{L})$$

Fix $L \in \mathcal{L}$ that meets $\Omega \cap B$. By (D) and by (3.14), L meets $B \cap b\Omega$ transversely in a compact convex curve that bounds the convex domain $L \cap B \cap \Omega \subset\subset Q$. Moreover, by (3.14) we have

(G) $F(L \cap B \cap \Omega) = F(L \cap B) \cap F(\Omega \cap Q) \subset \mathcal{P}$ is a domain in $F(L \cap B \cap Q)$ bounded by the simple closed curve $F(L \cap B \cap b\Omega) = F(L \cap B \cap Q) \cap F(b\Omega \cap Q) \subset \mathcal{P}$.

Recall that

$$P(T, \delta) \cap b\mathcal{D}(T, \delta) = P(T, \delta) \cap F(b\Omega \cap Q)$$

and

$$P(T, \delta) \cap F(Q \cap \Omega) = \mathcal{D}(T, \delta).$$

By (G) and by the fact that $\mathcal{P} \subset\subset P(T, \delta)$ it follows that $F(L \cap B \cap Q) \cap F(\Omega \cap Q) \subset \mathcal{D}(T, \delta) \cap \mathcal{P}$ and $F(L \cap B \cap Q) \cap F(b\Omega \cap Q) \subset b\mathcal{D}(T, \delta) \cap \mathcal{P} \subset S \cap P(T, \delta)$ which implies that

$$(3.15) \quad F(B \cap L \cap Q) \cap \mathcal{D}(T, \delta) = F(B \cap L \cap Q) \cap F(\Omega \cap Q)$$

and

$$(3.16) \quad F(B \cap L \cap Q) \cap b\mathcal{D}(T, \delta) = F(B \cap L \cap Q) \cap F(b\Omega \cap Q) \subset S \cap P(T, \delta).$$

Clearly, $F(B \cap L \cap Q)$ intersects S transversely.

Step 3. Choose $p \in F(L \cap B \cap Q) \cap F(b\Omega \cap Q)$. By (3.14) and by (F) the complex line T passing through p and \tilde{w}_0 belongs to \mathcal{T} and is contained in some $H \in \mathcal{H}$ so by (E), $F(L \cap B \cap Q)$ intersects each complex hyperplane H' parallel to H with $\text{dist}(H, H') < 2\delta$ at precisely one point and transversely. Lemma 3.8 now implies that there is a biholomorphic

map Φ from $H + 2\delta\mathbb{B}$ onto itself that fixes H and maps $F(B \cap L \cap Q) \cap (H + 2\delta\mathbb{B})$ onto $p + 2\delta\Delta q$ where q is a unit vector orthogonal to H . Note that $P(T, \delta) \subset\subset H + 2\delta\mathbb{B}$.

Let $U = \Phi(\mathcal{D}(T, \delta))$ and $\Sigma = \Phi(P(T, \delta) \cap S) = \Phi(b\mathcal{D}(T, \delta) \cap P(T, \delta))$. Now (3.15) and (3.16) imply that $\Phi(F(B \cap L \cap Q)) \cap (H + 2\delta\mathbb{B}) = p + 2\delta\Delta q$ meets \bar{U} in a domain bounded by a simple closed curve contained in $\Sigma \subset bU$, and the intersection is transverse. Moreover, since Φ fixes $H \supset T$ we have $p \in \Sigma \cap T$ and T meets \bar{U} in a domain bounded by a simple closed curve $\Sigma \cap T = S \cap T$ obtained as a transverse intersection of T with $\Sigma \subset bU$.

