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STRONGLY ADJACENCY-TRANSITIVE GRAPHS AND UNIQUELY SHIFT-TRANSITIVE GRAPHS

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Strongly adjacency-transitive graphs and uniquely shift-transitive graphs

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Abstract

An automorphism σ of a finite simple graph Γ is an *adjacency automorphism* if for every vertex $x \in V(\Gamma)$, either $\sigma x = x$ or σx is adjacent to x in Γ. An adjacency automorphism fixing no vertices is a *shift*. A connected graph Γ is *strongly adjacency-transitive* (*uniquely shift-transitive*, respectively) if there is, for every pair of adjacent vertices $x, y \in V(\Gamma)$, an adjacency automorphism (a unique shift, respectively) $\sigma \in \operatorname{Aut}\Gamma$ sending x to y. The *action graph* $\Gamma = \operatorname{ActGrph}(G, X, S)$ of a group G acting on a set X, relative to an inverse-closed nonempty subset $S \subseteq G$, is defined as follows: the vertex-set of Γ is X, and two different vertices $x, y \in V(\Gamma)$ are adjacent in Γ if and only if $y = sx$ for some $s \in S$.

A characterization of strongly adjacency-transitive graphs in terms of action graphs is given. A necessary and sufficient condition for cartesian products of graphs to be uniquely shift-transitive is proposed, and two questions concerning uniquely shift-transitive graphs are raised.

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1 Strongly adjacency-transitive graphs

The groups and graphs considered in this paper are finite, the graphs are simple and undirected. We refer the reader to [8] for the results on permutation groups.

An automorphism $\sigma \in \operatorname{Aut}\Gamma$ is an *adjacency automorphism* of a graph Γ if for every vertex $x \in V(\Gamma)$, one of the following holds: either $\sigma x = x$ or σx is adjacent to x in Γ.

The graph Γ is *adjacency-transitive* if there exists, for every pair of vertices $x, y \in V(\Gamma)$, a sequence of adjacency automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_k \in \operatorname{Aut}\Gamma$ such that $\sigma_1\sigma_2\cdots\sigma_k x = y$. It is equivalent to say that the adjacency automorphisms of Γ generate a transitive group on the vertex-set $V(\Gamma)$. If, in addition, for every pair of adjacent vertices $x, y \in V(\Gamma)$ there exists an adjacency automorphism $\sigma \in \operatorname{Aut}\Gamma$ sending x to y, then Γ is *strongly adjacency-transitive.* The notions of an adjacency automorphism and adjacency-transitivity in graphs were introduced and studied in [9].

In the sequel we denote by $\sim_\Gamma$ the vertex adjacency relation in a graph Γ.

Proposition 1.1 *Every arc-transitive graph having a nontrivial adjacency automorphism is strongly adjacency-transitive.*

PROOF. Let Γ be an arc-transitive graph: by definition, for every two ordered pairs of adjacent vertices (u, v) and (u', v') in Γ, there exists an automorphism $\sigma \in \operatorname{Aut}\Gamma$ such that $\sigma u = u'$ and $\sigma v = v'$. If $\rho \in \operatorname{Aut}\Gamma$ is a nontrivial adjacency automorphism, there exists a vertex $v \in V(\Gamma)$ for which $\rho v \neq v$, thus $v \sim_\Gamma \rho v$. For an arbitrary pair of adjacent vertices $x, y \in V(\Gamma)$ denote by σ the automorphism of Γ sending v to x and ρv to y. Then the conjugate $\sigma\rho\sigma^{-1}$ is an adjacency automorphism of Γ sending x to y. So Γ is strongly adjacency-transitive. ■

We characterize strongly adjacency-transitive graphs in terms of action graphs. The *action graph*

$$\Gamma = \operatorname{ActGrph}(G, X, S) \tag{1}$$

of a group G acting on a set X, relative to an inverse-closed nonempty subset $S \subseteq G$, is defined as follows (see [1]): the vertex-set of Γ is X, and two different vertices $x, y \in V(\Gamma)$ are adjacent in Γ if and only if $y = sx$ for some $s \in S$. We refer the reader to [4] for the notion of *action digraphs* and its application, and to [5] for an implementation of the action graph construction.

