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NAHM'S EQUATIONS AND
GENERALIZED C. NEUMANN'S
SYSTEM

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Abstract

In the paper a completely integrable Hamiltonian system on T^*M is constructed for every Riemannian symmetric space M . We show that the solutions of this system correspond to the solutions of Nahm's equations for suitably chosen maps. Nahm's equations were introduced by Nahm as a rewriting of Bogomolny equations for magnetic monopoles. We represent our system as a degeneration of a certain case of Hitchin's algebraically integrable system. We prove the complete integrability of our system by means of this representation. Some concrete examples of our Hamiltonian system on T^*M are described. When $M = S^n$ we get the classical C. Neumann system. If the configuration space M of our system is the n -dimensional hyperbolic space is the Minkowskian analogue of the C. Neumann system. Other examples that we describe are a many-body C. Neumann system, spherical pendulum and spherical pendulum with an additional magnetic force.

1 Introduction

In this paper we shall study an aspect of the system of ordinary differential equations

$$\dot{T}_i + \frac{1}{2} \sum \epsilon_{i,j,k} [T_j, T_k] = 0, \quad i = 1, 2, 3 ,$$

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where $T_i: I \rightarrow \mathfrak{g}$, $i = 1, 2, 3$ are functions from an interval into a Lie algebra \mathfrak{g} . These equations are called Nahm's equations and they originate in Yang-Mills theory. They first appeared in [22] as a rewriting of the Bogomolny equation for the magnetic monopoles on \mathbb{R}^3 and were subsequently used in the study of these monopoles by Hitchin in [13] and by Donaldson in [8]. There is also a more straightforward connection between Nahm's equations and Yang-Mills theory, namely Nahm's system is a three dimensional reduction of the anti-self-dual equation on a trivial G -bundle over \mathbb{R}^4 , where $Lie G = \mathfrak{g}$.

Here we shall consider an interpretation of Nahm's equations as a Hamiltonian system. In Section 1 we prove the following. Let $G^{\mathbb{C}}$ be a complex Lie group with the Lie algebra $Lie(G^{\mathbb{C}}) = \mathfrak{g}^{\mathbb{C}}$. The solutions of Nahm's equations for functions $T_i: I \rightarrow \mathfrak{g}^{\mathbb{C}}$ yield solutions of the Hamiltonian system $(T^*G^{\mathbb{C}}, \omega, H)$. The form ω is the holomorphic cotangent symplectic form on the cotangent bundle $T^*G^{\mathbb{C}}$. The Hamiltonian H is given by

$$H(q, p) = \frac{1}{2} \|p\|_{G^{\mathbb{C}}}^2 - \mathcal{K}(\text{Ad}_q^*(\alpha_0), \gamma_0) ,$$

where \mathcal{K} is the Killing form, $\|p\|_{G^{\mathbb{C}}}^2 = \mathcal{K}(p, p)$ and $\alpha_0, \gamma_0 \in (\mathfrak{g}^{\mathbb{C}})^*$ are constant. The correspondence between the solutions of Nahm's equations and the solutions of the system $(T^*G^{\mathbb{C}}, \omega, H)$ is one-to-one. The systems $(T^*G^{\mathbb{C}}, \omega, H)$, being complex, do not have an immediate mechanical meaning, but they do enable us to produce an interesting family of real Hamiltonian systems. In Section 3 we show that the imposition of suitable pairs of real involutions on $(T^*G^{\mathbb{C}}, \omega, H)$ and on Nahm's equations leads to the following result. Let M be an arbitrary Riemannian symmetric space, and $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ the corresponding Cartan decomposition of the Lie algebra \mathfrak{g} . This means that $M = G/U$ and $Lie(G) = \mathfrak{g}$, $Lie(U) = \mathfrak{u}$. Then every solution of Nahm's equations for the functions $T_1, T_3: I \rightarrow i\mathfrak{p}$, $T_2: I \rightarrow \mathfrak{u}$ yields a solution of the Hamiltonian system (T^*M, ω, H_M) which describes the motion of a particle on M under the influence of a certain force potential. We show in Section 5 that in the case where M is the standard n -sphere this system turns out to be C. Neumann's system describing harmonic motion constrained to the sphere. Therefore we call the family (T^*M, ω, H_M) generalized C. Neumann system. The case where M is the hyperbolic space \mathbb{H}^n yields a Minkowskian version of C. Neumann's system. This describes a particle in the Minkowski space $\mathbb{R}^{1,n}$ which moves under the influence of a quadratic force potential and is constrained to the unit sphere \mathbb{H}^n in $\mathbb{R}^{1,n}$. Other concrete examples of (T^*M, ω, H_M) that we mention are

a many-body Neumann problem, and motions on coadjoint orbits under the influence of certain potential and magnetic forces. In [8] Donaldson showed that the solutions of Nahm's equations for $T_i: I \rightarrow \mathfrak{su}(2)$ yield solutions of the mechanical system describing the motion on the hyperbolic space \mathbb{H}^3 under the influence of the potential $V(q) = \text{Tr}(\text{Ad}_q^*(\alpha_0) \cdot \alpha_0^*)$. This discovery served as the starting point for the work presented in this paper.

The systems $(T^*G^{\mathbb{C}}, \omega, H)$ and (T^*M, ω, H_M) turn out to be completely integrable in the Liouville-Arnold sense. For $(T^*G^{\mathbb{C}}, \omega, H)$ this means that there exist $n = \dim G^{\mathbb{C}}$ holomorphic functions $H_i: T^*G^{\mathbb{C}} \rightarrow \mathbb{C}$ which Poisson-commute with each other and with H and are functionally independent. We construct such functions in Section 2. Complete integrability of the systems (T^*M, ω, H_M) follows easily. The integrability of $(T^*G^{\mathbb{C}}, \omega, H)$ stems from the fact that the corresponding equation of motion can be expressed as a Lax equation. The use of the Lax equation in the description of integrable systems similar to ours was developed by Moser ([19], [20]), Adler and van Moerbeke ([4], [5]), Mishchenko and Fomenko ([18]) Mumford ([21]), Reyman and Semenov-Tian-Shansky ([24] and references therein), Adams, Harnad, Hurtubise, Previato ([3], [1], [2]) and many others.