With no loss of generality, assume that $p = 0$. Let M be the two-dimensional complex subspace spanned by T and $\mathbb{C}q$. Recall that $\mathbb{C}q$ meets \bar{U} in a domain bounded by a simple closed curve which is a transverse ointersection of $\mathbb{C}q$ with $\Sigma \subset bU$ and that T intersects \bar{U} in a domain bounded by a simple closed curve which is the transverse intersection of T with $\Sigma \subset bU$. Since both $T \cap bU$ and $\mathbb{C}q \cap bU$ are contained in Σ , an open subset of bU which is smooth, it follows that near $(T \cup bU) \cup (\mathbb{C}q \cap bU)$, bU is smooth and transverse to M . Thus $\tilde{U} = U \cap M$ is a bounded open set in M which has smooth boundary near $(T \cap bU) \cap (\mathbb{C}q \cap bU)$. The component \tilde{U}_0 of U containing $(T \cap U) \cup (\mathbb{C}q \cap U)$ is a bounded domain in M which has smooth boundary near $(T \cap b\tilde{U}_0) \cup (\mathbb{C}q \cap b\tilde{U}_0)$ and which T and $\mathbb{C}q$ meet in domains bounded by simple closed curves, which are transverse intersections of T and $\mathbb{C}q$ with $b\tilde{U}_0$. Now apply Lemma 3.5 to get a sequence $\tilde{A}_n \subset \tilde{U}_0$ of transversely embedded analytic discs whose boundaries $b\tilde{A}_n$, in the sense of that lemma, converge to $(T \cap b\tilde{U}_0) \cup (\mathbb{C}q \cap b\tilde{U}_0)$. It is now clear that by pulling back to $b\Omega \cap Q$ with $(\Phi \circ F)^{-1}$ that $A_n = (\Phi \circ F)^{-1}(\tilde{A}_n)$, $A = (\Phi \circ F)^{-1}(T \cap U_0)$, and $L \cap B \cap Q \cap b\Omega = (\Phi \circ F)^{-1}(\mathbb{C}q \cap b\tilde{U})$ satisfy (3.13) and $w_0 \in A$, $w_0 \in A_n (n \in \mathbb{N})$.

It remains to show that everything can be done in such a way that A_n ($n \in \mathbb{N}$) and A belong to \mathcal{D} . By transversality, for each $T \in \mathcal{T}$ the disc $T \cap F(Q \cap \Omega)$ is arbitrarily small perturbation of the intersection of $F(Q \cap \Omega)$ with z_1 -axis provided that \mathcal{T} is small enough. Further, our construction implies that the maps Φ are uniformly close to the identity provided that \mathcal{H} and δ are small enough. Since \mathcal{P} can be chosen arbitrarily small the reasoning from Discussion 3 following Lemma 3.3 applies to show that the discs \tilde{A}_n belong to an arbitrarily small neighbourhood of $T \cap F(Q \cap \Omega)$ provided that \mathcal{P} is small enough. It follows that A_n , $n \in \mathbb{N}$, and A can be chosen to belong to \mathcal{D} .

Theorem 1.1 is thus proved in the case of convexity. It remains to prove it in the strictly pseudoconvex case.

To this end, observe first that in the proof of Lemma 3.6 we never used the fact that the elements of \mathcal{L} are complex lines. What we needed was the following:

- (a) There is an open neighborhood \mathcal{W} of z_0 such that all $L \in \mathcal{L}_0$ are one-dimensional complex submanifolds of \mathcal{W} ,
- (b) the initial $\Lambda(z_0) \in \mathcal{L}_0$ is tangent to $\mathcal{W} \cap b\Omega$,
- (c) if $L \in \mathcal{L}_0$ is sufficiently close to $\Lambda(z_0)$ in the \mathcal{C}^1 -sense and if L meets $\mathcal{W} \cap \Omega$, then L meets $b\Omega$ transversely in a simple closed curve bounding the domain $\Omega \cap L$, and
- (d) given a neighborhood \mathcal{V} of z_0 , we have $L \cap \bar{\Omega} \subset \mathcal{V}$ provided that $L \in \mathcal{L}_0$ is sufficiently close to $\Lambda(z_0)$ in the \mathcal{C}^1 -sense.

Now the following lemma is proved in exactly the same way as Lemma 3.6.

