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**Preprint series, Vol. 36 (1998), 604**

FLEXIBILITY OF POLYHEDRAL  
EMBEDDINGS OF GRAPHS IN  
SURFACES

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ISSN 1318-4865

May 12, 1998

Ljubljana, May 12, 1998

## FLEXIBILITY OF POLYHEDRAL EMBEDDINGS OF GRAPHS IN SURFACES

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ABSTRACT. Whitney's theorem states that 3-connected planar graphs admit essentially unique embeddings in the plane. We generalize this result to embeddings of graphs in arbitrary surfaces by showing that there is a function  $\xi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that every 3-connected graph admits at most  $\xi(g)$  combinatorially distinct embeddings of face-width  $\geq 3$  into surfaces whose Euler genus is at most  $g$ .

### 1. INTRODUCTION

Whitney proved [14] that every 3-connected planar graph has an essentially unique embedding in the plane. This means that face boundaries and local rotations are uniquely determined. This result was obtained as a corollary of a stronger statement that any two embeddings of a 2-connected planar graph are Whitney equivalent, i.e., one can be obtained from the other by a sequence of simple local re-embeddings. (See, e.g., [9] for more details on Whitney equivalence.) Robertson and Vitray [12] extended that result to an arbitrary surface of genus  $g$  by assuming that the face-width of the embedding is at least  $2g + 3$ . Seymour and Thomas [13] and Mohar [8] improved the bound on the face-width to  $O(\log g / \log \log g)$ . Archdeacon [1] proved that an assumption on large face-width is necessary by showing that, for each integer  $k$ , there are graphs which admit distinct embeddings of face-width at least  $k$ . On the other hand, it has been noted in [5] that the finiteness of the number of irreducible triangulations for each fixed surface  $S$  implies that there is a bound  $b = b(S)$  such that every graph admits at most  $b$  triangular embeddings in  $S$ .

In this paper we show that for each surface  $S$ , there is a constant  $\xi = \xi(S)$  such that every 3-connected graph admits at most  $\xi$  embeddings of face-width  $\geq 3$  in  $S$ . The assumption on 3-connectivity is clearly necessary for such a result, and the following example shows that also the bound on the face-width cannot be weakened.

Let  $H_0$  be a 4-connected plane graph whose outer face is a 4-cycle  $v_1v_2v_3v_4$ . For  $n \geq 3$ , let  $G_n$  be the graph obtained by taking  $n$  copies  $H_1, \dots, H_n$  of  $H_0$  and, for  $i = 1, \dots, n$ , identifying the edge  $v_1v_2$  of  $H_i$  with the edge  $v_4v_3$  of  $G_{i+1}$  (indices modulo  $n$ ). The graph  $G_n$  is 4-connected and planar and has  $2^{n-1} - 1$  embeddings

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Received by the editors May 8, 1998.

1991 *Mathematics Subject Classification*. Primary 05C10.

*Key words and phrases*. Graph theory, surface, polyhedral embedding, face-width.

Supported in part by the Ministry of Science and Technology of Slovenia, Research Project P1-0210-101-97 and the SLO-US 0008 project on Minors and graphs on surfaces.

Research partially supported by NSF grant DMS 9701317.

of face-width 2 in the torus obtained by “flipping” one or more copies  $H_i$  “up or down” as shown by an example in Figure 1.

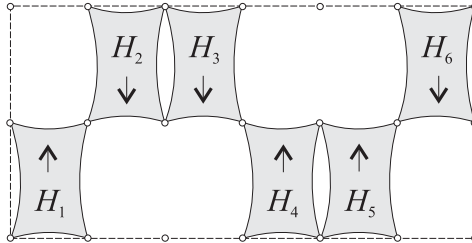


FIGURE 1. An embedding of  $G_6$  in the torus

This example can be easily transformed into a similar one where the graph  $G_n$  is nonplanar.

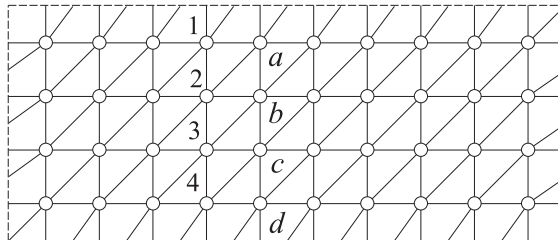


FIGURE 2. A 6-connected triangulation of the torus

The following example shows that also increasing connectivity to 6 does not help to get bounded flexibility. Let us observe that increasing connectivity to 7 or more does not make sense since for each surface  $S$ , there is only a finite number of 7-connected graphs that can be embedded in  $S$ . Let  $T_n$  be the 6-connected triangulation of the torus represented in Figure 2. If we replace the 8 triangular faces between the 4-cycles 1234 and  $abcd$  with the following four facial cycles:  $12b34d$ ,  $23c41a$ ,  $ab3cd1$ , and  $bc4da2$ , we get an embedding of face-width 2 in the orientable surface  $S_3$  of genus 3. Since such a change can be performed between any two consecutive vertical 4-cycles, this example gives rise to 6-connected graphs which admit arbitrarily many embeddings of face-width 2 in  $S_3$ .

## 2. PRELIMINARIES

All graphs in this paper are undirected, finite and simple. We follow standard terminology as used, for example, in [2]. A subgraph  $C$  of a graph  $G$  is *induced* if every pair of non-adjacent vertices in  $C$  is also non-adjacent in  $G$ . It is *non-separating* if  $G - V(C)$  is connected.

Let  $H$  be a subgraph of  $G$ . An *H-bridge* in  $G$  is a subgraph of  $G$  which is either an edge not in  $H$  but with both ends in  $H$ , or a connected component of  $G - V(H)$  together with all edges which have one end in this component and the other end in  $H$ . Let  $B$  be an  $H$ -bridge. The vertices of  $B \cap H$  are *vertices of attachment* of  $B$ , and each edge of  $B$  incident with a vertex of attachment is a *foot* of  $B$ .

Our treatment of graph embeddings follows essentially [9]. An *embedding* of a connected graph  $G$  is a pair  $\Pi = (\pi, \lambda)$  where  $\pi = \{\pi_v \mid v \in V(G)\}$  is a collection of *local rotations*, i.e.,  $\pi_v$  is a cyclic permutation of the edges incident with  $v$  ( $v \in V(G)$ ), and  $\lambda : E(G) \rightarrow \{+1, -1\}$  is a *signature*. The local rotation  $\pi_v$  describes the cyclic clockwise order of edges incident with  $v$  on the surface, and the signature  $\lambda(uv)$  of the edge  $uv$  is positive if and only if the local rotations  $\pi_u$  and  $\pi_v$  both correspond to the clockwise (or both to anticlockwise) rotations when traversing the edge  $uv$  on the surface. If we consider the graph  $G$  together with its embedding  $\Pi$ , we say that  $G$  is  $\Pi$ -*embedded*. The embedding  $\Pi$  determines a set of  $\Pi$ -*facial walks*. Facial walks are closed and are not distinguished if they differ only by choice of the initial vertex or by reversal of order of traversal. Each edge is either contained in two  $\Pi$ -facial walks or it appears twice in the same facial walk. If a  $\Pi$ -facial walk is a cycle, it is also called a  $\Pi$ -*facial cycle*. Two embeddings of  $G$  are *equivalent* if they have the same set of facial walks. A (contiguous, possibly closed) subwalk of a facial walk is called a *facial segment*.

The surface  $S$  of an embedding  $\Pi$  is given by attaching open discs (the faces) to the graph along the  $\Pi$ -facial walks.

The *Euler genus* of  $\Pi$  (or the  $\Pi$ -*genus* of  $G$ ) is the integer  $\varepsilon = \varepsilon(G, \Pi)$  defined by Euler's formula,  $\varepsilon = 2 - |V(G)| + |E(G)| - f$  where  $f$  denotes the number of  $\Pi$ -facial walks of  $G$ . The *Euler genus*  $\varepsilon(G)$  of the graph  $G$  is the minimum of Euler genera over all embeddings of  $G$ .

If  $G$  is a  $\Pi$ -embedded graph and  $H$  is a connected subgraph of  $G$ , then  $\Pi$  *induces* an embedding of  $H$  which is also denoted by  $\Pi$  and called the  $\Pi$ -*embedding* of  $H$ . Note that  $\varepsilon(H, \Pi) \leq \varepsilon(G, \Pi)$  and that strict inequality may occur.

