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THE GRAY GRAPH REVISITED

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Abstract

Certain graph-theoretic properties and alternative definitions of the Gray graph, the smallest known cubic edge- but not vertex-transitive graph, are discussed in detail.

1 Introduction, history

The smallest known cubic edge- but not vertex-transitive graph has 54 vertices and is known as the *Gray graph*, denoted hereafter by $\mathcal{G}$. The first published account on the Gray graph is due to Bouwer [1] who mentioned that this graph had in fact been discovered by Marion C. Gray in 1932, thus explaining its name. Bouwer [1] gives two ways of constructing $\mathcal{G}$. First, three copies of the complete bipartite graph $K_{3,3}$ are taken, and to a particular edge e of $K_{3,3}$ a vertex is inserted in the interior of e in each of the three copies of $K_{3,3}$, and the resulting three vertices are then joined to a new vertex. The second construction identifies a particular Hamilton cycle in $\mathcal{G}$ and the corresponding 27 chords (see Figure 1).

Some other ways of constructing $\mathcal{G}$, are presented in this note, thus shedding a new light on the structure of this remarkable graph. In the computations involved a usage of the Vega package [5] was essential.

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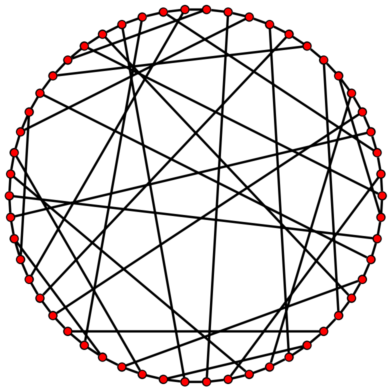

Figure 1: The Gray Graph with an identified Hamilton cycle as in [1].

2 Structural properties and alternative definitions

The Gray graph $\mathcal{G}$ is a cubic, bipartite, and edge- but not vertex-transitive graph. Its automorphism group $\mathrm{Aut}\,\mathcal{G}$ acts transitively on each of the bipartition sets. Since its girth is 8, it is the Levi graph of two dual, triangle-free, point-, line- and flag-transitive, non-self-dual 27_3-configurations.

There is a simple reason for intransitivity of $\mathrm{Aut}\,\mathcal{G}$ on the vertex set of $\mathcal{G}$. Two vertices have the same distance sequence if and only if they belong to the same bipartition set. More precisely, the two distance sequences are $(1,3,6,12,12,12,8)$ and $(1,3,6,12,16,12,4)$ and the respective vertices are henceforth called *black* and *white*. It follows that the diameter of $\mathcal{G}$ is 6. Note that $|\mathrm{Aut}\,\mathcal{G}| = 1296 = 2^4 3^4$. Let $S(3)$ denote an arbitrary Sylow 3-subgroup of $\mathrm{Aut}\,\mathcal{G}$. It may be seen that $S(3) \cong \mathbb{Z}_3^3 \rtimes \mathbb{Z}_3$ and that $S(3)$ acts transitively on the edge set of $\mathcal{G}$ as well as on the sets of black and white vertices.

We note further that there are a total of 81 octagons, that is induced cycles of length 8, in $\mathcal{G}$. Octagons in $\mathcal{G}$ play an essential role in our Construction 2.4 below. Let us also mention that each vertex of $\mathcal{G}$ is contained in 12 octagons and each edge of $\mathcal{G}$ is contained in 8 octagons.

Construction 2.1 This construction was pointed to us by Randic in a personal communication (but see also [4]) and gives $\mathcal{G}$, in the LCF notation [2], as the graph with code $[7, -7, 13, -13, 25, -25]^9$, thus identyfing a Hamilton cycle which admits a $\mathbb{Z}_9$-symmetry (see Figure 2).

