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A NOTE ON ADJACENCY-TRANSITIVITY OF A GRAPH AND ITS COMPLEMENT

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Abstract

We give a necessary and sufficient condition for a graphical regular representation to be adjacency-transitive, and provide an infinite family of finite simple undirected vertex-transitive graphs Γ, such that neither Γ nor Γ^c is adjacency-transitive.

1 Introduction

The graphs considered are finite, simple and undirected. We refer the reader to [2] for results on permutation groups.

An automorphism $\sigma \in \operatorname{Aut}\Gamma$ is called an *adjacency automorphism* of a graph Γ, if $\operatorname{dist}(x, \sigma(x)) \leq 1$ for every vertex $x \in V(\Gamma)$. If for every $x, y \in V(\Gamma)$ there exists a sequence of adjacency automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_k \in \operatorname{Aut}\Gamma$ such that $\sigma_1\sigma_2\cdots\sigma_k(x) = y$, then the graph Γ is said to be *adjacency-transitive*. In [3], where these two notions were introduced, examples of adjacency-transitive graphs and of vertex-transitive but not adjacency-transitive graphs are given. All of these examples have the property that either the graph itself or its complement is adjacency-transitive.

The purpose of this note is to give an infinite family of vertex-transitive graphs Γ, such that neither Γ nor Γ^c is adjacency-transitive (Proposition2.5). This is achieved by proving a necessary and sufficient condition for a graphical regular representation to be adjacency-transitive (Proposition 2.4). The fact that the complement of a disconnected vertex-transitive graph is adjacency-transitive (Corollary 2.2) is mentioned here just to indicate that the subject might be nontrivial.

2 The results

Recall that the lexicographic product $\Gamma[\Delta]$ of two graphs Γ and Δ has vertex set $V(\Gamma) \times V(\Delta)$, and two vertices (u, v), (u', v') are adjacent in $\Gamma[\Delta]$ if and only if (1) $u \sim_\Gamma u'$ or (2) $u = u'$ and $v \sim_\Delta v'$.

Proposition 2.1 *Let Γ be an adjacency-transitive graph of order at least 2 and let Δ be a vertex-transitive graph. Then the lexicographic product $\Gamma[\Delta]$ is adjacency-transitive.*

Proof. For every adjacency automorphism $\sigma \in \operatorname{Aut}\Gamma$ and every automorphism $\rho \in \operatorname{Aut}\Delta$ we define the following mapping $\tau_{\sigma,\rho}$ on $V(\Gamma[\Delta])$:

$$\tau_{\sigma,\rho}(u, v) = \begin{cases} (\sigma(u), \rho(v)), & \text{if } \sigma(u) \neq u; \\ (u, v), & \text{if } \sigma(u) = u. \end{cases}$$

Then it is a straightforward exercise to verify that $\tau_{\sigma,\rho}$ is an adjacency automorphism of $\Gamma[\Delta]$.

Given two arbitrary vertices $a = (u, v)$ and $b = (u', v')$ of $\Gamma[\Delta]$, let $\rho \in \operatorname{Aut}\Delta$ send v to v'. If $u \neq u'$, let $\sigma_1, \ldots, \sigma_k \in \operatorname{Aut}\Gamma$ be a sequence of adjacency automorphisms such that $(\sigma_1 \cdots \sigma_k)(u) = u'$. We may assume that $\sigma_k(u) \neq u$. Then the following is a sequence of adjacency automorphisms of $\Gamma[\Delta]$,

$$\tau_{\sigma_1,\mathrm{id}}\,, \ldots, \tau_{\sigma_{k-1},\mathrm{id}}\,, \tau_{\sigma_k,\rho},$$

and its product sends a to b. If $u = u'$, let σ be an adjacency automorphism of Γ not fixing u. Then $\tau_{\sigma^{-1},\mathrm{id}}\,\tau_{\sigma,\rho}$ is a product of adjacency automorphisms of $\Gamma[\Delta]$ sending a to b. Hence $\Gamma[\Delta]$ is adjacency-transitive. ■

Corollary 2.2 *The complement of a disconnected vertex-transitive graph is adjacency-transitive.*