Let  $G$  be a  $\Pi$ -embedded graph, and let  $C = x_1 \dots x_k$  be a  $k$ -cycle of  $G$ .  $C$  is said to be  $\Pi$ -*onesided* if the number of edges of  $C$  with negative signature is odd. Otherwise,  $C$  is  $\Pi$ -*twosided*. By passing to an equivalent embedding (for example, by reversing the clockwise local rotations to anticlockwise at some of the vertices  $x_i$  and changing the signature of all edges incident with  $x_i$ ), we may assume that all edges of  $C$  except possibly  $x_1 x_k$  have positive signature. Let  $e = x_i x_{i+1}$  ( $1 \leq i \leq k$ ) be an edge of  $C$ , and let  $d = \deg_G(x_i)$ . Then there is an  $l$  such that  $1 \leq l < d$  and  $\pi_{x_i}^l(e) \in E(C)$ . We say that the edges  $\pi_{x_i}(e), \dots, \pi_{x_i}^{l-1}(e)$  are incident with  $C$  on its *right side* and the edges  $\pi_{x_i}^{l+1}(e), \dots, \pi_{x_i}^{d-1}(e)$  are incident with  $C$  on its *left side*. By *cutting  $G$  along  $C$* , a new  $\Pi'$ -embedded graph  $G'$  is obtained from  $G$  as follows: If  $C$  is  $\Pi$ -twosided, we delete  $C$  and add instead two disjoint cycles  $C' = x'_1 \dots x'_k$  and  $C'' = x''_1 \dots x''_k$ . If  $C$  is  $\Pi$ -onesided, we replace  $C$  by a single  $2k$ -cycle  $C' = x'_1 \dots x'_k x''_1 \dots x''_k$ . In each case, the vertex  $x'_i$  (resp.,  $x''_i$ ) is adjacent to a vertex  $y \in V(G) \setminus V(C)$  if and only if  $x_i y \in E(G)$  and the edge  $x_i y$  is incident with  $C$  on its left side (resp., right side). We let  $\Pi'$  be the same as  $\Pi$  except that  $G'$  has no edges incident with  $C'$  (resp.,  $C''$ ) on its right side (resp., left side),  $i = 1, \dots, k$ . We say that the cycles  $C'$  and  $C''$  *correspond* to  $C$ . Clearly,  $C'$  and  $C''$  are  $\Pi'$ -facial cycles of  $G'$ . Let  $W'$  be a  $\Pi'$ -facial walk of  $G'$  different from  $C'$  and  $C''$ . If  $W'$  contains no point in  $V(C') \cup V(C'')$ , then  $W'$  is a  $\Pi$ -facial walk of  $G$ . Otherwise, the walk  $W$  obtained from  $W'$  by replacing  $x'_i$  and  $x''_i$  by  $x_i$  is a  $\Pi$ -facial walk of  $G$ . Conversely, if  $W$  is a  $\Pi$ -facial walk of  $G$ , then there is a  $\Pi'$ -facial walk  $W'$  of  $G'$  such that  $W$  is obtained from  $W'$  by replacing  $x'_i$  and  $x''_i$  by  $x_i$ . If  $G'$  is connected, then  $\varepsilon(G', \Pi) < \varepsilon(G, \Pi)$ . Otherwise,  $G' = G'_1 \cup G'_2$  where  $G'_1 \cap G'_2 = \emptyset$  and  $C' \subseteq G'_1, C'' \subseteq G'_2$ . In this case,  $\varepsilon(G, \Pi) = \varepsilon(G'_1, \Pi') + \varepsilon(G'_2, \Pi')$ . If

$\min\{\varepsilon(G'_1, \Pi'), \varepsilon(G'_2, \Pi')\} = 0$ , then  $C$  is said to be  $\Pi$ -contractible. If  $\varepsilon(G'_1, \Pi') = 0$ , then we write  $G'_1 = \text{Int}(C, \Pi)$  and  $G'_2 = \text{Ext}(C, \Pi)$ .

Disjoint cycles  $C, C'$  of  $G$  are (*freely*)  $\Pi$ -homotopic if either  $C$  and  $C'$  are both  $\Pi$ -contractible, or  $C$  and  $C'$  are  $\Pi$ -twosided and cutting along  $C$  and  $C'$  results in a graph which has a component  $D$  which contains precisely one copy of  $C$  and one copy of  $C'$  and whose  $\Pi$ -genus is zero. In the latter case we write  $D = \text{Int}(C, C', \Pi)$ , and we denote by  $\text{Ext}(C, C', \Pi)$  the other component(s) containing copies of  $C$  and  $C'$ .

If  $a, b$  are distinct vertices of  $G$  and  $P_1, P_2$  are internally disjoint paths from  $a$  to  $b$  in  $G$ , then  $P_1$  and  $P_2$  are said to be  $\Pi$ -homotopic if the cycle  $C = P_1 \cup P_2$  is  $\Pi$ -contractible. This definition extends to the case when  $P_1, P_2$  are cycles with the common vertex  $a = b$  or even cycles with a common edge (cf. [9]).

**Lemma 2.1.** *Let  $G$  be a  $\Pi$ -embedded graph and let  $\mathcal{C}$  be a set of noncontractible  $\Pi$ -homotopic cycles. Suppose that there is a path  $P$  (possibly  $P = \emptyset$ ) such that  $C \cap C' = P$  for any distinct cycles  $C, C' \in \mathcal{C}$ . Then the cycles in  $\mathcal{C}$  can be enumerated,  $\mathcal{C} = \{C_1, \dots, C_r\}$ , such that for each  $i$  and  $j$  ( $1 \leq i < j \leq r$ ),  $\text{Int}(C_i, C_j, \Pi) = \cup_{t=i}^{j-1} \text{Int}(C_t, C_{t+1}, \Pi)$ .*

*Proof.* We assume that  $P \neq \emptyset$ ; the case of pairwise disjoint cycles has similar proof and we leave the details to the reader. By contracting  $P$  to a point, we may assume that  $P = \{v\}$  is just a vertex.

The proof is by induction on  $r = |\mathcal{C}|$ . There is nothing to prove if  $r \leq 2$ , so assume  $r \geq 3$ . By removing an arbitrary cycle  $C \in \mathcal{C}$ , the remaining cycles can be enumerated, by the induction hypothesis, as  $C'_1, \dots, C'_{r-1}$  to satisfy the conclusion of the lemma. If  $C \subseteq \text{Int}(C'_t, C'_{t+1}, \Pi)$  for some  $t$ ,  $1 \leq t < r-1$ , then we insert  $C$  between  $C'_t$  and  $C'_{t+1}$  in the ordering for  $\mathcal{C}$ , and use the fact that  $\text{Int}(C'_t, C'_{t+1}, \Pi) = \text{Int}(C'_t, C, \Pi) \cup \text{Int}(C, C'_{t+1}, \Pi)$  to complete the proof. Otherwise,  $C \subseteq \text{Ext}(C'_1, C'_{r-1}, \Pi)$ . If  $\text{Int}(C, C'_{r-1}, \Pi)$  does not contain  $C'_1$ , then  $\text{Int}(C'_1, C, \Pi) = \text{Int}(C'_1, C'_{r-1}, \Pi) \cup \text{Int}(C'_{r-1}, C, \Pi)$  and we set  $C_t = C'_t$  for  $1 \leq t < r$ , and  $C_r = C$ . If  $C'_1 \subseteq \text{Int}(C, C'_{r-1}, \Pi)$ , then  $\text{Int}(C'_1, C'_{r-1}, \Pi) \subseteq \text{Int}(C, C'_{r-1}, \Pi)$ . Now, we set  $C_1 = C$  and  $C_t = C'_{t-1}$  for  $1 < t \leq r$ .  $\square$

We will refer to the natural ordering of  $\mathcal{C}$  as in Lemma 2.1 or in Corollary 2.2 below as the *linear nesting* of homotopic cycles.

**Corollary 2.2.** *Let  $G$  be a  $\Pi$ -embedded graph and let  $\mathcal{C}$  be a set of noncontractible  $\Pi$ -homotopic cycles such that the intersection of any two of them is either empty or a path. If the cycles in  $\mathcal{C}$  are  $\Pi$ -twosided, then they can be enumerated,  $\mathcal{C} = \{C_1, \dots, C_r\}$ , such that for each  $i$  and  $j$  ( $1 \leq i < j \leq r$ ),  $\text{Int}(C_i, C_j, \Pi) = \cup_{t=i}^{j-1} \text{Int}(C_t, C_{t+1}, \Pi)$ .*

*Proof.* Since the cycles in  $\mathcal{C}$  are  $\Pi$ -twosided, they can be separated by splitting vertices of their intersection. It is easy to see that by appropriate splitting of vertices, we can get a graph  $H$  that is  $\Pi'$ -embedded in the same surface and such that  $\mathcal{C}$  gives rise to a set of pairwise disjoint homotopic cycles. Now we apply Lemma 2.1 and observe that the  $\Pi$ -interior is obtained from the  $\Pi'$ -interior by contractions of edges in  $E(H) \setminus E(G)$ .  $\square$

We will make use of the following lemma which is a special case of the main theorem in [6].

**Lemma 2.3** (Juvan, Malnič, Mohar [6]). *For each Euler genus  $\varepsilon$  there is a positive integer  $c_1 = c_1(\varepsilon)$  such that for every  $\Pi$ -embedded graph  $G$  where  $\varepsilon(G, \Pi) = \varepsilon$ , and for every family of  $r$  cycles (paths) of  $G$  which pairwise intersect in at most two common segments, there is a subset of at least  $\lceil r/c_1 \rceil$  pairwise  $\Pi$ -homotopic cycles (paths).*

In the proof of Theorem 5.2 we shall use the following lemma of Fisk and Mohar [4]. We add its proof since in [4] it is formulated only for paths while here we allow walks.

**Lemma 2.4.** *Let  $k \geq 1$  and  $r \geq 1$  be integers. There exists an integer  $\varphi(k, r)$  such that the following holds: If a multigraph  $H$  contains  $\varphi(k, r)$  walks of length at most  $k$  joining vertices  $v_1$  and  $v_2$ , such that the edges of the walks incident with  $v_1$  are all distinct, then there is a vertex  $v \neq v_1$  of  $H$  such that  $v_1$  and  $v$  are joined by  $r$  internally disjoint subwalks of the given walks.*

*Proof.* It suffices to give the proof for simple graphs since  $H$  has a subdivision which is simple and such that each walk has length at most  $3k$ .