A very important and interesting family of finite dimensional integrable systems was provided by Hitchin in [13]. We show that the system $(T^*G^{\mathbb{C}}, \omega, H)$ can be represented as a certain degeneration of Hitchin's system. Let \mathcal{M} denote the moduli space of holomorphic structures on P , where $\pi: P \rightarrow C$ is a principal $G^{\mathbb{C}}$ -bundle over a complex curve C . Hitchin's integrable system is a system of $n = \dim \mathcal{M}$ holomorphic functionally independent functions $H_i: T^*\mathcal{M} \rightarrow \mathbb{C}$ which Poisson-commute with respect to the natural holomorphic symplectic structure on $T^*\mathcal{M}$. A rather straightforward generalization of this construction gives systems on $T^*\mathcal{M}_{par}$ and on $T^*\mathcal{M}_D$. Here \mathcal{M}_D denotes the moduli space of holomorphic structures with framings at the points $\{p_1, \dots, p_k\} = D \subset C$, and \mathcal{M}_{par} the moduli space of corresponding parabolic structures. These were discussed by Markman in [16], and by the author in [25]. The special case where $C = \mathbb{CP}^1$ was studied by Beauville in [7]. If we let some points in D coalesce, the framed bundles will give rise to bundles with higher order framings. We give a short description of moduli spaces of bundles with higher order framings in the Appendix of this paper. In Section 1 we show that our system $(T^*G^{\mathbb{C}}, \omega, H)$ is essentially Hitchin's system on $T^*\mathcal{M}_{D^d}$, where \mathcal{M}_{D^d} is the moduli space of $G^{\mathbb{C}}$ -bundles over \mathbb{CP}^1 with second-order

framings over two points $p_1, p_2 \in \mathbb{CP}^1$. The geometry of \mathcal{M}_{D^d} enables us to prove the integrability of $(T^*G^{\mathbb{C}}, \omega, H)$ in a fairly straightforward way.

Hitchin's systems on suitable moduli spaces \mathcal{M} of bundles over the curve C are algebraically integrable. This means that for a generic $\Phi \in \mathcal{M}$ the Liouville torus V containing Φ is an Abelian variety related to a certain algebraic curve. Usually functional independence of integrals H_i is established by proving that $\dim V = \dim \mathcal{M}$. This is feasible if $\mathfrak{g}^{\mathbb{C}}$ is a classical semi-simple Lie algebra or G_2 , (see [15]), but even then it requires a careful case-by-case approach for each type of algebra. In this paper we prove the independence of integrals in a different way using a result of Mishchenko and Fomenko [18] combined with an asymptotic argument. This approach works "in one go" for all semi-simple Lie algebras including the exceptional ones.

The family of integrable systems that we obtain in this paper is similar and indeed overlaps with the systems studied by Reyman and Semenov-Tian-Shansky in [24]. In their seminal paper Reyman and Semenov-Tian-Shansky obtain systems on Lie groups and on different coadjoint orbits. However, not all symmetric spaces are coadjoint orbits. The approach in [24] is also different from ours. In particular we use the representation of our system as a case of Hitchin's system to prove its integrability, while in [24] the authors use an R-matrix argument for this purpose. While working on the material presented in this paper the author was not aware of the work of Reyman and Semenov-Tian-Shansky.

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2 Nahm's equations as a Hamiltonian system

2.1 Nahm's equations and Hitchin's systems

In this section we will show how solutions of Nahm's equations yield solutions of a Hamiltonian system. We begin by representing Nahm's equations as a special case of Hitchin's system. First we have to provide the appropriate moduli space. Let $\pi: P \rightarrow C$ be a holomorphic $G^{\mathbb{C}}$ -bundle over a Riemann surface C and let $D = \{p_1, \dots, p_k\}$ be a divisor on C . A framing of P at p_i is a

$G^{\mathbb{C}}$ -equivariant holomorphic map $\phi_i: \pi^{-1}(p_i) \rightarrow G^{\mathbb{C}}$. This map can be thought of as a 0-jet of a trivialization of P at p_i . Let two points, say p_1, p_2 coalesce. Then the two 0-jets will give rise to a single 1-jet of a trivialization. We will call a k -jet of a holomorphic trivialization at a given point p a k -framing of the bundle at p . Let D^d be a divisor in C consisting of points $\{p_1, \dots, p_k\}$ with orders $\text{ord}(p_i) = d_i$. We will denote the moduli space of stable holomorphic bundles with d_i -framings over p_i , $i = 1, \dots, k$ by \mathcal{M}_{D^d} . Let $(A, \phi) \in \mathcal{M}_{D^d}$, where A is a complex structure on P and ϕ denotes a choice of a d_i -framing over each p_i holomorphic with respect to A . In the Appendix we prove the following proposition.

Proposition 1 *Let $\text{ad}P \rightarrow C$ denote the vector bundle with the fibre $\mathfrak{g}^{\mathbb{C}} = \text{Lie}(G^{\mathbb{C}})$ associated to P via the adjoint representation. Then $T_{(A, \phi)}^* \mathcal{M}_{D^d} = H_A^0(C; \text{ad}P \otimes K(D^d))$, where $K(D^d) = K \otimes [D^d]$ is the canonical bundle twisted by $[D^d]$. In other words, an element $\Phi \in T_{(A, \phi)}^* \mathcal{M}_{D^d}$ is a meromorphic section of $\text{ad}P \otimes K$ with poles of degrees d_i at the points p_i . In addition*

$$\sum_{i=1}^k \mathcal{K}(\text{Res}\Phi(p_i), \Psi(p_i)) = 0$$

for every holomorphic section in $\Psi \in H^0(C; \text{ad}P)$. Here \mathcal{K} is the metric on $\text{ad}P \otimes K$ induced by the Killing form on $\mathfrak{g}^{\mathbb{C}}$.

We will be interested in a very simple case of \mathcal{M}_{D^d} . Let $\pi: P \rightarrow \mathbb{CP}^1$ be a trivial $G^{\mathbb{C}}$ -bundle and let the divisor D consist of two double points p_1, p_2 . By Grothendieck's theorem there is only one trivial holomorphic $G^{\mathbb{C}}$ -bundle over \mathbb{CP}^1 . The 1-framings over a point are parametrized by the tangent bundle $TG^{\mathbb{C}}$, so we have

$$\mathcal{M}_{D^d} = (TG^{\mathbb{C}} \times TG^{\mathbb{C}})/G^{\mathbb{C}},$$

where $G^{\mathbb{C}} = \text{Aut}(P)$ acts diagonally from the left. It follows from Proposition (1) that for every $P \in \mathcal{M}_{D^d}$ the space $T_P^* \mathcal{M}_{D^d}$ consists of the elements of the form

$$\Phi = \frac{\alpha}{(z - p_1)^2} + \frac{\beta}{(z - p_1)} + \frac{-\beta}{(z - p_2)} + \frac{\gamma}{(z - p_2)^2} \quad (1)$$

for some $\alpha, \beta, \gamma \in \mathfrak{g}^{\mathbb{C}}$. Here $z \in \mathbb{CP}^1$ is an affine coordinate.

Let $\flat: \mathfrak{g}^{\mathbb{C}} \rightarrow (\mathfrak{g}^{\mathbb{C}})^*$ be the map defined by $\flat(\alpha) = \mathcal{K}(\alpha, -)$ and introduce a Lie bracket $[-, -]^*$ on $(\mathfrak{g}^{\mathbb{C}})^*$ such that the map \flat is a Lie algebra isomorphism. From now on we will work exclusively with the dual Lie algebra $((\mathfrak{g}^{\mathbb{C}})^*, [-, -]^*)$. Since no confusion can arise, we shall denote the bracket $[-, -]^*$ simply by $[-, -]$.