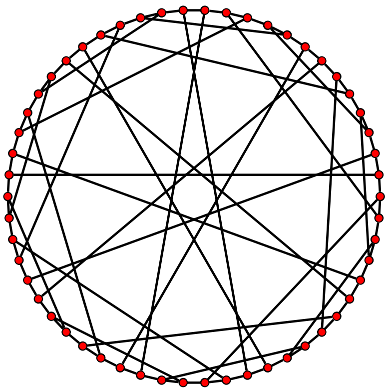

Figure 2: The Gray graph with an identified Hamilton cycle admitting a $\mathbb{Z}_9$.

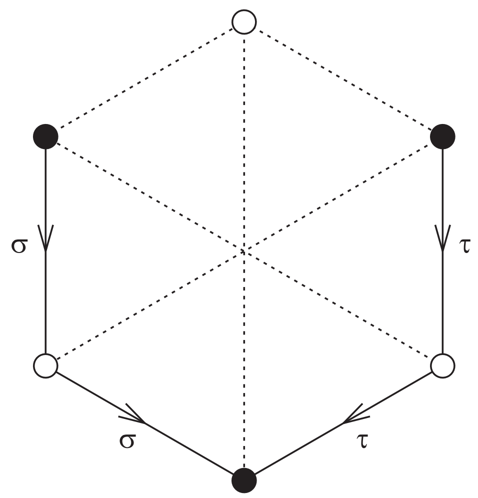

Figure 3: The Gray graph is a $\mathbb{Z}_3^2$-regular cover of $K_{3,3}$. The broken lines carry identity voltages $(0, 0)$, whereas $\sigma = (1, 0), \tau = (0, 1)$. Black and white vertices of $K_{3,3}$ lift to the vertices with distance sequences $(1, 3, 6, 12, 12, 12, 8)$ and $(1, 3, 6, 12, 16, 12, 4)$ respectively.

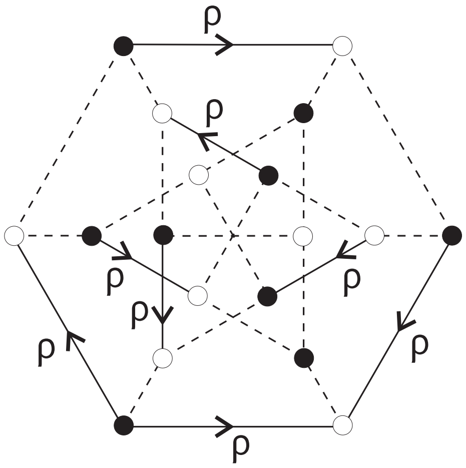

Figure 4: The Gray graph is a $\mathbb{Z}_3$-regular cover of the Pappus graph $\mathcal{P}$. The broken lines carry identity voltages 0, whereas $\rho = 1$. Black and white vertices of $\mathcal{P}$ lift to the vertices with distance sequences $(1, 3, 6, 12, 12, 12, 8)$ and $(1, 3, 6, 12, 16, 12, 4)$ respectively.

Construction 2.2 Following the construction using three copies of $K_{3,3}$ from the previous section it may be deduced that $\mathcal{G}$ is a regular cover of $K_{3,3}$ with $\mathbb{Z}_3^2$ as the group of covering transformations (see [3] for notation and terminology). More precisely, letting $\sigma = (1, 0)$ and $\tau = (0, 1)$ be the two generators of $\mathbb{Z}_3^2$, then Figure 3 gives $\mathcal{G}$ as a $\mathbb{Z}_3^2$-regular cover of $K_{3,3}$.

We note that by "averaging" the voltages, that is by replacing each σ and τ by $\sigma + \tau = (1, 1)$ and by selecting instead of the group $\langle \sigma, \tau \rangle \cong \mathbb{Z}_3^2$ its subgroup $\langle \sigma + \tau \rangle \cong \mathbb{Z}_3$ as a voltage group, the resulting voltage graph $K_{3,3}$ lifts to a 3-fold cover $\mathcal{P}$ on 18 vertices which is isomorphic to the Levi graph of the well-known Pappus configuration 9_3. Note that $\mathcal{P}$ is a 3-arc-transitive bipartite graph, and is the underlying graph of the voltage graph depicted in Figure 4. Furthermore, the voltage graph in Figure 4 lifts to the Gray graph $\mathcal{G}$. Let us also mention that the normalizer in $\operatorname{Aut} \mathcal{G}$ of the corresponding two groups of covering transformations $\mathbb{Z}_3^2$ and $\mathbb{Z}_3$ is a subgroup of order $324 = 2^2 3^4$.