For simple graphs we prove the lemma by induction on  $k + r$  and with  $\varphi(k, r) = r^{k-1}(k-1)!$ . For  $r = 1$ , there is nothing to prove. For  $k \leq 2$ , all the walks are disjoint paths, so we proceed to the induction step. Now, let  $H$  be a graph that has  $\varphi(k+1, r)$  walks from  $v_1$  to  $v_2$  of length at most  $k+1$  and with distinct initial edges. Pick a walk  $P$ . If there are at least  $k\varphi(k, r)$  walks intersecting  $P - \{v_1, v_2\}$ , then some  $\varphi(k, r)$  of these walks meet the same vertex of  $P - \{v_1, v_2\}$ , and so we obtain the desired walks by induction. Otherwise, let  $P_1, \dots, P_q$  be a maximal collection of internally disjoint walks from  $v_1$  to  $v_2$  (taken from the  $\varphi(k+1, r)$  walks). As each of the  $\varphi(k+1, r)$  walks (which is not the edge  $v_1v_2$ ) has an intermediate vertex in  $P_1 \cup \dots \cup P_q$ , we have  $\varphi(k+1, r) \leq qk\varphi(k, r)$ , and hence  $q \geq r$ .  $\square$

### 3. POLYHEDRAL EMBEDDINGS

Let  $G$  be a  $\Pi$ -embedded graph. If  $\varepsilon(G, \Pi) \geq 1$ , the *face-width* of  $\Pi$  (also called the *representativity*),  $\text{fw}(G, \Pi)$ , is the smallest integer  $r$  such that  $G$  has a  $\Pi$ -noncontractible cycle which is the union of  $r$  segments, each of which is contained in a  $\Pi$ -facial walk. If  $\varepsilon(G, \Pi) = 0$ , we let  $\text{fw}(G, \Pi) = \infty$ .

Let  $C_1$  and  $C_2$  be distinct  $\Pi$ -facial walks. We say that  $C_1$  and  $C_2$  *meet properly* if the intersection of  $C_1$  and  $C_2$  is either empty, a single vertex or an edge.  $\Pi$  is said to be a *polyhedral embedding* if every  $\Pi$ -facial walk is a cycle and any two  $\Pi$ -facial walks meet properly. The following results are due to Robertson and Vitray [12].

**Proposition 3.1.** *Let  $G$  be a connected  $\Pi$ -embedded graph. Then  $\Pi$  is a polyhedral embedding if and only if  $\text{fw}(G, \Pi) \geq 3$  and  $G$  is 3-connected.*

**Proposition 3.2.** *Let  $G$  be a 3-connected  $\Pi$ -embedded graph. If  $\text{fw}(G, \Pi) \geq 3$ , then every facial cycle is an induced nonseparating cycle.*

### 4. COMPARING DISTINCT EMBEDDINGS

**Lemma 4.1.** *Let  $\Pi$  and  $\Pi'$  be embeddings of a 3-connected graph  $G$  such that  $\text{fw}(\Pi) \geq 3$ . Suppose that  $C_1, \dots, C_r$  are distinct  $\Pi'$ -facial cycles such that any two of them meet properly. If  $C_1, \dots, C_r$  are all  $\Pi$ -noncontractible and  $\Pi$ -homotopic to each other, then  $\varepsilon(\Pi') \geq \frac{r}{13} - 1$ .*

*Proof.* Suppose first that  $C_1, \dots, C_r$  are  $\Pi$ -twosided. Then we may assume that  $C_1, \dots, C_r$  is a linear nesting, by Corollary 2.2. If  $C_i$  intersects  $C_1$  in  $\text{Ext}(C_1, C_i, \Pi)$ , then  $C_i$  and  $C_r$  do not intersect in  $\text{Ext}(C_i, C_r, \Pi)$  ( $1 \leq i \leq r$ ). Therefore, we may assume that for  $i = 1, \dots, t = \lceil r/2 \rceil$ , the cycles  $C_1$  and  $C_i$  do not intersect in  $\text{Ext}(C_1, C_t, \Pi)$ . Let  $H = \text{Int}(C_1, C_t, \Pi)$ . By inserting a new vertex into each of the faces  $C_1, C_t$  of  $H$  and joining each of them to all vertices of  $C_1$  and  $C_t$ , respectively, we get a plane graph  $H'$  without vertices of degree 2 whose facial cycles meet properly. By Proposition 3.1,  $H'$  is 3-connected. By Menger's Theorem there are three disjoint  $(C_1, C_t)$ -paths  $P'_1, P'_2, P'_3$  in  $H$ . These paths determine disjoint paths  $P_1, P_2, P_3$  in  $G$  since  $C_1$  and  $C_t$  do not intersect in  $\text{Ext}(C_1, C_t, \Pi)$ . Each of  $P_1, P_2, P_3$  intersects all cycles  $C_i$ ,  $i = 1, \dots, t$ . Let  $v_{ik}$  be a vertex of  $P_k \cap C_i$  ( $k = 1, 2, 3$ ;  $i = 1, \dots, t$ ). Let  $G''$  be the graph obtained from  $G$  by adding new vertices  $v_1, \dots, v_r$  and joining each  $v_i$  with the vertices  $v_{i1}, v_{i2}, v_{i3}$ ,  $i = 1, \dots, t$ . Since  $C_1, \dots, C_t$  are  $\Pi'$ -facial,  $\Pi'$  can be extended to an embedding  $\Pi''$  of  $G''$  in the same surface as  $\Pi'$ . Clearly, the subgraph of  $G''$  consisting of  $P_1, P_2, P_3$  and the stars of vertices  $v_i$  ( $i = 1, \dots, t$ ) contains  $K_{3,t}$  as a minor. Therefore,

$$(4.1) \quad \varepsilon(\Pi') = \varepsilon(G'', \Pi'') \geq \varepsilon(K_{3,t}) \geq \left\lceil \frac{t-2}{2} \right\rceil \geq \frac{r}{4} - 1.$$

(The second inequality in (4.1) is an easy corollary of Euler's formula and biparticity of  $G$ . Ringel [11] proved that this is indeed an equality.)

Suppose now that  $C_1, \dots, C_r$  are  $\Pi$ -onesided. Then any two cycles intersect (and cross each other locally in  $\Pi$ ). If  $p = \lceil 2r/13 \rceil$  of the cycles intersect in the same point, then those cycles can be enumerated as concluded in Lemma 2.1. (The details are left to the reader.) In that case, the same conclusion as in (4.1) gives. As in (4.1) we get the inequality:

$$(4.2) \quad \varepsilon(\Pi') \geq \varepsilon(K_{3,p}) \geq \frac{r}{13} - 1.$$

So, we may assume that no  $p$  of the cycles intersect in the same point. Let  $v_{ij} \in V(C_i) \cap V(C_j)$ . For each vertex  $v \in \{v_{ij} \mid 1 \leq i < j \leq r\}$ , select a pair  $(i, j)$  such that  $v = v_{ij}$ . Now, we define a graph  $G''$  obtained from  $G$  by adding  $r$  new vertices  $u_1, \dots, u_r$ , and joining  $u_l$  ( $1 \leq l \leq r$ ) to all vertices  $v_{il}$  and  $v_{lj}$  whose selected pair contains  $l$ . Clearly,  $\Pi'$  can be extended to an embedding of  $G''$  in the same surface. The new vertices and edges form a subgraph of  $G''$  which is a subdivision of a simple graph  $H$  with  $r$  vertices and at least  $\binom{r}{2}/(p-1)$  edges. Euler's formula implies that

$$\begin{aligned} \varepsilon(\Pi') &\geq \varepsilon(H) \geq 2 - |V(H)| + \frac{1}{3}|E(H)| \\ &\geq 2 - r + \frac{13r(r-1)}{6(2r-1)} \geq \frac{r+11}{12} > \frac{r}{13}. \end{aligned}$$

□

Let  $\Pi$  and  $\Pi'$  be embeddings of a graph  $G$ . A closed walk in  $G$  is said to be  $(\Pi, \Pi')$ -unstable if it is  $\Pi$ -facial and is not  $\Pi'$ -facial.

**Lemma 4.2.** *Let  $G$  be a 3-connected graph with embeddings  $\Pi$  and  $\Pi'$  such that  $\text{fw}(\Pi) \geq 3$  and  $\text{fw}(\Pi') \geq 3$ . If  $r$  is the number of  $(\Pi, \Pi')$ -unstable cycles, then*

$$r \leq 13(\varepsilon + 1)c_1$$

where  $\varepsilon = \varepsilon(G, \Pi)$  and  $c_1 = c_1(\varepsilon)$  is the constant from Lemma 2.3.