Let $T_i: I \rightarrow (\mathfrak{g}^{\mathbb{C}})^*$, $i = 1, 2, 3$ be maps from an interval into a complex semi-simple Lie algebra which satisfy Nahm's system

$$\dot{T}_i + \frac{1}{2} \sum \epsilon_{ijk} [T_j, T_k] = 0, \quad i = 1, 2, 3. \quad (2)$$

Define new maps $\alpha, \beta, \gamma: I \rightarrow (\mathfrak{g}^{\mathbb{C}})^*$ by $\alpha = T_2 + iT_3$, $\beta = -2iT_1$, and $\gamma = T_2 - iT_3$. The system

$$\dot{\alpha} = \frac{1}{2}[\beta, \alpha], \quad \dot{\gamma} = \frac{1}{2}[-\beta, \gamma], \quad \dot{\beta} = [\gamma, \alpha] \quad (3)$$

is equivalent to the system (2). The solutions of (3) can be interpreted as paths

$$\Phi(t) = \frac{\alpha(t)}{(z - p_1)^2} + \frac{\beta(t)}{(z - p_1)} + \frac{-\beta(t)}{(z - p_2)} + \frac{\gamma(t)}{(z - p_2)^2} : I \rightarrow T^*\mathcal{M}_{D^d}$$

in $T^*\mathcal{M}_{D^d}$. Let $r = \text{rk}(\mathfrak{g}^{\mathbb{C}})$ be the rank of $\mathfrak{g}^{\mathbb{C}}$, and let $\{Q_1, \dots, Q_r\}$ be a basis of the ring of $\text{Ad}_{G^{\mathbb{C}}}$ -invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$. For every $i = 1, \dots, r$ we have $Q_i(\Phi) \in H^0(\mathbb{CP}^1; K(D)^{d_i}) = H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$, where $d_i = \text{deg}(Q_i)$. Hitchin's map

$$\mathbf{H} : T^*\mathcal{M}_{D^d} \rightarrow \bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d_i)) \quad (4)$$

is given by $\mathbf{H}(\Phi) = (Q_1(\Phi), \dots, Q_r(\Phi))$.

Proposition 2 *Let $\Phi(t): I \rightarrow T^*\mathcal{M}_{D^d}$ be a solution of the system (3). Then $\mathbf{H}(\Phi(t))$ is constant with respect to $t \in I$. In other words, the components of Hitchin's map are the first integrals of the system (3).*

Proof: Multiplying by $(z - p_1)^2(z - p_2)^2$ gives a one-to-one correspondence between the meromorphic maps of the form (1) and the elements of $H^0(\mathbb{CP}^1; \mathcal{O}(2) \otimes \mathfrak{g}^{\mathbb{C}})$ represented as polynomials of degree two with coefficients

in $\mathfrak{g}^{\mathbb{C}}$. The projective transformation $z \mapsto (z - p_2)/(z - p_1)$ sends the marked points p_1 and p_2 into 0 and ∞ respectively, and the element $\Phi(z)$ assumes the form

$$\Phi(z) = \alpha + z\beta + z^2\gamma. \quad (5)$$

In terms of (5) the system (3) becomes the Lax equation

$$\dot{\Phi}_t(z) = \frac{1}{2} \left[\frac{d}{dz} \Phi_t(z), \Phi_t(z) \right]. \quad (6)$$

This tells us that for every $z_0 \in \mathbb{CP}^1$ and every $t_0 \in I$ the velocity $\dot{\Phi}_{t_0}(z_0)$ is tangent to the coadjoint orbit $\mathcal{O}_{\Phi_t(z_0)}$ through $\Phi_t(z_0) \in \mathfrak{g}^{\mathbb{C}*}$. Therefore the solution path $\Phi_t(z_0)$ lies on that orbit and thus for every $\text{Ad}_{G^{\mathbb{C}}}$ -invariant Q_i the map

$$Q_i(\Phi_t(z)) : I \longrightarrow H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$$

is constant along the solutions of (6). In other words, the components of Hitchin's map are the first integrals of the system (3). \square

2.2 A Hamiltonian system on $T^*G^{\mathbb{C}}$

Using left trivializations we see that

$$T^*\mathcal{M}_{D^d} = T^*(TG^{\mathbb{C}} \times TG^{\mathbb{C}}/G^{\mathbb{C}}) \cong T^*(\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}) \times T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}.$$

Let $\pi: T^*\mathcal{M}_{D^d} \rightarrow T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}$ be the projection, and let

$$q: T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}} \rightarrow T^*G^{\mathbb{C}}$$

the lifting of the map $[g_1, g_2] \mapsto g_2^{-1}g_1$ to the cotangent bundles. Let $t \mapsto \Phi(t)$ be a solution of the system (3). In this subsection we shall describe a Hamiltonian system on $T^*G^{\mathbb{C}}$ whose solutions are the paths $t \mapsto \gamma(t) = (q \circ \pi)(\Phi(t))$.

First we rewrite the system (3) in such a form that part of it is the canonical system for H . Define new maps

$$\begin{aligned} \mathcal{A} &= (\alpha, \gamma) = (T_2, T_2) + (iT_3, iT_3) & : I &\longrightarrow (\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})^* \\ \mathcal{B} &= (\beta, -\beta) = (-2iT_1, 2iT_1) & : I &\longrightarrow (\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})^*. \end{aligned}$$

Then (3) becomes

$$\dot{\mathcal{A}} = \frac{1}{2}[\mathcal{B}, \mathcal{A}] \quad (7)$$

$$\dot{\mathcal{B}} = [\tau(\mathcal{A}), \mathcal{A}], \quad (8)$$

where $\tau: (\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})^* \rightarrow (\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})^*$ is given by $\tau(x, y) = (-y, -x)$. The general solution of (7) is

$$\mathcal{A} = \text{Ad}_{(g_1, g_2)}^*(\alpha_0, \gamma_0) \quad , \quad \mathcal{B} = \frac{1}{2}(\dot{g}_1 g_1^{-1}, \dot{g}_2 g_2^{-1}) = \frac{1}{2}(\dot{g}_1 g_1^{-1}, -\dot{g}_1 g_1^{-1}),$$

where $\alpha_0, \gamma_0 \in (\mathfrak{g}^{\mathbb{C}})^*$ are constant. Here $\dot{g}_i g_i^{-1}$, for $i = 1, 2$, denotes appropriate elements in $(\mathfrak{g}^{\mathbb{C}})^*$. The paths $\Phi_t: I \rightarrow T^*\mathcal{M}_{D^d}$ that satisfy (7) are of the form

$$\Phi_t = \text{Ad}_{g_1(t)}^*(\alpha_0) + z \dot{g}_1(t) g_1^{-1}(t) + z^2 \text{Ad}_{g_2(t)}^*(\gamma_0). \quad (9)$$