Theorem 1.2 *A graph* Γ *is strongly adjacency-transitive if and only if it is (isomorphic to) an action graph* $\mathrm{ActGrph}(G, X, S)$*, where* G *is a group acting transitively and faithfully on the set* X *and* S *is a subset in* G *satisfying the following conditions:* $S = S^{-1}$*,* S *generates* G *and* S *is a union of conjugacy classes in* G*.*

Proof. Let Γ be a strongly adjacency-transitive graph. Define $X = V(\Gamma)$ and let $S \subseteq \mathrm{Aut}\,\Gamma$ be the set of adjacency automorphisms of Γ. Obviously, $S = S^{-1}$ holds. The subgroup $\langle S\rangle \leq \mathrm{Aut}\,\Gamma$ generated by S acts transitively and faithfully on X, and two different vertices $x, y \in X$ are adjacent in Γ if and only if there is an adjacency automorphism $s \in S$ such that $y = sx$. So we have

$$\Gamma = \mathrm{ActGrph}(\langle S\rangle, X, S),$$

and S is a union of conjugacy classes in $\langle S\rangle$ since the set S is closed under conjugation in $\mathrm{Aut}\,\Gamma$ by [9, Proposition 2.4].

To prove the opposite assertion, let G be a group acting transitively and faithfully on X and let G be generated by a subset $S \subseteq G$, where S is a union of conjugacy classes in G such that $S = S^{-1}$. Denote by Γ the action graph $\Gamma = \mathrm{ActGrph}(G, X, S)$. First we prove that G is a group of automorphisms of Γ: for an arbitrary $g \in G$ we have

$$\begin{aligned} x \sim_\Gamma y &\iff x \neq y \text{ and } y = sx,\ s \in S \\ &\iff gx \neq gy \text{ and } gy = gsx,\ s \in S \\ &\iff gx \neq gy \text{ and } gy = (gsg^{-1})gx,\ s \in S \\ &\iff gx \neq gy \text{ and } gy = tgx,\ t \in S \\ &\iff gx \sim_\Gamma gy. \end{aligned}$$

So g is an automorphism of Γ and $G \leq \mathrm{Aut}\,\Gamma$. Besides, if $s \in S$ and $x \in V(\Gamma)$, then either $sx = x$, or $sx \neq x$ implying $x \sim_\Gamma sx$. Thus S is a set of adjacency automorphisms of Γ. Since S generates the vertex-transitive subgroup G in $\mathrm{Aut}\,\Gamma$, the graph Γ is strongly adjacency-transitive. ■

Defining the action graph (1) to be *quasiabelian* if S is a union of conjugacy classes, one may rephrase Theorem 1.2.

Proposition 1.3 *A graph is strongly adjacency-transitive if and only if it is (isomorphic to) a connected vertex-transitive quasiabelian action graph.*

We give two corollaries of Theorem 1.2. Recall that the *Cayley graph* $\Gamma = Cay(G, S)$ is defined for an arbitrary group G and a subset $S \subseteq G$ satisfying

$1 \notin S$ and $S = S^{-1}$: the vertex-set of Γ is G, and adjacency in Γ is given by $g \sim_\Gamma gs$ for all $g \in G$ and all $s \in S$. A *quasiabelian* Cayley graph is a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, where S is a union of conjugacy classes in G. (See [7] and [9, 10] for results on quasiabelian Cayley graphs, and [3] under the equivalent notion of *normal Cayley graphs.*)

Corollary 1.4 *[9, Proposition 2.1] Every connected quasiabelian Cayley graph is strongly adjacency-transitive.*

Corollary 1.5 *Every connected Cayley graph of an abelian group is strongly adjacency-transitive.*

We conclude this section with the following observations.

Proposition 1.6 *There are quasiabelian Cayley graphs that are not Cayley graphs of abelian groups.*

PROOF. Define the quasiabelian Cayley graph $\Gamma = \mathrm{Cay}(S_4, T)$ of the symmetric group S_4 relative to the conjugacy class T of all 4-cycles in S_4. One can check that $\mathrm{Aut}\,\Gamma \simeq S_4 \wr S_2$, and that no abelian subgroup in $\mathrm{Aut}\,\Gamma$ is regular. (For instance, compute the order $|\mathrm{Aut}\,\Gamma| = 1152$ using [5], and proceed with elementary group-theoretic arguments.) Thus Γ is not a Cayley graph of an abelian group. ■