*Proof.* By Proposition 3.1, the unstable cycles meet properly. By Lemma 2.3,  $\lceil r/c_1 \rceil$  of them are pairwise  $\Pi'$ -homotopic. Proposition 3.2 implies that they are  $\Pi'$ -noncontractible. By Lemma 4.1,  $\varepsilon(G, \Pi) \geq r/(13c_1) - 1$ . This proves the lemma.  $\square$

**Lemma 4.3.** *Let  $G$  be a 3-connected graph with distinct embeddings  $\Pi$  and  $\Pi'$  such that  $\text{fw}(\Pi) \geq 3$  and  $\text{fw}(\Pi') \geq 3$ . Suppose that  $C$  is a  $\Pi$ -facial cycle and that  $C'$  is a  $\Pi'$ -facial cycle. Let  $p$  denote the number of connected components of  $C \cap C'$ . Then  $p$  is smaller than the number of  $(\Pi, \Pi')$ -unstable cycles.*

*Proof.* We may assume that  $C'$  is not  $\Pi$ -facial since otherwise  $p \leq 1$  (and there are at least two unstable cycles). For each edge  $e \in E(C')$  there is a  $(\Pi, \Pi')$ -unstable cycle  $C(e)$  which contains  $e$ . Therefore each connected component  $P$  of  $C \cap C'$  intersects the  $\Pi$ -facial cycle  $C(e) \neq C$  where  $e$  is the edge of  $C'$  following  $P$ . Since  $C$  and  $C(e)$  meet properly, these cycles  $C(e)$  are distinct. Since  $C$  is also unstable,  $p$  is smaller than the number of  $(\Pi, \Pi')$ -unstable cycles.  $\square$

Let  $H$  be a graph with  $k$  connected components. The number  $\beta(H) = |E(H)| - |V(H)| + k$  is called the *Betti number* (or the *cyclomatic number*) of  $H$ .

**Lemma 4.4.** *Let  $G$  be a 3-connected graph with distinct embeddings  $\Pi$  and  $\Pi'$  such that  $\text{fw}(\Pi) \geq 3$  and  $\text{fw}(\Pi') \geq 3$ . Let  $H$  be the union of all  $(\Pi, \Pi')$ -unstable cycles. Then*

$$\beta(H) < 85(\varepsilon + 1)^2 c_1^2$$

where  $\varepsilon = \varepsilon(G, \Pi)$  and  $c_1 = c_1(\varepsilon)$  is from Lemma 2.3.

*Proof.* Let  $C_1, \dots, C_r$  be the  $(\Pi, \Pi')$ -unstable cycles. We prove by induction on  $t$  that  $\beta_t = \beta(C_1 \cup \dots \cup C_t) \leq 1 + t(t-1)/2$ ,  $t = 1, \dots, r$ . Clearly,  $\beta_1 = 1$ . So assume that  $t > 1$ . Let  $S_1, \dots, S_q$  be the maximal segments of  $C_t$  which are edge-disjoint from  $C_1 \cup \dots \cup C_{t-1}$ . Since the  $\Pi$ -facial cycles meet properly,  $q \leq t-1$ . By the induction hypothesis,  $\beta_t = \beta_{t-1} + q \leq 1 + (t-1)(t-2)/2 + t-1 = 1 + t(t-1)/2$ .

Finally,  $r \leq 13c_1(\varepsilon + 1)$  by Lemma 4.2. This shows that  $\beta(H) < 85c_1^2(\varepsilon + 1)^2$ .  $\square$

Let  $C$  and  $C'$  be cycles of a  $\Pi'$ -embedded graph  $G$ . Suppose that one of the following holds:

- (a)  $C \cap C' = \{u\}$  where  $u \in V(G)$  and the edges of  $C$  and  $C'$  incident with  $u$  interlace in the  $\Pi'$ -clockwise ordering around  $u$  (cf. Fig. 3(a)).
- (b)  $C \cap C'$  is the edge  $uv$  and the edges of  $C$  and  $C'$  incident with  $u$  and  $v$  interlace in the  $\Pi'$ -clockwise ordering around  $u$  and  $v$  as shown in Fig. 3(b).

Then we say that  $C$  and  $C'$  *interlace* in  $\Pi'$ .

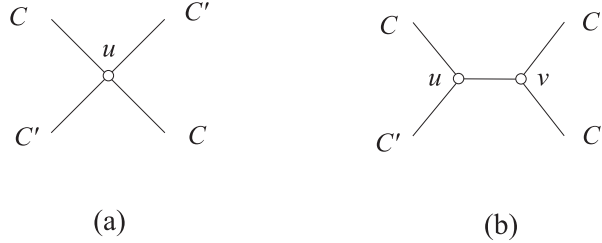


FIGURE 3. The cycles  $C$  and  $C'$  interlace in  $\Pi'$



**Lemma 4.5.** *Let  $G$  be a graph and let  $\Pi$  and  $\Pi'$  be polyhedral embeddings of  $G$ . Let  $C$  be a  $\Pi$ -facial cycle and let  $S$  be a segment of  $C$ . Suppose that in the embedding  $\Pi'$ , there is an interior vertex of  $S$  which has an edge on the right and there is an interior vertex of  $S$  with an edge on the left side of  $C$ . Then there are interior vertices  $u, v$  of  $S$  and a  $\Pi$ -facial cycle  $C'$  such that  $C \cap C' = \{u\}$  or  $C \cap C' = \{uv\}$  and  $C$  and  $C'$  interlace in  $\Pi'$ .*

*Proof.*  $S$  contains an interior vertex  $u$  which has an edge on the right and contains an interior vertex  $v$  with an edge on the left side of  $C$ . Since each vertex of  $C$  has either an edge on the left or on the right side of  $C$ , we may assume that either  $u = v$ , or that  $u$  and  $v$  are adjacent on  $C$ . We assume that  $u = v$  since the proof of the case when  $uv \in E(C)$  proceeds in the same way. In the  $\Pi$ -clockwise ordering around  $u$ , there are consecutive edges  $e, f \notin E(C)$  such that  $e$  is on the left side of  $C$  and  $f$  is on the right side in  $\Pi'$ . Now, we let  $C'$  be the  $\Pi$ -facial walk containing  $e$  and  $f$ .  $\square$

### 5. FLEXIBILITY OF EMBEDDINGS OF FACE-WIDTH 3

Suppose that  $G$  is a 3-connected graph and that  $\Pi_0, \dots, \Pi_N$  are distinct embeddings of  $G$  each of which has face-width at least 3. For distinct integers  $i, j \in \{0, \dots, N\}$  we introduce the following notation. We let  $\mathcal{C}_{ij}$  be the set of all  $(\Pi_i, \Pi_j)$ -unstable cycles, and let  $c_{ij} = |\mathcal{C}_{ij}|$ . The subgraph  $U_{ij} = \cup \mathcal{C}_{ij}$  of  $G$  is called the  $(\Pi_i, \Pi_j)$ -unstable part of  $G$ . Clearly,  $U_{ij} = U_{ji}$  and  $c_{ij} - \varepsilon(G, \Pi_i) = c_{ji} - \varepsilon(G, \Pi_j)$ . In particular, if  $\varepsilon(G, \Pi_i) = \varepsilon(G, \Pi_j)$ , then  $c_{ij} = c_{ji}$ .

**Lemma 5.1.** *Suppose that  $\Pi_0, \Pi_1, \Pi_2$  are embeddings of  $G$  in the same surface and that  $c_{01} = c_{02} = c_{12}$ . Suppose, moreover, that  $\mathcal{C}_{01} \cap \mathcal{C}_{02} = \emptyset$ . Then  $U_{01} = U_{02} = U_{12}$ .*

*Proof.* Since  $\mathcal{C}_{01} \cap \mathcal{C}_{02} = \emptyset$ , we have  $\mathcal{C}_{02} \subseteq \mathcal{C}_{12}$ . Now,  $c_{02} = c_{12}$  implies that  $\mathcal{C}_{02} = \mathcal{C}_{12}$ . Similarly,  $\mathcal{C}_{01} = \mathcal{C}_{21}$ . Hence,  $U_{02} = U_{12} = U_{21} = U_{01}$ .  $\square$

Now we turn to the main theorem of this paper.

**Theorem 5.2.** *There is a function  $\xi: \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that every 3-connected graph admits at most  $\xi(\varepsilon)$  embeddings of face-width  $\geq 3$  into surfaces whose Euler genus is at most  $\varepsilon$ .*

The rest of this section is devoted to the proof of Theorem 5.2. The proof is by induction on  $\varepsilon$ . Clearly,  $\xi(0) = 1$  by Whitney's Theorem. So, we let  $\varepsilon \geq 1$ . We now assume (reductio ad absurdum) that there is no upper bound on the number of distinct embeddings of face-width  $\geq 3$  of 3-connected graphs  $G$  in the surface  $S$  of Euler genus  $\varepsilon$ . Let  $\Pi_0, \dots, \Pi_N$  be such embeddings, where  $\varepsilon(G, \Pi_i) = \varepsilon$ ,  $i = 1, \dots, N$ . We assume that  $G$  can be chosen so that  $N$  is as large as we want. During the proof, we will occasionally select and continue working with a subset of  $\Pi_0, \dots, \Pi_N$  but we will always be able to argue that the new set of embeddings is still as large as we want. Our main concern will be a smooth flow of the proof, and we have no intention to derive good bounds on  $\xi(\varepsilon)$ .

**Claim 5.3.** *There is an integer function  $r(N, \varepsilon)$  such that, for each fixed  $\varepsilon$ ,  $\lim_{N \rightarrow \infty} r(N, \varepsilon) = \infty$ , and there is an integer  $c$  and a subset  $I$  of  $\{0, \dots, N\}$  of cardinality  $r(N, \varepsilon)$  such that  $c_{ij} = c$  for any distinct elements  $i, j \in I$ .*

*Proof.* By Lemma 4.2,  $c_{ij} = c_{ji}$  are bounded by a constant depending only on  $\varepsilon$ . Now, the existence of  $r(N, \varepsilon)$  follows by Ramsey's Theorem (see, e.g., [10, Theorem 1.1]).  $\square$

By using Claim 5.3 and by passing to the subset of embeddings  $\Pi_i, i \in I$ , we may assume henceforth that  $c_{ij} = c_{ji} = c$  for  $0 \leq i < j \leq N$ , and that  $N$  is still as large as we want.