Recall that Hitchin's map is given by $\mathbf{H}(\Phi) = (Q_1(\Phi), \dots, Q_r(\Phi))$. We can take $Q_1(\Phi) = \mathcal{K}(\Phi, \Phi)$. Write again $\Phi(z) = \alpha + z\beta + z^2\gamma$. Taking the coefficient of z^2 in $\mathcal{K}(\Phi, \Phi)$ gives $\tilde{H}(\Phi) = \mathcal{K}(\beta, \beta) + 2\mathcal{K}(\alpha, \gamma)$. This yields the function $\tilde{H}: T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}} \rightarrow \mathbb{C}$ given by

$$\tilde{H}([g_1, g_2], (\dot{g}_1 g_1^{-1}, \dot{g}_2 g_2^{-1})) = \frac{1}{2}\mathcal{K}(\dot{g}_1 g_1^{-1}, \dot{g}_1 g_1^{-1}) - \mathcal{K}(\text{Ad}_{g_1}^*(\alpha_0), \text{Ad}_{g_2}^*(\gamma_0)).$$

Write $g = g_2^{-1} g_1$. Then the map $q: T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}} \rightarrow T^*G^{\mathbb{C}}$ gives

$$q([g_1, g_2], (\dot{g}_1 g_1^{-1}, \dot{g}_2 g_2^{-1})) = (g_2^{-1} g_1, \text{Ad}_{g_2^{-1}}^*(\dot{g}_1 g_1^{-1} - \dot{g}_2 g_2^{-1})) = (g, \dot{g} g^{-1}). \quad (10)$$

The Ad^* -invariance of \mathcal{K} yields the function $H: T^*G^{\mathbb{C}} \rightarrow \mathbb{C}$ which satisfies the condition $\tilde{H} = H \circ q$, and is given by the formula

$$H(g, \dot{g} g^{-1}) = \frac{1}{2}\|\dot{g} g^{-1}\|_{G^{\mathbb{C}}}^2 - \mathcal{K}(\text{Ad}_g^*(\alpha_0), \gamma_0), \quad (11)$$

where $\|\alpha\|_{G^{\mathbb{C}}}^2 = \frac{1}{4}\mathcal{K}(\alpha, \alpha)$ for $\alpha \in \mathfrak{g}^{\mathbb{C}*}$.

Proposition 3 *Let ω be the natural complex symplectic form on the cotangent bundle $T^*G^{\mathbb{C}}$ and let the Hamiltonian $H: T^*G^{\mathbb{C}} \rightarrow \mathbb{C}$ be given by (11). Rewritten in terms of $(g, \dot{g} g^{-1}) \in T^*G^{\mathbb{C}}$, equation (8) assumes the form*

$$(\dot{g} g^{-1}) = [\text{Ad}_{g^{-1}}^*(\gamma_0), \alpha_0]. \quad (12)$$

*This equation is the non-trivial part of the canonical system of equations for the Hamiltonian system $(T^*G^{\mathbb{C}}, \omega, H)$.*

Proof: We get the above form of equation (8) immediately from (10). Let U be a neighbourhood of $g \in G^{\mathbb{C}}$ equipped with local coordinates (q_1, \dots, q_n) , centered at g and let $(q_1, \dots, q_n, p_1, \dots, p_n)$ be canonical coordinates on T^*U with respect to the natural symplectic structure ω . Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of $(\mathfrak{g}^{\mathbb{C}})^* = T_e^*G^{\mathbb{C}}$ such that $\mathcal{K}(\beta g^{-1}, \alpha_i) = p_i(\beta)$ for every $\beta \in T_g^*G^{\mathbb{C}}$. Choose a curve $t \mapsto x(t) \in G^{\mathbb{C}}$ given by $q_j(x(t)) = 0$ for $j \neq i$ and $q_i(x(t)) = t$. Then $\mathfrak{b}(\frac{d}{dt}|_{t=0} x_i(t)) = g \alpha_i$. The i -th equation of the system (12) has the form

$$\mathcal{K}((\dot{g}g^{-1}), \alpha_i) = \mathcal{K}([\text{Ad}_{g^{-1}}^*(\gamma_0), \alpha_0], \alpha_i) \quad (13)$$

On the other hand we have

$$\begin{aligned} \frac{\partial H}{\partial q_i} &= \frac{d}{dt}|_{t=0} \mathcal{K}(\text{Ad}_{x(t)}^*(\alpha_0), \gamma_0) \\ &= \mathcal{K}([\text{Ad}_g^*(\alpha_i), \text{Ad}_g^*(\alpha_0)], \gamma_0) = -\mathcal{K}([\text{Ad}_{g^{-1}}^*(\gamma_0), \alpha_0], \alpha_i) \end{aligned}$$

from which we see that the equation (13) is indeed the i -th non-trivial canonical equation

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

□

As a corollary of the above proposition and the connection between Nahm's equations and our Hamiltonian system we get the following theorem.

Theorem 1 *Let $T_1, T_2, T_3: I \rightarrow (\mathfrak{g}^{\mathbb{C}})^*$ be a solution of Nahm's system (2) and let this solution be given in the form*

$$(T_2 + iT_3)(t) = \text{Ad}_{g_1(t)}^*(\alpha_0), \quad (T_2 - iT_3)(t) = \text{Ad}_{g_2(t)}^*(\gamma_0)$$

$$iT_1(t) = \dot{g}_1 g_1^{-1}(t) = -\dot{g}_2 g_2^{-1}(t)$$

via the rewriting (3). Then the path $\gamma: I \rightarrow T^*G^{\mathbb{C}}$ given by

$$t \longrightarrow \gamma(t) = \left(g_2^{-1}(t) g_1(t), 2\dot{g}_1 g_1^{-1}(t) \right)$$

is a solution of the complex Hamiltonian system $(T^*G^{\mathbb{C}}, \omega, H)$.

□

3 Integrability

In this section we shall prove that the Hamiltonian system $(T^*G^{\mathbb{C}}, \omega, H)$ introduced above is integrable in the Liouville-Arnold sense. To this end we will use the representation of our system as a Hitchin system on $T^*\mathcal{M}_{D^d}$. First we define functions on $T^*G^{\mathbb{C}}$ which will turn out to be the Poisson-commuting integrals of our system. Let the element $\Phi \in T^*\mathcal{M}_{D^d}$ be of the form

$$\Phi = \frac{\text{Ad}_{g_1}^*(\alpha_0)}{(z-p_1)^2} + \frac{\dot{g}_1 g_1^{-1}}{(z-p_1)} + \frac{\dot{g}_2 g_2^{-1}}{(z-p_2)} + \frac{\text{Ad}_{g_2}^*(\gamma_0)}{(z-p_2)^2}. \quad (14)$$

Hitchin's map $\mathbf{H}: T^*\mathcal{M}_{D^d} \rightarrow \bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$ is $\text{Ad}_{G^{\mathbb{C}}}^*$ -invariant. Therefore formula (10) and the fact that $\dot{g}_1 g_1^{-1} = -\dot{g}_2 g_2^{-1}$ yield the map

$$\mathbf{H}_G: T^*G^{\mathbb{C}} \longrightarrow \bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$$

defined by

$$\mathbf{H}_G(g, \dot{g}g^{-1}) = \mathbf{H}(\Phi) = \mathbf{H}\left(\frac{\text{Ad}_g^*(\alpha_0)}{(z-p)^2} + \frac{\frac{1}{2}\dot{g}g^{-1}}{(z-p_1)} - \frac{\frac{1}{2}\dot{g}g^{-1}}{(z-p_2)} + \frac{\gamma_0}{(z-p_2)^2}\right). \quad (15)$$