Proposition 1.7 *There exist strongly adjacency-transitive Cayley graphs that are not quasiabelian Cayley graphs.*

PROOF. By [9, p. 325], the triangle graph T_7 is an adjacency-transitive Cayley graph that is not a quasiabelian Cayley graph. Since its automorphism group has rank 3, the graph T_7 is arc-transitive. Proposition 1.1 implies T_7 is strongly adjacency-transitive. ■

2 Uniquely shift-transitive graphs

A *shift* of a graph is an adjacency automorphism fixing no vertices (see [3, Definition 3.4]). Shifts are easily found in Cayley graphs of abelian groups, as in the wider class of quasiabelian Cayley graphs: if $\Gamma = \mathrm{Cay}(G, S)$, where S is a union of conjugacy classes in G, then the right multiplication by $s \in S$ of elements in G induces a shift of Γ (see [9, Proof of Proposition 2.1]), and the same holds for the left multiplication by s.

We call a graph Γ *shift-transitive* if there exists, for every pair of vertices $x, y \in V(\Gamma)$, a sequence of shifts $\sigma_1, \sigma_2, \ldots, \sigma_k \in \operatorname{Aut}\Gamma$ such that $\sigma_1\sigma_2\cdots\sigma_k x = y$. If, in addition, for every pair of adjacent vertices $x, y \in V(\Gamma)$ there exists exactly one (at least one, resp.) shift $\sigma \in \operatorname{Aut}\Gamma$ sending x to y, then Γ is *uniquely shift-transitive* (*strongly shift-transitive*, resp.). Observe that the valency of a uniquely shift-transitive graph Γ equals the number of shifts in $\operatorname{Aut}\Gamma$.

The cycle C_4 is not uniquely shift-transitive. Written as $P_2 \times P_2$, it presents the fundamental obstruction for the cartesian product to preserve uniquely shift-transitivity, as stated in the following theorem. (We refer the reader to [6] for the theorem on unique prime cartesian factorization of connected graphs.)

Theorem 2.1 *A graph Γ is uniquely shift-transitive if and only if in the prime cartesian factorization of Γ, all factors are uniquely shift-transitive and at most one factor is isomorphic with P_2.*

To prove Theorem 2.1 we need some auxiliary results. We omit the justifications of the first two.

Lemma 2.2 *If $\gamma \in \operatorname{Aut}\Gamma$ and $\delta \in \operatorname{Aut}\Delta$ are shifts of Γ and Δ, respectively, then the automorphisms $\gamma \times \mathrm{id}_\Delta$ and $\mathrm{id}_\Gamma \times \delta$ are shifts of the cartesian product $\Gamma \times \Delta$.*

The automorphisms $\gamma \times \mathrm{id}_\Delta$ and $\mathrm{id}_\Gamma \times \delta$ in Lemma 2.2 are called *cartesian shifts along the factors* Γ and Δ, respectively.

Proposition 2.3 *Let Γ and Δ be shift-transitive graphs. Then*

(a) the cartesian product $\Gamma \times \Delta$ is shift-transitive;

(b) if Γ or Δ is not uniquely shift-transitive then neither is $\Gamma \times \Delta$.

Lemma 2.4 *Let Γ and Δ be connected graphs and let $\sigma \in \operatorname{Aut}(\Gamma \times \Delta)$ be a shift. Then*

(a) if σ fixes setwise one of the fibers $\{c\} \times V(\Delta)$, $c \in V(\Gamma)$, then σ fixes setwise each one of them, and $\sigma = \mathrm{id}_\Gamma \times \delta$ for some shift $\delta \in \operatorname{Aut}\Delta$;

(b) if $\sigma(\{c\} \times V(\Delta)) \cap (\{c\} \times V(\Delta)) = \emptyset$ for some $c \in V(\Gamma)$, then σ fixes setwise each of the fibers $V(\Gamma) \times \{d\}$, $d \in V(\Delta)$, and $\sigma = \gamma \times \mathrm{id}_\Delta$ for some shift $\gamma \in \operatorname{Aut}\Gamma$;

(c) *if there is a vertex u in a fiber $\{c\} \times V(\Delta)$, $c \in V(\Gamma)$, such that $u, \sigma u, \sigma^2 u \in \{c\} \times V(\Delta)$, then $\sigma = \mathrm{id}_\Gamma \times \delta$ for some shift $\delta \in \mathrm{Aut}\,\Delta$.*

PROOF. Denote by ρ the canonical projection of $V(\Gamma) \times V(\Delta)$ onto $V(\Delta)$.