**Claim 5.4.** *Suppose that  $\log_2 N \geq \log_2^2(2c)$ . Then there is a number  $\alpha > 0$  which depends only on  $\varepsilon$ , and there is a subset  $I \subseteq \{0, \dots, N\}$  such that  $|I| \geq \alpha \sqrt{\log_2 N}$  and such that for each  $i \in I$ , there is a  $\Pi_i$ -facial cycle  $C_i$  which is  $\Pi_j$ -nonfacial for every  $j \in I \setminus \{i\}$ .*

*Proof.* Suppose that each  $(\Pi_0, \Pi_1)$ -unstable cycle is  $\Pi_i$ -facial where  $i \geq 2$ . Then  $\mathcal{C}_{01} \cap \mathcal{C}_{0i} = \emptyset$ . By Lemma 5.1,  $U_{01} = U_{0i} = U_{1i}$ . By Lemma 4.4, the Betti number of  $U_{01}$  is bounded by  $c^2/2$ . Since  $\beta(U_{01})$  is the dimension of the cycle space of  $U_{01}$  over  $GF(2)$ ,  $U_{01}$  contains less than  $2^{\beta(U_{01})}$  cycles. Hence less than  $\alpha_1 = 2^{\beta(U_{01})c}$  embeddings  $\Pi_i$  ( $i \geq 1$ ) have their unstable part  $U_{0i}$  contained in  $U_{01}$ . We remove all such embeddings  $\Pi_i$ . In each of the remaining  $N - \alpha_1$  embeddings  $\Pi_i$  ( $i \geq 2$ ), one of the cycles in  $\mathcal{C}_{01}$  is  $\Pi_i$ -nonfacial. Since  $|\mathcal{C}_{01}| = c$ , there is a  $\Pi_0$ -facial cycle  $C_0$  which is nonfacial in at least  $N_1 = (N - \alpha_1)/c$  embeddings. Clearly,  $N_1 \geq N/2c$  if  $N \geq 2\alpha_1$  (which we may assume).

By passing to the subset of the remaining embeddings and continuing the process, let us assume that  $1 \leq i \leq \alpha \sqrt{\log_2 N}$  where  $\alpha$  will be determined below. Suppose that we have cycles  $C_0, \dots, C_{i-1}$  as claimed, and now we want to find  $C_i$ . We are left with  $N_i \geq N/(2c)^i$  embeddings  $\Pi_i, \dots, \Pi_{N_i+i-1}$ .

Let  $U = U_{01} \cup \dots \cup U_{0, i-1}$ . The proof of Lemma 4.4 shows that the Betti number of  $U$  is bounded above by  $r^2/2$  where  $r = ic$  is an upper bound on the number of cycles in  $\mathcal{C}_{01} \cup \dots \cup \mathcal{C}_{0, i-1}$ . Then  $U$  contains less than  $2^{\beta(U)}$  cycles, and hence less than  $\alpha_2 = 2^{\beta(U)c}$  embeddings  $\Pi_j$  ( $j \geq i$ ) have their unstable part  $U_{0j}$  contained in  $U$ . We will prove below that  $N_i - \alpha_2 \geq N_i/2$ . Hence, we may assume that  $U_{0i} \not\subseteq U$ . Denote by  $Q_1, \dots, Q_p$  ( $1 \leq p \leq c$ ) the  $(\Pi_0, \Pi_i)$ -unstable cycles which are not contained in  $U$ . Let  $U' = U \cup Q_1 \cup \dots \cup Q_p$ . Since  $r$  is also an upper bound on the number of  $\Pi_0$ -facial cycles forming  $U'$ ,  $\beta(U') \leq r^2/2$  and hence at least  $N_i - 2^{cr^2/2}$  embeddings  $\Pi_j$  ( $j > i$ ) satisfy  $U_{0j} \not\subseteq U'$ . Assuming  $2^{cr^2/2} < N_i/2$ , we get at least  $N_i/2p \geq N_i/2c$  remaining embeddings and a  $\Pi_i$ -facial cycle  $C_i$  (where  $C_i = Q_s$  for some  $1 \leq s \leq p$ ) which is nonfacial in all other embeddings. By retaining only those embeddings, we can continue the process. The reader can verify that the choice  $\alpha = \frac{1}{2}c^{-3/2}$  guarantees that  $2^{cr^2/2} < N_i/2$  for  $i \leq \alpha \sqrt{\log_2 N}$ .  $\square$

By Claim 5.4 we may assume that for each  $i \in \{0, \dots, N\}$ , there is a  $\Pi_i$ -facial cycle  $C_i$  which is  $\Pi_j$ -nonfacial for every  $j \in \{0, \dots, N\} \setminus \{i\}$ . In particular, the cycles  $C_0, \dots, C_N$  are distinct.

The cycle  $C_i \in \mathcal{C}_{i0}$  is contained in  $U_{0i} = \cup \mathcal{C}_{0i}$ . Since  $\mathcal{C}_{0i}$  contains  $c$  cycles, any two of which meet properly,  $C_i$  can be written as the union of no more than  $c^2$   $\Pi_0$ -facial segments. Let  $S_{i1}, \dots, S_{i, \kappa_i}$  ( $\kappa_i \leq c^2$ ) be these  $\Pi_0$ -facial segments.

**Claim 5.5.** *Suppose that there are pairwise disjoint  $\Pi_0$ -facial segments  $A_i \subseteq S_{i1}$  ( $i = 1, \dots, N$ ), and suppose that there are distinct vertices  $v_{ij} \in A_i \cap C_j$  such that an edge  $e_{i,j}$  of  $C_j$  incident with  $v_{ij}$  is not in the same  $\Pi_0$ -facial walk as  $A_i$  ( $1 \leq i < j \leq N$ ). Then  $N < \kappa$  where  $\kappa$  is an integer which depends only on  $\varepsilon$ .*

*Proof.* Let  $1 \leq p(i, j) \leq c^2$  be the index of the segment of  $C_j$  such that  $v_{ij} \in S_{i1} \cap S_{j, p(i, j)}$ . If  $N$  is large enough, then Ramsey's Theorem (cf. [10, Theorem 1.1]) implies that there is a set  $I \subseteq \{1, \dots, N\}$  with  $k_0 = 2 \lceil 9\sqrt{\varepsilon} + 18 \rceil + 2c$  elements

and there exists an integer  $p$  such that  $p(i, j) = p$  for all  $i, j \in I$ ,  $i < j$ . We may assume that  $I = \{1, \dots, k_0\}$ . Let  $D_i$  be the  $\Pi_0$ -facial cycle containing  $A_i$ , and let  $D'_i$  be the  $\Pi_0$ -facial cycle containing  $S_{ip}$ ,  $i = 1, \dots, k_0$ . Suppose that for  $1 \leq a < b < c < k_0 - 3$ ,  $D_a = D_b = D_c$ . Then  $v_{a,k_0}$ ,  $v_{b,k_0}$ , and  $v_{c,k_0}$  all belong to  $D_a \cap S_{k_0,p}$ , hence  $D_a = D'_{k_0}$ . Similarly we prove that  $D_a = D'_{k_0-1} = D'_{k_0-2}$ . The vertices  $v_{1,k_0-2}$ ,  $v_{1,k_0-1}$ , and  $v_{1,k_0}$  all belong to  $D_1 \cap D'_{k_0}$ , so  $D_1 = D'_{k_0}$ . Similarly,  $D_1 = D_2 = \dots = D_{2c+1} = D'_{k_0}$ . Now, the edges  $e_{1,k_0}, e_{2,k_0}, \dots, e_{2c+1,k_0}$  show that  $C_{k_0} \cap D_1$  consists of more than  $c$  components, a contradiction to Lemma 4.3. This proves that there is a subset of  $k = \lceil 9\sqrt{\varepsilon} + 18 \rceil$  cycles, say  $C_1, \dots, C_k$ , such that  $D_1, \dots, D_k$  are all distinct. Now we distinguish two cases.

CASE 1:  $p = 1$ . We can extend the embedding  $\Pi_0$  to an embedding in the same surface of a graph  $\tilde{G} \supseteq G$  which contains a subdivision of the complete graph  $K_k$  as follows. We insert a new vertex  $x_i$  into each  $\Pi_0$ -face  $D_i$  and add edges inside  $D_i$  from  $x_i$  to  $v_{ij}$  and inside  $D_j$  from  $v_{ij}$  to  $x_j$ ,  $1 \leq i < j \leq k$ . Since  $\varepsilon(K_k) \geq (k-3)(k-4)/6$ , we get a contradiction to the fact that  $k \geq 9\sqrt{\varepsilon} + 18$ .