Let now $\{e_i\}$ be an arbitrary basis of $\bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$ and let the functions $H_i: T^*G^{\mathbb{C}} \rightarrow \mathbb{C}$ be given by

$$H_i(g, \dot{g}g^{-1}) = \langle \mathbf{H}_G(g, \dot{g}g^{-1}), e_i \rangle. \quad (16)$$

Theorem 2 *The Hamiltonian system $(T^*G^{\mathbb{C}}, \omega, H)$ is completely integrable in the Liouville-Arnold sense. The $n = \dim G^{\mathbb{C}}$ Poisson commuting integrals of this system can be chosen from the functions defined in (16).*

The proof of this theorem is given in the following two subsections, more precisely in Corollary 1 and in Proposition 19.

3.1 Poisson-commutation of integrals

First we observe that the space $T^*\mathcal{M}_{D^d}$ is a symplectic quotient of the space $T^*(TG^{\mathbb{C}} \times TG^{\mathbb{C}})$ with respect to the natural action of $G^{\mathbb{C}}$. Trivializing by the

right translations we can write

$$T^*(TG^{\mathbb{C}} \times TG^{\mathbb{C}}) = \left\{ \left((g_1, g_2), (a_1, a_2) \right), (t_1, t_2, t_3, t_4) \right\},$$

where $\left\{ \left((g_1, g_2), (a_1, a_2) \right) \right\} = TG^{\mathbb{C}} \times TG^{\mathbb{C}}$, and t_i are the coordinates of cotangents. In this set of coordinates the moment map of the $G^{\mathbb{C}}$ -action is given by

$$\mu \left(\left((g_1, g_2), (a_1, a_2), (t_1, t_2, t_3, t_4) \right) \right) = \text{Ad}_{g_1}^*(t_1) + \text{Ad}_{g_2}^*(t_2),$$

so the elements $\Phi \in T^*\mathcal{M}_{D^d} = \mu^{-1}(0)/G^{\mathbb{C}}$ are of the form

$$\Phi = \frac{\text{Ad}_{g_1}^*(t_3)}{(z-p_1)^2} + \frac{\text{Ad}_{g_1}^*(t_1)}{(z-p_1)} + \frac{\text{Ad}_{g_2}^*(t_2)}{(z-p_2)} + \frac{\text{Ad}_{g_2}^*(t_4)}{(z-p_2)^2},$$

where $\text{Ad}_{g_1}^*(t_1) = -\text{Ad}_{g_2}^*(t_2)$. The paths satisfying the equation (7) lie in the subspace of $T^*\mathcal{M}_{D^d}$ which contains the elements $\Phi \in T^*\mathcal{M}_{D^d}$ with t_3 and t_4 fixed. In addition they do not depend on the variables (a_1, a_2) . Therefore these paths actually lie in the subspace

$$\left\{ \left(\left([g_1, g_2], (0, 0) \right), (t_1, t_2, \alpha_0, \gamma_0) \right) \right\} \subset T^*\mathcal{M}_{D^d}.$$

The space $T^*\mathcal{M}_{D^d}$ can be written as

$$T^*\mathcal{M}_{D^d} = T^*(\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}) \times T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}} = T^*\mathcal{N}_1 \times T^*\mathcal{N}_2,$$

and the symplectic structure ω on $T^*\mathcal{M}_{D^d}$ is the sum of cotangent symplectic structures ω_1 and ω_2 on $T^*\mathcal{N}_1$ and $T^*\mathcal{N}_2$, respectively. Let $\tilde{H}_i: T^*\mathcal{M}_{D^d} \rightarrow \mathbb{C}$ be the components of Hitchin's map (4) with respect to the basis $\{e_i\}$ used in (16) and let $T^*\mathcal{M}_{D^d} = T^*\mathcal{N}_1 \times T^*\mathcal{N}_2 = \{(x, y)\}$. Then

$$\begin{aligned} \{\tilde{H}_i, \tilde{H}_j\}(x_0, y_0) &= \{\tilde{H}_i(x, y_0), \tilde{H}_j(x, y_0)\}_1(x_0) \\ &+ \{\tilde{H}_i(x_0, y), \tilde{H}_j(x_0, y)\}_2(y_0). \end{aligned} \quad (17)$$

Define the map $\mathbf{K}: T^*(TG^{\mathbb{C}} \times TG^{\mathbb{C}}) \rightarrow \bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d))$ by

$$\mathbf{K} \left(\left((g_1, g_2)(a_1, a_2) \right), (t_3, t_1, t_2, t_4) \right) = (Q_1(\Psi), \dots, Q_r(\Psi)),$$

where $\Psi = \frac{t_3}{(z-p_1)^2} + \frac{t_1}{(z-p_1)} + \frac{t_2}{(z-p_2)} + \frac{t_4}{(z-p)^2}$ and let K_i be the components of \mathbf{K} with respect to the basis $\{e_i\}$. The functions K_i do not depend on

the base space variables, so they Poisson commute. But since the functions H_i are induced from K_i by the symplectic quotient construction $T^*\mathcal{M}_{D^d} = \mu^{-1}(0)/G^{\mathbb{C}}$, they also Poisson commute. In addition $\{\tilde{H}_i(x, y_0), \tilde{H}_j(x, y_0)\}_1 = 0$, since the functions $\tilde{H}_i(x, y_0)$ do not depend on the relevant base coordinates. Thus we get

$$\{\tilde{H}_i(x_0, y), \tilde{H}_j(x_0, y)\} = 0$$

for all pairs i, j . This proves the following proposition.

Proposition 4 *The functions $\tilde{H}_i(x_0, y): T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}} \rightarrow \mathbb{C}$ Poisson commute with respect to the canonical symplectic form ω on $T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}$.*

□

Corollary 1 *The functions $H_i: T^*G^{\mathbb{C}} \rightarrow \mathbb{C}$ defined by (16) Poisson-commute with respect to the natural symplectic structure on $T^*G^{\mathbb{C}}$.*

Proof: The map $q: T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}} \rightarrow T^*G^{\mathbb{C}}$ is symplectic, since it is a lifting of a map defined on the base spaces. From the definition we have $\tilde{H}_i(x_0, y) = \tilde{H}_i(y) = q^*(H_i)(y)$ and therefore

$$\{\tilde{H}_i, \tilde{H}_j\} = \{q^*(H_i), q^*(H_j)\} = q^*({H_i, H_j}) = 0$$

for the appropriate Poisson brackets. From the formula (10) it is easily seen that $q^*(f) = 0$ if and only if $f = 0$. □