Construction 2.3 Another interesting construction identifies $\mathcal{G}$ as the anti-line graph of a certain Cayley graph of the Sylow 3-subgroup $S(3)$ of $\operatorname{Aut} \mathcal{G}$. Each element of $S(3)$ can be represented as a quadruple (i, j, k, l), where $i, j, k, l \in \mathbb{Z}_3$ and the multiplication obeys the following rules.

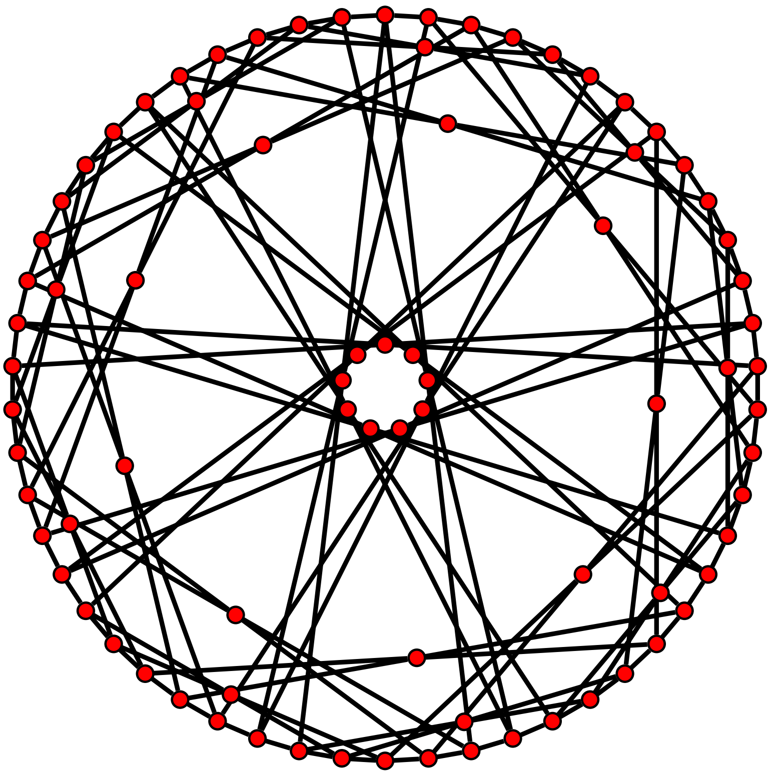

Figure 5: The line graph $L(\mathcal{G})$ is a Cayley graph for the Sylow 3-subgroup $S(3) = \mathbb{Z}_3^3 \rtimes \mathbb{Z}_3 \leq \mathrm{Aut}\,\mathcal{G}$. Note that $S(3)$ acts transitively on the edges and the two bipartition sets of $\mathcal{G}$.

$$(i,j,k,0)(r,s,t,w) = (i+r, j+s, k+t, w)$$

$$(i,j,k,1)(r,s,t,w) = (i+t, j+r, k+s, w+1)$$

$$(i,j,k,2)(r,s,t,w) = (i+s, j+t, k+r, w+2)$$

Note that the normal subgroup in $S(3)$ isomorphic to $\mathbb{Z}_3^3$ consists of all the elements $(i,j,k,0) \in S(3), i,j,k \in \mathbb{Z}_3$. Let $a = (1,0,0,0)$ and $b = (0,0,0,1)$. Then the Cayley graph $Cay(S(3), \{a, a^{-1}, b, b^{-1}\})$ of $S(3)$ with respect to the set of generators $\{a, a^{-1}, b, b^{-1}\}$, is the line graph $L(\mathcal{G})$ of the Gray graph (see Figure 5).