CASE 2:  $p \neq 1$ , say  $p = 2$ . Suppose that  $2 < i < j < k$  and  $D'_i = D'_j = D'_k = D'$ . Then  $D'$  intersects  $D_1$  in three distinct vertices. Since the  $\Pi_0$ -facial cycles meet properly,  $D' = D_1$ . Similarly we see that  $D' = D_2$ , a contradiction. This implies that we may assume that  $D'_3, \dots, D'_{k_1}$  are all distinct, where  $k_1 = \lfloor k/2 \rfloor$ . Let  $k_2 = \lfloor k_1/3 \rfloor > 1.5\sqrt{\varepsilon} + 2$ . Then we may assume that  $D'_{k_2+1}, \dots, D'_{2k_2}$  are distinct  $\Pi_0$ -facial walks which are distinct from each of  $D_1, \dots, D_{k_2}$ . We now extend the embedding  $\Pi_0$  to an embedding in the same surface of a graph  $\tilde{G} \supseteq G$  which contains a subdivision of the complete bipartite graph  $K_{k_2, k_2}$  in the same way as in Case 1 by using vertices in the  $\Pi_0$ -faces  $D_1, \dots, D_{k_2}$  and  $D'_{k_2+1}, \dots, D'_{2k_2}$ , respectively, and joining them through the vertices  $v_{ij}$ ,  $i = 1, \dots, k_2$ ,  $j = k_2 + 1, \dots, 2k_2$ . Since  $\varepsilon(K_{k_2, k_2}) \geq (k_2 - 2)^2/2$ , we get a contradiction to the fact that  $k_2 > 1.5\sqrt{\varepsilon} + 2$ .  $\square$

**Claim 5.6.** *Let  $A, B$  be segments of  $\Pi_0$ -facial cycles. Suppose that for  $i = 1, \dots, N$ , the cycle  $C_i$  contains a segment  $S_i$  joining  $A$  and  $B$ . If  $S_1, \dots, S_N$  are pairwise internally disjoint, then  $N < 6480c_1(c_1 + 1)^4$ .*

*Proof.* Suppose that  $N \geq 6480c_1(c_1 + 1)^4$ . Since  $\Pi_0$ -facial cycles meet properly,  $B - V(A)$  consists of at most two facial subsegments of  $B$ . Hence, if  $2N/5$  or more of the segments end up in  $B - V(A)$ , then  $B$  has a subsegment disjoint from  $A$  such that at least  $N/5$  of the segments  $S_i$  end up in that subsegment. Similarly, if at least  $2N/5$  of the segments end up in  $A - V(B)$ . Otherwise, at least  $N/5$  of the segments start and end in  $A \cap B$ . Therefore, we may assume that either  $A = B$  or  $A \cap B = \emptyset$ , and that  $N \geq 6^4 c_1 (c_1 + 1)^4$ . We may as well assume that the interior vertices of each segment  $S_i$  are not in  $A \cup B$ . For  $i = 1, \dots, N$ , let  $a_i \in A$  and  $b_i \in B$  be the ends of  $S_i$ , and let  $S'_i = S_i - \{a_i, b_i\}$ .

By imagining that  $A$  and  $B$  are contracted to point(s), we may speak of homotopy of the segments  $S_i$ . By Lemma 2.3, there is a set  $I_0$  of  $6^4(c_1 + 1)^4$  segments which are  $\Pi_0$ -homotopic. Clearly, there is a subset  $I_1$  of  $I_0$ , where  $|I_1| \geq |I_0|^{1/2}$ , such that the ends  $a_i$  ( $i \in I_1$ ) are either all distinct or all the same. Similarly, there is a subset  $I_2$  of  $I_1$ , where  $|I_2| \geq |I_1|^{1/2}$ , such that the vertices  $b_i$  ( $i \in I_2$ ) are either all distinct or all the same. Since  $|I_2| \geq 6(c_1 + 1)$ , we may assume that  $S_1, \dots, S_k$  ( $k = 6c_1 + 5$ ) are  $\Pi_0$ -homotopic segments, their ends  $a_i$  (resp.  $b_i$ ),  $i = 1, \dots, k$ , are either all distinct or all the same, and they are enumerated in the same way as

concluded in Lemma 2.1. For  $1 \leq i < j \leq k$ , let  $A_{i,j}$  (resp.  $B_{i,j}$ ) be the segment of  $A$  (resp.  $B$ ) from  $a_i$  to  $a_j$  (resp.  $b_i$  to  $b_j$ ).

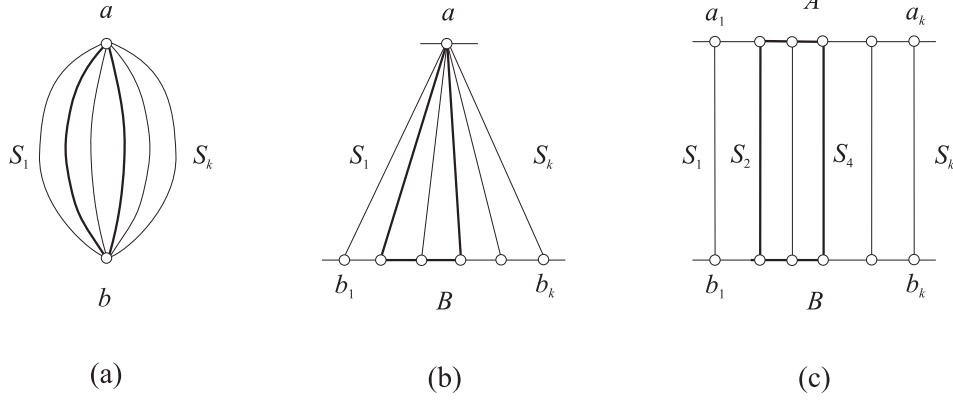


FIGURE 4. The  $\Pi_0$ -homotopic segments  $S_1, \dots, S_k$

Since  $S_i$  and  $S_j$  are  $\Pi_0$ -homotopic,  $D_{ij} = S_i \cup A_{i,j} \cup S_j \cup B_{i,j}$  is  $\Pi_0$ -contractible. We will denote  $\text{Int}(D_{ij}, \Pi_0)$  by  $\overline{D}_{ij}$ . If  $1 < i < j < k$ , then  $D_{ij}$  is a cycle unless  $A_{1,k} = B_{1,k}$  is a single vertex. We have one of the cases shown in Figure 4 where  $D_{24}$  is drawn by thicker lines. In the case of Figure 4(a), it is possible that  $a = b$ .

Suppose that  $1 < i < j < k$  and  $j \neq i + 1$ . Suppose first that  $A_{1,k} = \{a\}$  and  $B_{1,k} = \{b\}$  are just vertices. Since  $S_i$  is a  $\Pi_i$ -facial segment, Proposition 3.2 implies that  $G - S_i$  is connected. The same holds for  $S_j$ . In particular, this implies that no  $\Pi_0$ -facial walk in  $\overline{D}_{1k}$  contains both  $a$  and  $b$ . Therefore, there is a path  $P \subseteq \overline{D}_{1k} - \{a, b\}$  which joins  $S'_1$  and  $S'_k$ . No edge connects  $S'_i$  and  $S'_j$ ; such an edge would be either in  $\overline{D}_{ij}$  (in which case it would cross  $S_{i+1}$ ) or not (in which case it would cross  $S_1$ ), yielding a contradiction in each case. Since  $S_i$  and  $S_j$  are induced subgraphs of  $G$ , no  $(S_i \cup S_j)$ -bridge in  $G$  is just an edge, except possibly the edge  $ab$ . Suppose now that  $Q$  is an  $(S_i \cup S_j)$ -bridge in  $\overline{D}_{ij}$ , and  $v \in V(Q) \setminus (S_i \cup S_j)$ . If  $v \in V(\overline{D}_{i,i+1})$ , then there is a path in  $G - S_i$  from  $v$  to  $S'_j$  since  $G - S_i$  is connected. Such a path intersects  $S_{i+1}$  before it reaches  $S'_j$ . Therefore  $Q \supseteq S_{i+1}$ . A similar argument shows that  $Q \supseteq S_{i+1}$  if  $Q$  contains a vertex in  $\overline{D}_{i+1,j}$ . This shows that  $Q$  is the only  $(S_i \cup S_j)$ -bridge in  $\overline{D}_{ij}$ . If  $Q$  is an  $(S_i \cup S_j)$ -bridge which is not in  $\overline{D}_{ij}$ , then we similarly see that this is the only such bridge. This shows that there are precisely two or three  $(S_i \cup S_j)$ -bridges, and if there are three, one of them is just the edge  $ab$ , which is not in  $\overline{D}_{ij}$ .

Suppose now that  $B_{1,k}$  is not just a vertex. Then we can use similar arguments as above to prove that there are precisely two  $(S_i \cup S_j)$ -bridges which are not edges. Also, there are no edges joining  $S_i \setminus S_j$  with  $S_j \setminus S_i$ . Since  $C_i$  is an induced cycle, if  $a_i b_i \in E(G)$ , then  $C_i = S_i + a_i b_i$  must lie in  $\overline{D}_{1k}$ , so  $C_i$  would be  $\Pi_0$ -contractible, a contradiction. Similarly for  $S_j$ . This shows that there are precisely two  $(S_i \cup S_j)$ -bridges in  $G$ .

Let  $i = 3c_1 + 3$ . By Lemma 2.3, there are indices  $2 \leq p < q < r \leq 3c_1 + 1$  such that  $S_p, S_q, S_r$  are  $\Pi_i$ -homotopic. We claim that  $\overline{D}'_{pr} := \text{Int}(D_{pr}, \Pi_i) = \overline{D}_{pr}$ . Since

there are at least two  $D_{pr}$ -bridges in  $G$ ,  $D_{pr}$  is not a  $\Pi_i$ -facial cycle. Denote by  $Q$  and  $R$  the  $(S_p \cup S_r)$ -bridges in  $G$  (distinct from  $ab$ ) where  $Q \subseteq \overline{D}_{pr}$ .