3.2 Completeness of integrals

First we shall count the functions $H_i: T^*G^{\mathbb{C}} \rightarrow \mathbb{C}$. Let the element $\Phi \in T^*\mathcal{M}_{D^d}$ be written in the form $\Phi(z) = \text{Ad}_{g_1}^*(\alpha_0) + z \dot{g}_1 g_1^{-1} + z^2 \text{Ad}_{g_2}^*(\gamma_0)$, where z is the affine coordinate on \mathbb{CP}^1 , and let $Q_k: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathbb{C}$ be an element of a basis of the Ad_G -invariant polynomials with $\deg Q_k = d_k$. Then we can define the functions H_i by

$$\begin{aligned} Q_k(\Phi(z)) &= Q_k(\text{Ad}_{g_2^{-1}g_1}^*(\alpha_0) + z \text{Ad}_{g_2^{-1}}^*(\dot{g}_1 g_1^{-1}) + \gamma_0) \\ &= \sum_{i=0}^{2d_k} H_{i+l}(g, \dot{g}g^{-1}) z^i, \end{aligned} \tag{18}$$

where $l = \sum_{j=1}^{k-1} (2d_j + 1)$. This definition amounts to a particular choice of the basis $\{e_i\}$ mentioned previously. From the well known equation which states $\sum_{j=1}^r (2d_j - 1) = \dim \mathfrak{g}^{\mathbb{C}}$ (see e.g. [14]) we get that the number of functions H_i is $\sum_{j=1}^r (2d_j + 1) = \dim \mathfrak{g}^{\mathbb{C}} + 2r$. For every $k = 1, \dots, r$ we have

$$Q_k(\Phi(0)) = Q_k(\text{Ad}_g^*(\alpha_0)) = \text{const.} \quad , \quad Q_k(\Phi(\infty)) = Q_k(\text{Ad}_{g^{-1}}^*(\gamma_0)) = \text{const.}$$

which means that $2r$ of our functions are constant. Below we will prove that the remaining integrals $H_i: G^{\mathbb{C}} \rightarrow \mathbb{C}$ are functionally independent, that is, $dH_1 \wedge \dots \wedge dH_n \neq 0$ generically. This is obviously equivalent to the following proposition.

Proposition 5 *Let $\mathbf{H}_G: T^*G^{\mathbb{C}} \rightarrow \bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$ be the map given by (15). Then its derivative*

$$d(\mathbf{H}_G)_{(g, \dot{g}g^{-1})}: T_{(g, \dot{g}g^{-1})}(T^*G^{\mathbb{C}}) \rightarrow \bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d_i)) \quad (19)$$

has rank $n = \dim \mathfrak{g}^{\mathbb{C}}$.

Proof: We will show that already the restriction of $d(\mathbf{H}_G)_{(g, \dot{g}g^{-1})}$ to the vertical subspace $T_{\dot{g}g^{-1}} \mathfrak{g}^{\mathbb{C}*} = \mathfrak{g}^{\mathbb{C}*}$ of $T_{(g, \dot{g}g^{-1})}(T^*G^{\mathbb{C}})$ has rank n for a generic choice of $(g, \dot{g}g^{-1})$. Let $\Phi \in T^*\mathcal{M}_{D^d}$ be written in the affine coordinate in the form $\Phi = \frac{1}{z}\alpha + \beta + z\gamma$. Clearly, for a suitable choice of Φ , the functions $H^{i,j}$ given by the expansions

$$Q_j(\Phi(z)) = \sum_{i=-d_i}^{d_i} H^{i,j}(\alpha, \beta, \gamma) z^i \quad (20)$$

are the (reindexed) system of functions H_j given by (18). Let us introduce another indeterminate w and define

$$\Psi(z, w) = \frac{1}{w}\alpha + \beta + z\gamma.$$

Then we have $\Phi(z) = \Psi(z, z)$. For every $\text{Ad}_{G^{\mathbb{C}}}^*$ -invariant Q_i the expansion gives

$$Q_i(\Psi(z, w)) = \sum_{j=0}^{d_i} \left(\sum_{k=0}^{d_i-j} h_{i,k}^j \cdot w^{-k} \right) \cdot z^j \quad (21)$$

$$Q_i(\Psi(z, w)) = \sum_{j=0}^{d_i} \left(\sum_{k=0}^{d_i-j} h_{i,j}^k \cdot z^k \right) w^{-j}. \quad (22)$$

Let $Q_i(\delta+z\epsilon) = \sum_{i=0}^{d_i} f_\epsilon^{i,j}(\delta)z^i$, where $\epsilon, \delta \in \mathfrak{g}^{\mathbb{C}}$ and ϵ is constant. In their paper [18] Mishchenko and Fomenko proved that 1-forms $df_\delta^{i,j}$ (the differentiation is taken with respect to δ) span a subspace of dimension $b = \frac{1}{2}(\dim \mathfrak{g}^{\mathbb{C}} + r)$ in $\mathfrak{g}^{\mathbb{C}*}$. Since $\Psi(z, \infty) = \beta + z\gamma$ and $\Psi(0, w) = \beta + \frac{1}{w}\alpha$, this tells us that the forms $dh_{i,j}^0$ and $dh_{i,0}^j$ span b -dimensional subspaces $E_{(\beta,\gamma)}$ and $F_{(\beta,\alpha)}$ of $\mathfrak{g}^{\mathbb{C}*}$, respectively. For generic choices of α, β and γ these two spaces intersect transversally and therefore span $\mathfrak{g}^{\mathbb{C}*}$. The r -dimensional intersection is spanned by $dh_{i,0}^0$. From $Q_i(\Psi(z, z)) = Q_i(\Phi(z))$ and from (22) we get

$$dH^{i,j} = \sum_{k=0}^{d_i-j} dh_{i,j}^{k-j} \quad \text{for } j \leq 0 \quad , \quad dH^{i,j} = \sum_{k=0}^{d_i-j} dh_{i,k+j}^k \quad \text{for } j > 0 \quad , \quad (23)$$

where the $H^{i,j}$ are given by (20). Let τ be the real structure of $\mathfrak{g}^{\mathbb{C}}$, corresponding to the compact real form, and let, as usual, \mathcal{K} denote the Killing form on $\mathfrak{g}^{\mathbb{C}}$. Then $\alpha \rightarrow \mathcal{K}(\alpha, \tau(\alpha))$ defines a norm $\|\cdot\|$ on $\mathfrak{g}^{\mathbb{C}}$ and it also induces one on $(\mathfrak{g}^{\mathbb{C}})^*$. The forms $dh_{i,j}^k$ are polynomial functions of $(\alpha, \beta, \gamma) \in ((\mathfrak{g}^{\mathbb{C}})^*)^3$. More precisely, components of α occur in $dh_{i,j}^k$ with the power j , those of γ with the power k , and the components of β have the power $d_i - j - k$. Therefore, in each sum in (23) the first summand has the highest degree in β . In all other summands β occurs with degree at least two less than in the first. From this we see

$$\lim_{\|\beta\| \rightarrow \infty} \frac{dH^{i,j}}{\|dH^{i,j}\|} = \frac{dh_{i,0}^j}{\|dh_{i,0}^j\|} \quad \text{for } j \leq 0 \quad , \quad \lim_{\|\beta\| \rightarrow \infty} \frac{dH^{i,j}}{\|dH^{i,j}\|} = \frac{dh_{i,j}^0}{\|dh_{i,j}^0\|} \quad \text{for } j > 0 \quad .$$