(a) Let the shift σ fix setwise the fiber $F = \{c\} \times V(\Delta)$ for some $c \in V(\Gamma)$, and let $c' \in V(\Gamma)$ be adjacent to c in Γ.

Choose an arbitrary vertex $d \in V(\Delta)$. Then $\sigma(c,d) = (c,d')$ for some neighbour $d' \in V(\Delta)$ of d. Let $\sigma(c',d) = (c'',d'')$, where $c'' \in V(\Gamma)$ and $d'' \in V(\Delta)$. Suppose $\sigma(c',d) \notin F' = \{c'\} \times V(\Delta)$. Then, since σ is a shift of $\Gamma \times \Delta$, we have $d'' = d$ and the vertex c'' is adjacent to c' in Γ. The vertices (c,d) and (c',d) are adjacent in $\Gamma \times \Delta$, and so are their images $\sigma(c,d) = (c,d')$ and $\sigma(c',d) = (c'',d)$. Thus $c = c''$, giving $\sigma(c',d) \in F$, a contradiction to $\sigma F = F$. So

$$\sigma(c',d) \in F', \tag{2}$$

whence $c'' = c'$ and d'' is adjacent to d in Δ. The same adjacency argument as above gives $d' = d''$, i.e.

$$\rho\sigma(c,d) = \rho\sigma(c',d). \tag{3}$$

One infer from (2) that the shift σ fixes setwise the fiber F'. The connectedness of Γ implies the shift σ fixes setwise all fibers over Γ. Adding (3), one gets $\sigma = \mathrm{id}_\Gamma \times \delta$ for some shift $\delta \in \mathrm{Aut}\,\Delta$.

(b) Let $F = \{c\} \times V(\Delta)$ and suppose $\sigma F \cap F = \emptyset$. Verify that the shift σ moves every fiber $F' = \{c'\} \times V(\Delta)$ over Γ into a fiber over Γ, and that $\sigma F' \cap F' = \emptyset$.

(c) Denote by F the fiber $V(\Gamma) \times \{\rho\sigma u\}$. Check that the intersection $\sigma(F) \cap F$ is empty, then apply (b).

■

Proposition 2.5 *Let Γ and Δ be connected graphs and let $\sigma \in \mathrm{Aut}\,(\Gamma \times \Delta)$ be a shift which is not cartesian along Γ or Δ. Then each of the graphs Γ and Δ is isomorphic to a cartesian product with a P_2 factor.*

PROOF. Suppose the shift $\sigma \in \mathrm{Aut}\,(\Gamma \times \Delta)$ is not cartesian along Γ or Δ. Then Γ and Δ have order at least 2. Fix an arbitrary vertex $c \in V(\Gamma)$, then define the c-fiber

$$F = \{c\} \times V(\Delta)$$

and its subset

$$A = \{u \in F \mid \sigma u \in F\}.$$

By Lemma 2.4(a,b) we have $A \neq \emptyset$ and $A \neq F$. We will show that Δ is isomorphic to the cartesian product $\Omega \times P_2$, where Ω is the subgraph in $\Gamma \times \Delta$ induced by A. Define

$$B = F \setminus A = \{u \in F \mid \sigma u \notin F\}.$$

Then $\sigma A \subseteq B$ by Lemma 2.4(c). Moreover, there are no edges between A and B except the matching amongst A and σA induced by the shift σ: if $v \in A$ is adjacent to $w \in B$ and $w \neq \sigma v$, then the vertices $\sigma v \in F \setminus \{w\}$ and $\sigma w \notin F$ are not adjacent in $\Gamma \times \Delta$, a contradiction.

We now prove $B = \sigma A$. Suppose $B \neq \sigma A$ and fix a vertex $w \in B \setminus \sigma A$ adjacent to a vertex $v \in \sigma A$. Then $\sigma^{-1} w \notin F$, so the image $\sigma^{-1} w$ is not adjacent to $\sigma^{-1} v \in A$, a contradiction. Therefore $B = \sigma A$, and $\Delta \simeq \Omega \times P_2$.