If  $A_{1,k} = \{a\}$  and  $B_{1,k} = \{b\}$  are just vertices and  $ab \in E(G)$ , then  $C_p = S_p + ab$ . We already argued above that  $ab \notin E(\overline{D}_{pr})$  and, similarly,  $ab \notin E(\overline{D}'_{pr})$ . Thus, to prove the above claim, it suffices to show that  $Q \subseteq \overline{D}'_{pr}$  and that  $R \not\subseteq \overline{D}'_{pr}$ .

Suppose that  $R \subseteq \overline{D}'_{pr}$ . Since  $R$  contains  $S_i$  and since  $C_i$  is  $\Pi_i$ -facial,  $C_i \subseteq \overline{D}'_{pr}$ . If  $D_{pr}$  is a cycle, then by using the embeddings of  $\overline{D}_{pr}$  and  $\overline{D}'_{pr}$ , we easily construct an embedding of  $G$  of genus 0. This gives a contradiction since planar graphs have no nonplanar embeddings of face-width 3 or more (cf. [12]). This shows that  $D_{pr}$  is not a cycle, i.e., we have  $A_{1,k} = B_{1,k} = \{a\}$  where  $a \in V(G)$ . Then the  $\Pi_i$ -noncontractible cycles  $C_{i-1} = S_{i-1}$  and  $C_{i+1} = S_{i+1}$  are both in  $R$  and hence they are  $\Pi_i$ -homotopic. It is easy to see that  $C_i \subseteq \overline{D}'_{i-1,i+1} = \overline{D}_{i-1,i+1}$ . By Lemma 4.5 (applied on  $S = S_i$ ,  $\Pi = \Pi_i$ ,  $\Pi' = \Pi_0$ ) we see that there is a  $\Pi_i$ -facial cycle  $C'$  which intersects  $C_i$  in a vertex or an edge disjoint from  $a$  and which interlaces with  $C_i$  in  $\Pi_0$ . Since  $C' \subseteq \overline{D}_{1k}$ ,  $C'$  and  $C_i$  have another point of intersection. This contradiction to Proposition 3.1 proves that  $R \not\subseteq \overline{D}'_{pr}$ .

Suppose now that  $Q \not\subseteq \overline{D}'_{pr}$ . Then there is a  $D_{pr}$ -bridge  $Q' \subseteq Q$  which is not in  $\overline{D}'_{pr}$ . As we proved above,  $\overline{D}'_{pr} \subseteq D_{pr} \cup Q$ . This implies that  $\overline{D}'_{pr}$  contains a  $\Pi_i$ -facial walk  $D'$  which contains all vertices of attachment of  $Q'$ . This, in particular, implies that  $Q'$  is not just an edge. Moreover, each  $\Pi_i$ -facial walk that contains a foot of  $Q'$ , contains precisely one other foot of  $Q'$ . Hence, if  $f$  is the number of feet of  $Q'$ , then there are precisely  $f$   $\Pi_i$ -facial walks  $\overline{Q}_1, \dots, \overline{Q}_f$  containing feet of  $Q'$ . Let  $Q_j = \overline{Q}_j \cap Q'$ ,  $j = 1, \dots, f$ . Each  $Q_j$  intersects  $D'$  in its end(s)  $x, y$ . If  $x \neq y$ , then (because  $\overline{Q}_j$  and  $D'$  meet properly)  $xy \in E(D')$  and  $\overline{Q}_j = Q_j + xy$ . Thus  $\overline{Q}_j$  is an induced nonseparating cycle which is contained in the disk  $\overline{D}_{pr}$ . Therefore, it is also  $\Pi_0$ -facial. It is easy to see that this is not possible for  $j = 1, \dots, f$ . This proves that  $Q'$  does not exist and hence  $\overline{D}'_{pr} = \overline{D}_{pr}$ .

Similarly we see that there are  $\Pi_i$ -homotopic segments  $S_s, S_t, S_u$  where  $i+1 < s < t < u < 6c_1 + 5$ .

Suppose that  $A_{1,k} = \{a\}$  and  $B_{1,k} = \{b\}$  are just vertices (possibly  $a = b$ ). As proved above, no  $\Pi_0$ -facial walk in  $\overline{D}_{1k}$  contains  $a$  and  $b$ . Therefore  $S_i$  has an intermediate vertex in which there are edges on the left side of  $S_i$ , and an intermediate vertex with an edge on the right side of  $S_i$  in the embedding  $\Pi_0$ . We say that  $S_i$  has the *left-right property*.

Suppose now that  $B_{1,k}$  is not just a vertex. Then  $\overline{D}_{1k}$  is a disk and hence  $C_i \not\subseteq \overline{D}_{1k}$ . We claim that the segment  $\overline{S}_i$  of  $C_i$  from  $a_i$  through  $S_i$  until  $S_r \cup S_s$  has the left-right property. If not, then one of the  $\Pi_0$ -facial walks containing an edge  $e$  of  $S_i$  contains the entire  $\overline{S}_i$ . In particular, it contains  $e$  and also a vertex of  $(S_r \cup S_s) \setminus \{a_i\}$ . Clearly, this is not possible, hence the claim.

Since  $S_i$  (or  $\overline{S}_i$ ) has the left-right property, Lemma 4.5 (applied on this segment as  $S$ ,  $\Pi = \Pi_i$ ,  $\Pi' = \Pi_0$ ) shows that there is a  $\Pi_i$ -facial cycle  $C'$  which intersects  $S_i$  (or  $\overline{S}_i$ ) in an internal vertex or edge  $x$  and interlaces with  $C_i$  in  $\Pi_0$ . Therefore  $C'$  is not  $\Pi_0$ -facial. In particular,  $C'$  and  $C_i$  are  $\Pi_0$ -noncontractible. This implies that  $C' \not\subseteq \overline{D}_{1k}$ . (In the case when  $A_{1,k} = B_{1,k} = \{a\}$  is the same vertex,  $\overline{D}_{1k}$  contains  $\Pi_0$ -noncontractible cycles. Then we use the fact that  $C' \cap C_i = x$ , so  $a \notin V(C')$ .) Since  $\overline{D}_{pr} = \overline{D}'_{pr}$  and  $\overline{D}_{su} = \overline{D}'_{su}$ , we have  $C' \cap \overline{D}_{pr} \subseteq A \cup B$  and  $C' \cap \overline{D}_{su} \subseteq A \cup B$ .

Since  $C_i$  and  $C'$  interlace around  $x$  in  $\Pi_0$ , we may assume that  $C' \cap \overline{D}_{pr} = A_{p,r}$  and  $C' \cap \overline{D}_{su} = B_{s,u}$ . In particular, neither  $A_{1,k}$  nor  $B_{1,k}$  is just a vertex. Similarly,  $C_i \cap \overline{D}_{pr} = B_{p,r}$  and  $C_i \cap \overline{D}_{su} = A_{s,u}$ . Now, let  $S_i''$  be the segment of  $C_i$  from the edge of  $S_i$  incident with  $a_i$  to  $a_s$  (so that  $S_i' \cap S_i'' = e$ ). Then  $S_i''$  has the left-right property, and Lemma 4.5 shows that there is a  $\Pi_i$ -facial walk  $C''$  which intersects  $C_i$  in the interior of  $S_i''$  and which interlaces with  $C_i$  in  $\Pi_0$ . As above, we conclude that  $C'' \cap \overline{D}_{pr} = A_{p,r}$  and  $C'' \cap \overline{D}_{su} = B_{s,u}$ . Thus, the  $\Pi_i$ -facial cycles  $C'$  and  $C''$  do not meet properly. This contradiction completes the proof.  $\square$

Now, we will apply Lemma 2.4 to prove

**Claim 5.7.** *Let  $A_0, \dots, A_{p-1}$  ( $p \geq 1$ ) be pairwise disjoint  $\Pi_0$ -facial segments. Suppose that each of the cycles  $C_i$  ( $1 \leq i \leq N$ ) intersects  $A_0$  and leaves  $A_0$  by an edge  $e_i$  such that  $e_1, \dots, e_N$  are all distinct. Suppose also that  $N \geq \varphi(k, 6480c_1(c_1 + 1)^4)$  where  $\varphi$  is the function of Lemma 2.4. Then there is a cycle  $C_i$  and a  $\Pi_0$ -facial segment  $A_p$  contained in  $C_i$  and disjoint from  $A_0, \dots, A_{p-1}$  such that at least  $k/(4pc^2) - 1$  other cycles  $C_j$  ( $1 \leq j \leq N$ ) intersect and leave  $C_i$  in distinct vertices of the segment  $A_p$ .*

*Proof.* Let  $H$  be the graph obtained from  $C_1 \cup \dots \cup C_N \cup A_0 \cup \dots \cup A_{p-1}$  by first contracting the segments  $A_0 \cup \dots \cup A_{p-1}$  and then splitting the vertex resulting from  $A_0$  into two vertices  $v_1, v_2$  so that each cycle  $C_i$  is split at  $A_0 \cap e_i$  into a path, and the edge  $e_i$  is incident with  $v_1$ . Let  $\overline{C}_i$  be the walk in  $H$  from  $v_1$  to  $v_2$  corresponding to  $C_i$ . Suppose that no  $\overline{C}_i$  has more than  $k$  vertices of intersection with other walks  $\overline{C}_j$ . Then we apply Lemma 2.4 to get a set of  $6480c_1(c_1 + 1)^4$  internally disjoint subwalks of the walks  $\overline{C}_i$  between  $v_1$  and another vertex  $v$  of  $H$ . These subwalks determine internally disjoint paths (or cycles) in  $G$  joining  $v_1$  and  $v$  (or the corresponding  $\Pi_0$ -facial segments). Now, we get a contradiction by applying Claim 5.6.