Therefore for a generic choice of (α, β, γ) , and provided β is large enough, the forms $dH^{i,j}$ span the same space as the forms $\{dh_{i,0}^j, dh_{i,j}^0\}$, that is, the whole $(\mathfrak{g}^{\mathbb{C}})^*$. Choose a basis in $(\mathfrak{g}^{\mathbb{C}})^*$ and n 1-forms $dH^{i,j}$ that span $(\mathfrak{g}^{\mathbb{C}})^*$. Form the $n \times n$ matrix which has the $dH^{i,j}$ as columns, and denote its determinant by $F(\alpha, \beta, \gamma)$. Then F is a polynomial function of (α, β, γ) which is different from zero on an open set in $((\mathfrak{g}^{\mathbb{C}})^*)^3$. Therefore it is different from zero for a generic choice of (α, β, γ) . \square

4 Integrable systems on symmetric spaces

4.1 Symmetric spaces and real structures on $G^{\mathbb{C}}$

Our aim in this subsection is to represent an arbitrary Riemannian symmetric space as the common fixed point set of two involutions on a complex Lie group $G^{\mathbb{C}}$. First we recall some facts about symmetric spaces. Details and proofs can be found in [11]

A Riemannian manifold M is called a globally symmetric Riemannian space if every point $p \in M$ is a fixed point of an involutive isometry of M which takes any geodesic through p into itself as a curve, but reverses its parametrization. Every symmetric space is homogeneous. Let $M = G/U$, where $U \subset G$ is a subgroup of the Lie group G and let $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{u} = \text{Lie}(U)$ be the respective Lie algebras. Then M is symmetric if and only if there exists a vector subspace $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ and

$$[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u} \quad , \quad [\mathfrak{u}, \mathfrak{p}] \subset \mathfrak{p} \quad , \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{u} .$$

A Direct sum decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ satisfying the above condition is called a symmetric decomposition. It can be seen immediately that $\tilde{\mathfrak{g}} = \mathfrak{u} \oplus i\mathfrak{p}$ is a Lie algebra if and only if $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ is a symmetric decomposition. Clearly, $\tilde{\mathfrak{g}} = \mathfrak{u} \oplus i\mathfrak{p}$ is a symmetric decomposition as well.

Every Riemannian symmetric space is a cartesian product of irreducible Riemannian symmetric spaces. These fall into four types and were classified by E. Cartan. Let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ be a symmetric decomposition of a simple real algebra \mathfrak{g} . Then either \mathfrak{g} or $\tilde{\mathfrak{g}}$ is compact. The compact member of the pair $(M = G/U, \tilde{M} = \tilde{G}/U)$, where $\text{Lie}(\tilde{G}) = \tilde{\mathfrak{g}}$, is an irreducible symmetric space of type I, while the non-compact one is called its non-compact dual and is of type III. An elementary example of this duality is the pair (S^2, \mathcal{H}^2) , where S^2 is the 2-sphere, and \mathcal{H}^2 the hyperbolic 2-plane. Let now $\mathfrak{g} = \mathfrak{f} \times \mathfrak{f}$ for some compact simple real Lie algebra \mathfrak{f} , and let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ be the decomposition of \mathfrak{g} into +1 and -1 eigenspaces of the involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ given by $\theta(x, y) = (y, x)$. Denote again $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus i\mathfrak{p}$ and let F be the simple Lie group having \mathfrak{f} as its Lie algebra. Then $M = G/U \cong F$ is a symmetric space of type II, while $\tilde{M} = \tilde{G}/U \cong F^{\mathbb{C}}/F$ is its non-compact dual of type IV. The procedures described above yield all the irreducible symmetric spaces. Let $G^{\mathbb{C}}$

G , and \tilde{G} be the Lie groups corresponding to the Lie algebras $\mathfrak{g}^{\mathbb{C}}$, \mathfrak{g} and $\tilde{\mathfrak{g}}$, respectively. Here $\mathfrak{g}^{\mathbb{C}}$ denotes the common complexification of \mathfrak{g} and $\tilde{\mathfrak{g}}$. Let $\tau_G, \tau_{\tilde{G}} : G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ be the real structures of the complex Lie group $G^{\mathbb{C}}$ having G and \tilde{G} as their real forms. Call the Lie groups G and \tilde{G} related with respect to U if their respective Lie algebras are of the form $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ and $\tilde{\mathfrak{g}} = \mathfrak{u} \oplus i\mathfrak{p}$ and these decompositions are symmetric.

Proposition 6 *Let G be a real semi-simple Lie group and $M = G/U$ an irreducible Riemannian globally symmetric space and let \tilde{G} be related to G with respect to U . Then*

$$M = \{a \in G ; \tau_{\tilde{G}}(a^{-1}) = a\} .$$

In other words, the symmetric space M is the simultaneous fixed-point set of two involutions in $G^{\mathbb{C}}$, namely τ_G and $\sigma_{\tilde{G}}$, where $\sigma_{\tilde{G}}(g) = \tau_{\tilde{G}}(g^{-1})$.

Proof: Let $\tilde{\mathcal{H}} = G^{\mathbb{C}}/\tilde{G}$. Observe that $\tilde{\mathcal{H}} = \{g \in G^{\mathbb{C}}; \tau_{\tilde{G}}(g^{-1}) = g\}$. Suppose first that M is of type I and let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ be the corresponding symmetric decomposition. It is proved in [11] that $M = \exp(\mathfrak{p}) \subset \tilde{\mathcal{H}}$. Since G is compact, every element $g^c \in G^{\mathbb{C}}$ has a unique factorization $g^c = g \cdot h$, where $g \in G$ and $h \in \mathcal{H} \cong G^{\mathbb{C}}/G$. In the case of $g^c \in \tilde{\mathcal{H}}$ this factorization gives $g^c = g_{\mathfrak{p}} \cdot h_{i\mathfrak{u}}$ for $g_{\mathfrak{p}} \in M$ and $h_{i\mathfrak{u}} \in \exp(i\mathfrak{u})$. Since $\tau_G(g^c) = g_{\mathfrak{p}} \cdot h_{i\mathfrak{u}}^{-1}$, the fixed-point set of $\tau_G/\tilde{\mathcal{H}}$ is indeed M .