One shows similarly that $\Gamma \simeq \Pi \times P_2$ for some (connected) graph Π. ■

Remark. It follows from the proof of Proposition 2.5 that

$$\Gamma \times \Delta \simeq (\Omega \times \Pi) \times C_4. \tag{4}$$

The shift σ is cartesian along the factors of the right factorization in (4),

$$\sigma = \mathrm{id}_{\Omega \times \Pi} \times \phi,$$

where $\phi \in \mathrm{Aut}\, C_4$ is a shift of order 4.

Proposition 2.6 *The cartesian product $\Gamma \times \Delta$ of two graphs Γ and Δ is shift-transitive if and only if Γ and Δ are shift-transitive.*

Proof. Proposition 2.3(a) settles the "if" implication. We prove the "only if" part. Let $\Gamma \times \Delta$ be shift-transitive. Factorize $\Delta \simeq (P_2)^s \times \Delta'$, where $s \geq 0$ and Δ' has no P_2 factor in its cartesian factorization. Then

$$\Gamma \times \Delta \simeq (\Gamma \times (P_2)^s) \times \Delta' = \Sigma.$$

Put $\Gamma' = \Gamma \times (P_2)^s$. If Δ' is not shift-transitive, then the shift-transitive graph Σ has a shift which is not cartesian along the factors Γ' or Δ'. By Proposition 2.5, Δ' is a cartesian product with a P_2 factor, a contradiction. Thus Δ' is shift-transitive, and so is Δ by Proposition 2.3(a). ■

Theorem 2.1 is a corollary of Propositions 2.5 and 2.6. It leads to abundance of uniquely shift-transitive graphs among Cayley graphs of abelian groups.

Corollary 2.7 *The cartesian product of cycles $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$ is uniquely shift-transitive if and only if there are no 4-cycles involved.*

Corollary 2.8 *Every abelian group of order not divisible by 4 admits a uniquely shift-transitive Cayley graph.*

3 Two questions

Examples of uniquely shift-transitive graphs can be found among Cayley graphs of abelian groups: besides the cartesian product of cycles as in Corollary 2.7 we have, for instance, the Möbius ladder M_n, $n \geq 4$ (see also [9, Section 3]).

Proposition 3.1 *Let $\Gamma = \mathrm{Cay}(G, S)$ be a quasiabelian Cayley graph of a nonabelian group G, where the generating set S is an inverse-closed union of conjugacy classes and $1 \notin S$. Then Γ is not uniquely shift-transitive.*

PROOF. For $s \in S \setminus Z(G)$, the left and the right multiplication by s induce two different shifts of Γ sending the vertex 1 to s. ■

Thus a uniquely shift-transitive quasiabelian Cayley graph must be a Cayley graph of an abelian group.

Question 1 *Is every uniquely shift-transitive Cayley graph a Cayley graph of an abelian group?*

If the answer to Question 1 is positive, then every shift σ of a uniquely shift-transitive Cayley graph Γ arises from the multiplication by a fixed element of an abelian group. Hence σ must be *semiregular*, i.e. all cycles in its cyclic decomposition have same length. This fact may prove useful in approaching the problem.

Question 2 *Does there exist a uniquely shift-transitive non-Cayley graph?*

The answer to Question 2 is negative in case we omit the uniqueness of the shift acting along an arbitrary edge, according to the following result.

Proposition 3.2 *There exist strongly shift-transitive non-Cayley graphs.*

PROOF. Let $\Gamma = (K(n,k))^c$ be the complement of *Kneser's graph* $K(n,k)$, where n and k are two positive integers such that $n = 2k+1$ and $k \geq 3$. The vertices of Γ are the k-elements subsets in $I_n = \{1, 2, \ldots, n\}$, and two such k-subsets are adjacent in Γ if and only if they have nontrivial intersection. By [2], every automorphism of the graph Γ arises from a permutation in S_n acting naturally on the k-subsets of I_n. One can check that the automorphism of Γ induced by any s-cycle in S_n, $k+2 \leq s \leq 2k-1$, is a shift of Γ, and any vertex of Γ may be moved to a neighbour by at least two shifts of this kind. However, Γ is not a Cayley graph by [2]. ∎

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