Proof. If Γ is a disconnected vertex-transitive graph, then it is the disjoint union of, say, $n > 2$ isomorphic (connected) vertex-transitive graphs Δ. The complement Γ^c is then isomorphic with the lexicographic product $K_n[\Delta^c]$, where K_n denotes the complete graph on n vertices. As K_n is adjacency-transitive, the result follows from Proposition 2.1. ■

Let G be a finite group and R a subset of G not containing the identity and satisfying $R^{-1} = R$. Then the *Cayley graph* $\Gamma = \mathrm{Cay}(G, R)$ has vertex set G, and for every $g \in G$ the set of neighbours of g in Γ is gR. The following lemma is a slight generalization of Proposition 2.1 in [3].

Lemma 2.3 *Let $U \subseteq S \subseteq G$, where U is a union of conjugacy classes that generates the finite group G, $S^{-1} = S$ and $1 \notin S$. Then the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is adjacency-transitive.*

Proof. For every $a \in G$ denote by σ_a the operation of left multiplication by a on G. Then σ_a is an automorphism of Γ. If $a \in U$, then σ_a is an adjacency automorphism of Γ: for every $g \in G$ we have

$$\sigma_a(g) = ag = g(g^{-1}ag)$$

and $g^{-1}ag$ belongs to S, since U is closed under conjugacy. Since U generates G, it follows that Γ is adjacency-transitive. ∎

A *graphical regular representation* (or in short, a *GRR*) of a finite group G is a graph Γ whose automorphism group acts regularly on its vertices and is isomorphic with G.

Proposition 2.4 *Let G be a finite group and $\Gamma = \mathrm{Cay}(G, S)$ be a GRR of G, where $S \subseteq G$, $S = S^{-1}$ and $1 \notin S$. Then Γ is adjacency-transitive if and only if S contains a union of conjugacy classes that generates G.*

Proof. Since $\mathrm{Aut}\,\Gamma \simeq G$ we have $\mathrm{Aut}\,\Gamma = \{\sigma_a \mid a \in G\}$. If σ_a is a nontrivial adjacency automorphism of Γ, then for every $g \in G$ there exists an $s_g \in S$ satisfying $ag = gs_g$, so $g^{-1}ag = s_g$. Thus the conjugacy class of a is contained in S. So if Γ is adjacency-transitive, then S contains a union of conjugacy classes that generates G.

The opposite assertion holds by Lemma 2.3. ∎

Proposition 2.5 *Let $D_{2n} = \langle a, b \mid a^n = b^2 = (ba)^2 = 1 \rangle$ be the dihedral group of order $2n$, where $n \geq 7$. Then the Cayley graph*

$$\Gamma_n = \mathrm{Cay}(D_{2n}, \{a, a^{-1}, b, ab, a^3b\})$$

is not adjacency-transitive, and neither is its complement $(\Gamma_n)^c$.

Proof. First we show that Γ_n is a GRR of D_{2n}. It suffices to see that the stabilizer of the vertex $1 \in V(\Gamma_n)$ is trivial. Observe that the edges of Γ_n which are labelled by $\{a, a^{-1}\}$ belong to exactly one triangle in Γ_n, whereas those labelled by $\{b\}$ or $\{ab\}$ belong to two, and those labelled by $\{a^3b\}$

to none. Further, a^3b has four neighbours in common with a but only two neighbours in common with a^{-1}, and each of b and ab is adjacent to just one of a and a^{-1}. It follows that any automorphism $\sigma \in \operatorname{Aut}\Gamma_n$ fixing the vertex 1 fixes each of its five neighbours a^3b, a, a^{-1}, b and ab, and by connectedness and vertex-transitivity of Γ_n, is therefore trivial.

The conjugacy class of a^kb in D_{2n} is $\{a^{2i+k}b \mid i \in \mathbb{Z}\}$. Hence the set $S = \{a, a^{-1}, b, ab, a^3b\}$ contains no union of conjugacy classes that generates D_{2n}, and the same holds for the set $D_{2n} \setminus (S \cup \{1\})$. By Proposition 2.4, neither Γ_n nor $(\Gamma_n)^c$ is adjacency-transitive. ■

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