Lemma 3.9. *Let Ω be a bounded domain in \mathbb{C}^N with $b\Omega$ of class \mathcal{C}^2 . Suppose that $D_0 \subset \Omega$ is a transversely embedded analytic disc. Let $w_0 \in D_0$ and $z_0 \in bD_0$. Let $\mathcal{W}, \Lambda(z_0)$ and \mathcal{L}_0 be as above. Let \mathcal{D} be a neighbourhood of D_0 in the space of transversely embedded analytic discs. There is an $\mathcal{L} \subset \mathcal{L}_0$, a \mathcal{C}^1 -neighbourhood of $\Lambda(z_0)$ in \mathcal{L}_0 , such that for each $L \in \mathcal{L}$ that meets $\Omega \cap \mathcal{W}$*

(a) $L \cap b\Omega$ is a compact convex curve, and

(b) there are $A \in \mathcal{D}$ and a sequence $\{A_n\}_{n=1,2,\dots} \subset \mathcal{D}$ such that $w_0 \in A, w_0 \in A_n$ ($n \in \mathbb{N}$) and such that for each smooth 1-form α on \mathbb{C}^2

$$\lim_{n \rightarrow \infty} \int_{bA_n} \alpha = \int_{bA} \alpha + \int_{L \cap b\Omega} \alpha.$$

To conclude the proof, we suppose that $b\Omega$ is strictly pseudoconvex at z_0 . There exist a neighborhood \mathcal{W} of z_0 and a biholomorphic map $\Psi : \mathcal{W} \rightarrow \mathcal{W}$, \mathcal{W} a neighborhood of $0 = \Psi(z_0)$ such that Ψ is a polynomial map of degree two and such that $\Psi(\Omega \cap \mathcal{W})$ is strictly convex. Let \tilde{L}_0 be the complex tangent line to $b\Psi(\Omega \cap \mathcal{W})$ at 0. Let $\tilde{\mathcal{L}}_0$ be a neighborhood of \tilde{L}_0 in the space of complex lines, and for each $\tilde{L} \in \tilde{\mathcal{L}}_0$ let $L = \Phi^{-1}(\tilde{L} \cap \tilde{\mathcal{W}})$. If \mathcal{W} and $\tilde{\mathcal{L}}_0$ are sufficiently small then $\mathcal{L}_0 = \{L : \tilde{L} \in \tilde{\mathcal{L}}_0\}$ has the properties (a)-(d). Recall that by our assumptions, $\int_{bD} f\omega = 0$ for every form $\omega \in \mathbb{C}^{1;1}[dz]$ and for each transversely embedded analytic disc $D \in \mathcal{D}$. By Lemma 3.9 it follows that there is a neighborhood $\tilde{\mathcal{L}} \subset \tilde{\mathcal{L}}_0$ of \tilde{L}_0 in the space of complex lines and a neighborhood Q of z_0 such that for each $L \in \mathcal{L} = \{L : \tilde{L} \in \tilde{\mathcal{L}}\}$ which meets $\Omega \cap \mathcal{W}$,

$$(3.18) \quad \int_{L \cap b\Omega} f\omega = 0$$

for each $\omega \in \mathbb{C}^{1;1}[dz]$. Since Ψ is a polynomial map of degree two, it follows that for all $\alpha \in \mathbb{C}^{1;0}[dz]$ the form $\Psi^*\alpha$ lies in $\mathbb{C}^{1;1}[dz]$.

If $\tilde{L} \in \tilde{\mathcal{L}}$, then given a form $\alpha \in \mathbb{C}^{1;0}[dz]$ (3.18) implies that

$$\int_{\tilde{L} \cap \Psi(b\Omega \cap \mathcal{W})} (f \circ \Psi^{-1})\alpha = \int_{\Psi(L \cap b\Omega)} (f \circ \Psi^{-1})\alpha = \int_{L \cap b\Omega} (f \circ \Psi^{-1} \circ \Psi)\Psi^*\alpha = 0.$$

By [GIS, Th.3.2.1] it follows that $f \circ \Psi^{-1}$ is a CR-function in a neighborhood of 0 in $\Psi(b\Omega \cap \mathcal{W})$, which implies that f is a CR-function in a neighborhood of z_0 .