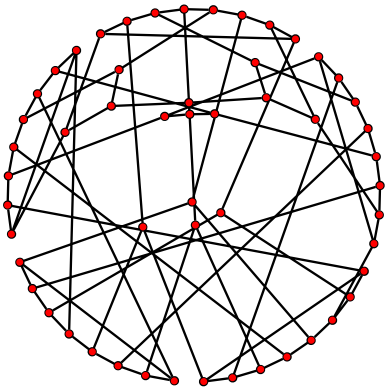

Figure 6: The Gray graph and five disjoint octagons.

Construction 2.4 Finally we want to discuss by far the most interesting rule for constructing the Gray graph which shows that $\mathcal{G}$ possesses a surprising feature of containing "itself within itself". In what follows we shall first discuss some graph-theoretic distinction between the two kinds of vertices and then gradually build our way towards a construction of $\mathcal{G}$ identifying the *Gray graph within the Gray graph* feature.

Now let x be an arbitrary vertex in $\mathcal{G}$ and let $\mathcal{G}(x; i_1, \ldots, i_r)$, where $1 \leq i_1 \leq \ldots \leq i_r \leq 6$, denote the subgraph of $\mathcal{G}$ induced by all the vertices at distance $i_1, \ldots, i_r$ from x. Of course, since the vertex orbits of $\mathrm{Aut}\,\mathcal{G}$ coincide with the color classes, the graph $\mathcal{G}(x; i_1, \ldots, i_r)$ may only depend on the color of x.

An unordered triple of octagons may be associated with an arbitrary vertex in $\mathcal{G}$ in the following way. It transpires that for a black vertex b the graph $\mathcal{G}(b; 3, 4)$ is isomorphic to a union of three octagons $3C_8$; these octagons give