Therefore, some  $\overline{C}_i$  intersects other walks in at least  $k$  distinct vertices, and so at least  $k - p$  other cycles intersect and leave  $C_i$  in distinct vertices disjoint from  $A_0, \dots, A_{p-1}$ . The cycle  $C_i$  is the union of at most  $c^2$   $\Pi_0$ -facial segments. Hence, there is a  $\Pi_0$ -facial segment  $A$  of  $C_i$  such that at least  $(k - p)/c^2$  other cycles intersect and leave  $A$  in distinct vertices. Since  $A$  intersects each  $\Pi_0$ -facial segment  $A_j$  ( $0 \leq j < p$ ) at most once, there is a subsegment  $A_p$  of  $A$  disjoint from  $A_0, \dots, A_{p-1}$  such that at least  $(k - p)/(c^2(p + 1)) \geq k/(2pc^2) - 1$  other cycles  $C_j$  intersect and leave  $A_p$  in distinct vertices.  $\square$

Now we have all the main assumptions and main ingredients to conclude the proof of Theorem 5.2. Define the function  $\Phi : \mathbb{N} \rightarrow \mathbb{N}$  inductively as follows. Set  $\Phi(0) = 1$ , and  $\Phi(k) = \varphi(2\kappa c^2(\Phi(k - 1) + 1), 6480c_1(c_1 + 1)^4)$ , where  $\kappa$  is the integer from Claim 5.5, if  $k \geq 1$ .

Now, assume that  $N > 6c_1(c_1 + 1)\Psi$ , where  $\Psi = (c^2\Phi(\kappa))^{c^2\Phi(\kappa)}$ . Let us first assume that each cycle  $C_i$  intersects less than  $\Psi$  other cycles. Then there is a subset of  $6c_1(c_1 + 1)$  disjoint cycles. By Lemma 2.3, there is a subset of  $6(c_1 + 1)$  disjoint  $\Pi_0$ -homotopic cycles, say  $C_1, \dots, C_{6c_1+6}$ . Similarly as in the proof of Claim 5.6, we set  $i = 3c_1 + 3$  and take  $\Pi_i$ -homotopic cycles  $C_p, C_q, C_r$  and  $C_s, C_t, C_u$  where  $1 < p < q < r < i < s < t < u < 6c_1 + 5$ . Clearly,  $\text{Int}(C_p, C_r, \Pi_i) = \text{Int}(C_p, C_r, \Pi_0)$  and  $\text{Int}(C_s, C_u, \Pi_i) = \text{Int}(C_s, C_u, \Pi_0)$ . This shows that every  $\Pi_i$ -facial walk which intersects  $C_i$  is contained in  $\text{Int}(C_r, C_s, \Pi_0)$ . In particular, this

holds for the  $\Pi_i$ -facial walk  $C'$  obtained by Lemma 4.5 which interlaces with  $C_i$  in  $\Pi_0$ , a contradiction.

Suppose now that there is a  $\Pi_0$ -facial segment  $A_0$  in which at least  $\Phi(\kappa)$  cycles intersect and leave  $A_0$  using distinct edges. In such a case, let  $\mathcal{C}$  be the set of those cycles. We may assume that  $\mathcal{C} = \{C_1, \dots, C_{\Phi(\kappa)}\}$ . Now we successively apply Claim 5.7 as follows. By Claim 5.7, one of the cycles, say  $C_1$ , contains a  $\Pi_0$ -facial segment  $A_1$  which is disjoint from  $A_0$  such that at least  $\Phi(\kappa-1)$  other cycles  $C_j \in \mathcal{C}$  intersect and leave  $A_1$  in distinct vertices. Inductively, we find cycles  $C_1, \dots, C_\kappa$  such that each  $C_i$  contains a  $\Pi_0$ -facial segment  $A_i$  disjoint from  $A_0, \dots, A_{i-1}$  in which all cycles  $C_j$ ,  $i < j \leq \kappa$ , intersect in distinct vertices. This is a contradiction to Claim 5.5.

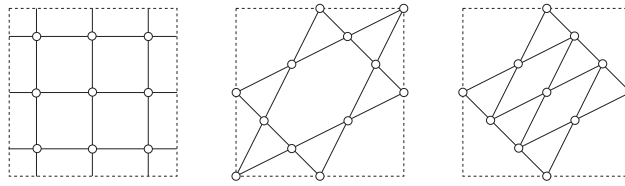
In what it remains, we may assume that  $C_1$  intersects  $\Psi$  other cycles and that, for  $i = 1, \dots, N$ , the cycles intersecting  $C_i$  leave  $C_i$  in at most  $c^2(\Phi(\kappa) - 1)$  distinct edges (since  $C_i$  is composed of at most  $c^2$   $\Pi_0$ -facial segments). Starting with  $C_1$ , there is an edge  $e_1$  such that a set  $\mathcal{C}_1$  of at least  $(\Psi - 1)/(c^2(\Phi(\kappa) - 1)) \geq \Psi/(c^2\Phi(\kappa))$  cycles leave  $C_1$  through  $e_1$ . Let  $C_2 \in \mathcal{C}_1$ . For each  $C_i \in \mathcal{C}_1 \setminus \{C_2\}$ , let  $f_i$  be the first edge after  $e_1$  at which  $C_i$  leaves  $C_2$ . Then there is an edge  $e_2$  such that  $f_i = e_2$  for at least  $\Psi/(c^2\Phi(\kappa))^2$  cycles  $C_i \in \mathcal{C}_1$ . Now we define  $\mathcal{C}_2$  as the set of all these cycles  $C_i$ . We select  $C_3 \in \mathcal{C}_2$ , find the next edge  $e_3$ , etc. Eventually, we end up with a sequence of cycles  $C_1, C_2, \dots, C_{c^2\Phi(\kappa)}$ . The construction shows that the cycles  $C_1, C_2, \dots, C_{c^2\Phi(\kappa)-1}$  leave  $C_{c^2\Phi(\kappa)}$  using distinct edges. This contradiction to the above assumption completes the proof of Theorem 5.2.

## 6. SOME EXAMPLES

If  $K_n$  ( $n \geq 7$ ) triangulates the surface  $S$ , then every embedding  $\Pi$  of  $K_n$  in  $S$  is triangular and hence of face-width 3. It is easy to see that precisely  $2n(n-1)$  automorphisms of  $K_n$  preserve the embedding  $\Pi$ . Therefore, by taking all  $n!$  automorphisms of  $K_n$  we end up with  $\frac{1}{2}(n-2)!$  nonequivalent embeddings of  $K_n$  of face-width 3 in  $S$ . Bonnington et al. [3] constructed for all values of  $n$  congruent to 7 or 19 modulo 36 at least  $2^{n^2/54 - O(n)}$  nonisomorphic triangular embeddings of  $K_n$  in orientable surfaces. This shows that  $K_n$  (for these restricted values of  $n$ ) admits at least  $2^{n^2/54 - O(n)}(n-2)!$  nonequivalent embeddings of face-width 3 in the orientable surface of Euler genus  $\varepsilon = (n-3)(n-4)/6$ .

However, unless the number of nonisomorphic triangular embeddings of  $K_n$  can be proved to be much larger, there are even better candidates for maximum flexibility of embeddings of face-width 3 in the same surface. Let  $G_0$  be a triangulation of the 2-sphere with at least  $k$  facial triangles  $T_1, \dots, T_k, \dots$ . For each  $T_i$  ( $1 \leq i \leq k$ ), add a new copy of the complete graph  $K_4$  and join its vertices completely with the three vertices of  $T_i$ . Denote the resulting graph by  $G_k$ . Since  $K_7$  has 48 nonequivalent embeddings into the torus such that a fixed triangle is a face, these embeddings used on each of the added graphs result in  $48^k$  distinct embeddings of  $G_k$  in the orientable surface  $S_k$  of Euler genus  $2k$ . This shows that  $\xi(2g) \geq 48^g$ . Similarly we see that  $\xi(2g+1) \geq 6 \cdot 48^g$ , by using 6 embeddings of  $K_6$  with a fixed facial triangle in the projective plane to get embeddings of odd Euler genus. This is better than the aforementioned bound for  $K_n$  if  $k$  is large enough.

Another interesting aspect of flexibility of embeddings of face-width 3 is the following. For a fixed surface  $S$ , there are finitely many graphs without vertices of degree 2 which are embedded with face-width 3 but the removal of every edge gives

FIGURE 5. Three embeddings of the line graph of  $K_{3,3}$ 

an embedding of face-width 2 [7] (cf. also [9]). Such graphs (and their embeddings) are said to be *minimal of face-width 3*. If  $G$  is embedded with face-width 3 or more, then it contains a subdivision of an embedded graph  $H$  which is minimal of face-width 3. It is easy to see that the embedding of  $H$  uniquely extends to an embedding of  $G$  (if  $G$  is 3-connected). Such an observation was used in [5] to show that triangulations of a fixed surface have bounded flexibility. Unfortunately, this does not yield a simple proof of Theorem 5.2 since the subgraph  $H$  may have embeddings of smaller face-width or even smaller genus. Figure 5 shows three embeddings of the line graph of  $K_{3,3}$  in the torus having face-width 3, 2, and 1, respectively (the first one being minimal of face-width 3).

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