Theorem 1.1 is finally proved.

4. Concluding Remarks.

In this final section we note a consequence of the main theorem.

Let Ω be as in Theorem 1.1 and let $D \subset \Omega$ be a transversely embedded analytic disc. If a continuous function f on bD has a continuous extension to \overline{D} which is holomorphic on D then $\int_{bD} f\omega = 0$ for all $(1,0)$ -forms with linear coefficients. The holomorphic extendibility is invariant with respect to biholomorphic maps. Accordingly the following theorem holds in Stein manifolds, which we view as closed complex submanifolds of \mathbb{C}^M .

Theorem 4.1. *Let Ω be a relatively compact domain in a closed complex submanifold \mathcal{M} , $\dim \mathcal{M} \geq 2$ of \mathbb{C}^M that has \mathcal{C}^2 boundary. Let $D_0 \subset \Omega$ be a transversely embedded analytic disc, and let Ω be strictly pseudoconvex at $z_0 \in bD_0$. Let \mathcal{D} be a neighbourhood of D_0 in the space of transversely embedded analytic discs $D \subset \Omega$. Suppose that f is a continuous function on a neighbourhood of bD_0 in $b\Omega$ such that for each $D \in \mathcal{D}$ satisfying $w_0 \in D$,*

$$(4.1) \quad f|_{bD} \text{ has a continuous extension to } D \text{ which is holomorphic on } D.$$

Then f is a CR function in a neighbourhood of z_0 . If (4.1) holds for all $D \in \mathcal{D}$ then there are a neighbourhood W of D_0 in \mathcal{M} and a continuous function \tilde{f} on $W \cap \overline{\Omega}$ such that $\tilde{f} = f$ on $W \cap b\Omega$.

Proof. Let $\dim \mathcal{M} = N$. The set $\overline{D} \subset \mathcal{M}$ is a compact, polynomially convex set in \mathbb{C}^M and thus has a Stein neighborhood basis in \mathbb{C}^M . The intersections of the elements of this basis with \mathcal{M} form a Stein neighborhood basis of \overline{D} in \mathcal{M} . Thus, by the result of Docquier and Grauert and the holomorphic triviality of holomorphic vector bundles on discs as used above and by the fact that the extendibility assumptions we make are invariant under biholomorphic maps, we can assume that Ω is a domain in \mathbb{C}^N and that D is the intersection of Ω with the z_1 -axis. The preceding theorem now implies that f is a CR-function in a neighborhood of z_0 in $b\Omega$. This proves the first part of the theorem. As $b\Omega$ is strictly pseudoconvex at z_0 , there are a neighborhood P of z_0 and a continuous function \tilde{f} on $P \cap \overline{\Omega}$ that is holomorphic on $P \cap \Omega$ and that satisfies $\tilde{f} = f$ on $P \cap b\Omega$. Our assumptions imply that there is $\delta > 0$ such that if \mathcal{T} denotes the set of all complex lines parallel to the z_1 -axis and at distance not exceeding δ from it, then each $T \in \mathcal{T}$ meets $P \cap b\Omega$ transversely (and thus also meets $P \cap \Omega$), T meets $b\Omega$ in a simple closed curve bounding $T \cap \Omega$, and $f|(T \cap b\Omega)$ has a continuous extension f_T to $T \cap \overline{\Omega}$ that is holomorphic in $T \cap \Omega$. Since $T \cap b\Omega \cap P$ contains an arc, f_T coincides with $\tilde{f}|_T$ near $b\Omega \cap P$. Thus, by Hartogs's lemma, the function \tilde{f} defined by $\tilde{f}|(T \cap \overline{\Omega}) = f_T$ has all the required properties on $\mathcal{W} = \cup\{T : T \in \mathcal{T}\}$.

This completes the proof.

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Institute of Mathematics, Physics and Mechanics, University of Ljubljana
Department of Mathematics, University of Washington