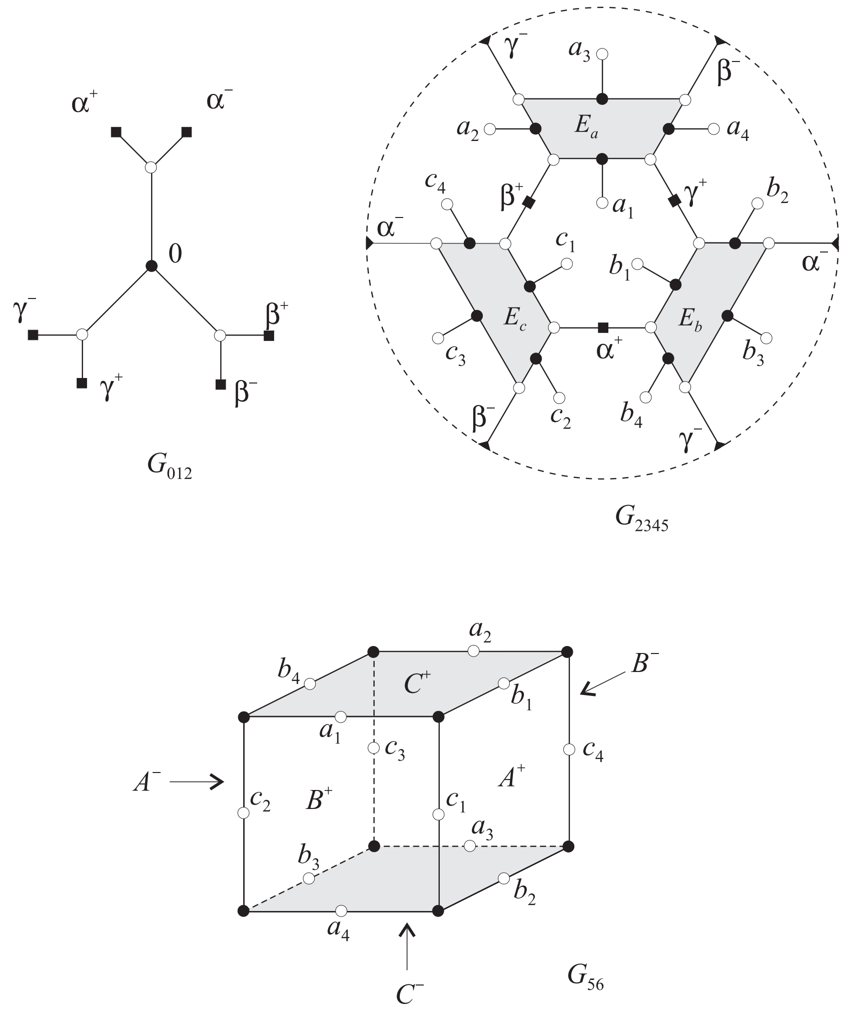

Figure 7: The Gray graph as a union of G_{012}, G_{2345} and G_{56}.

rise to the above triple. Similarly, for a white vertex w the graph $\mathcal{G}(w;4,5)$ is isomorphic to a union of three disjoint octagons and four isolated vertices $3C_8+4K_1$. Again, the three octagons give rise to the above mentioned triple. Further, it may be seen that the black and white triples have at most one octagon in common, which happens if and only if the corresponding two vertices are neighbors in $\mathcal{G}$. It thus follows that the graph whose vertex set consists of all the white and black triples of octagons, with the adjacency meaning that two triples have non-empty intersection, is isomorphic to the Gray graph. Consequently, the graph whose vertex set consists of all octagons in $\mathcal{G}$, with two octagons adjacent if and only if they both belong to one of the above triples, is isomorphic to the line graph $L(\mathcal{G})$ of the Gray graph. Figure 6 identifies five disjoint octagons, the upper three octagons define a black triple, whereas the top octagon together with the two bottom octagons define a white triple.

Based on the above discussion the rule for constructing $\mathcal{G}$ may be given via three auxiliary graphs $G_{012} = \mathcal{G}(b; 0, 1, 2), G_{2345} = \mathcal{G}(b; 2, 3, 4, 5), G_{56} = \mathcal{G}(b; 5, 6)$ shown in Figure 7. The root black vertex in G_{012} is labeled 0. The six vertices $\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+$, and γ^- at distance 2 from 0 are glued to the corresponding vertices of G_{2345}, which is depicted in the projective plane. The twelve vertices $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4$ at distance 5 (from 0) are identified with the corresponding twelve midpoints of the subdivided cube G_{56}. Three disjoint grey octagons E_a, E_b, E_c are visible on the projective plane. The faces of the subdivided cube give rise to three opposite disjoint octagonal pairs: $A^+, A^-; B^+, B^-; C^+, C^-$.

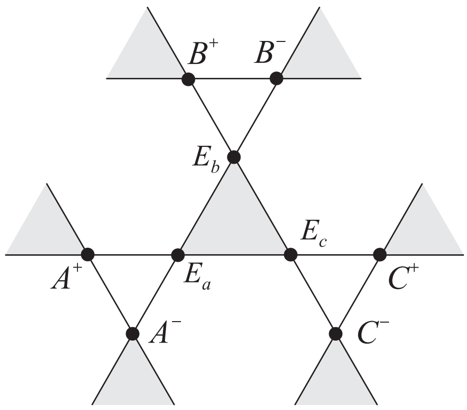

Figure 8: The local situation in the octagon graph isomorphic to $L(\mathcal{G})$.

Figure 8 depicts the local situation in the above mentioned graph of octagons, isomorphic to $L(\mathcal{G})$, where as seen from the graph G_{56} in Figure 7 the adjacency corresponds to two octagons being at distance 2 in $\mathcal{G}$. For instance, the octagon E_a is at distance 2 from E_b and E_c as well as from octagons C^+ and C^-; see also the five shaded octagons in Figure 7.

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