Let now M be of type III. Again $M = \exp(\mathfrak{p}) \subset \tilde{\mathcal{H}}$. In this case the map $\exp : (i\mathfrak{u} \oplus \mathfrak{p}) \rightarrow \tilde{\mathcal{H}}$ is a diffeomorphism. Define an involution $\vartheta : \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$ by $\vartheta(h) = \exp(-\alpha_{i\mathfrak{u}} + \alpha_{\mathfrak{p}})$, where $\exp^{-1}(h) = \alpha_{i\mathfrak{u}} + \alpha_{\mathfrak{p}}$ is the decomposition of $\exp^{-1}(h)$ into its $i\mathfrak{u}$ and \mathfrak{p} components. The fixed-point set of ϑ is M . Since $d\tau_{G_e} = d\vartheta_e$, we have $\vartheta = \tau_G/\tilde{\mathcal{H}}$ which proves the proposition for M of type III. Similar considerations provide the proof for types II and IV. \square

4.2 Real structures on $(T^*G^{\mathbb{C}}, \omega, H)$

Using Proposition 6, we shall now construct an integrable system on an arbitrary symmetric space $M = G/U$. We will also specify the form of Nahm's equations whose solutions give rise to the solutions of our new systems.

Let $\tau: G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ be the real structure corresponding to the real form $G \subset G^{\mathbb{C}}$ and let $C: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ be the real form of \mathbb{CP}^1 which in the affine coordinate z is given by conjugation $C(z) = \bar{z}$. If $p_1, p_2 \in \text{supp}(D)$ are real, we can define an involution

$$\mathbf{C} = C \otimes \tau : T^* \mathcal{M}_{D^d} \longrightarrow T^* \mathcal{M}_{D^d} \quad (24)$$

which on the subspace $(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}$ is given by $\mathbf{C}[g_1, g_2] = [\tau(g_1), \tau(g_2)]$. Let $T^* \mathcal{M}_{D^d}^{\mathbf{C}} \subset T^* \mathcal{M}_{D^d}$ be the fixed-point set of \mathbf{C} . Its elements satisfy the condition $\mathbf{C}(\Phi(z)) = d\tau(\Phi(\bar{z}))$, where $d\tau: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$ is the derivative of τ at $e \in G^{\mathbb{C}}$. If in addition the element Φ lies on some solution of the equation (7), then it is of the form

$$\Phi = \text{Ad}_{g_1}^*(\alpha_0) + z \dot{g}_1 g_1^{-1} + z^2 \text{Ad}_{g_2}^*(\gamma_0),$$

where $g_1, g_2 \in G$ and $\alpha_0, \gamma_0, (\dot{g}_1 g_1^{-1}) \in \mathfrak{g} = \text{Lie}(G)$.

Next, we consider a different real involution of $T^* \mathcal{M}_{D^d}$. Let p_1, p_2 be antipodal, for example, $p_1 = 0, p_2 = \infty$ and denote by $A: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ the antipodal map. Then we can define the involution

$$\mathbf{A} = A \otimes \tilde{\tau} : T^* \mathcal{M}_{D^d} \longrightarrow T^* \mathcal{M}_{D^d},$$

where $\tilde{\tau}: G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ is a real structure corresponding to another real form $\tilde{G} \subset G^{\mathbb{C}}$. On the subspace $(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}$ we now have $\mathbf{A}[g_1, g_2] = [(g_2^*)^{-1}, (g_1^*)^{-1}]$, where $g^* = \tilde{\tau}(g^{-1})$. In this case a solution Φ of (7) satisfies $\mathbf{A}(\Phi) = \Phi$ if it is of the form

$$\Phi = \text{Ad}_g^*(\alpha_0) + z \dot{g} g^{-1} + z^2 \text{Ad}_{(g^*)^{-1}}(d\tilde{\tau}(\alpha_0))$$

and $\dot{g} g^{-1} = -d\tilde{\tau}(\dot{g} g^{-1}) = (\dot{g} g^{-1})^*$. Choose now a Riemannian symmetric space $M = G/U$. Let the real forms G and \tilde{G} of $G^{\mathbb{C}}$ be related with respect to U , as defined in the Subsection 4.1, and let $\tau, \tilde{\tau}: G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ be the corresponding real structures with derivatives $d\tau, d\tilde{\tau}: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$. In the previous subsection we have seen that the symmetric space M is determined by the Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$.

Theorem 3 *Let the Hamiltonian function $H_M: T^*M \rightarrow \mathbb{R}$ be given by*

$$H_M(h, p) = \frac{1}{2} \|p\|_M^2 + \mathcal{K}(\text{Ad}_h^*(\alpha_0), \alpha_0^*), \quad (25)$$

where $p \in T_h^*M$ and $\|\cdot\|_M$ is the natural norm on M , and $\alpha \in \mathfrak{g}$.

(i) Let a solution $T_1, T_2, T_3: I \rightarrow (\mathfrak{g}^{\mathbb{C}})^*$ of Nahm's system yield a solution $\gamma(t): I \rightarrow T^*G^{\mathbb{C}}$ of the Hamiltonian system $(T^*G^{\mathbb{C}}, \omega, H)$, as explained in Theorem (1). Then $\gamma(t)$ is a solution of the Hamiltonian system (T^*M, ω, H_M) if and only if

$$T_1(I) \subset i\mathfrak{p} \quad , \quad T_2(I) \subset \mathfrak{u} \quad , \quad T_3(I) \subset i\mathfrak{p} .$$

(ii) The Hamiltonian system (T^*M, ω, H_M) is completely integrable in the Liouville-Arnold sense.

Proof: Denote by $T^*\mathcal{M}_{D^d}^{\mathbf{M}}$ the simultaneous fixed-point set of the involutions $\mathbf{C} = C \otimes \tau$ and $\mathbf{A} = A \otimes \tilde{\tau}$ defined above. First we will show that the system (T^*M, ω, H_M) is obtained from the system on $T^*\mathcal{M}_{D^d}^{\mathbf{M}}$ in the same way as the system $(T^*G^{\mathbb{C}}, \omega, H)$ was obtained from Hitchin's system on $T^*\mathcal{M}_{D^d}$. As before let $\pi: T^*\mathcal{M}_{D^d} \rightarrow T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}$ be the natural projection and $q: T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})G^{\mathbb{C}} \rightarrow T^*G^{\mathbb{C}}$ the map given by (10). Consider the restriction of $(q \circ \pi): T^*\mathcal{M}_{D^d} \rightarrow T^*G^{\mathbb{C}}$ to the subspace $T^*\mathcal{M}_{D^d}^{\mathbf{M}} \subset T^*\mathcal{M}_{D^d}$. We observe that $\pi(T^*\mathcal{M}_{D^d}^{\mathbf{M}})$ is the subspace of $T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}$ consisting of the elements of the form $([g, (g^*)^{-1}], (\dot{g}g^{-1}, -\dot{g}g^{-1}))$ with $\dot{g}g^{-1} = (\dot{g}g^{-1})^*$. From this we get

$$q\left([g, (g^*)^{-1}], (\dot{g}g^{-1}, (\dot{g}g^{-1})^*)\right) = \left(g^*g, \text{Ad}_{g^*}^*(\dot{g}g^{-1} + (\dot{g}g^{-1})^*)\right) = (h, p)$$

Since $g^*g \in G$ and since $\tau(g^*g) = \tau(g\tilde{\tau}(g^{-1})) = g^*g$, Proposition 6 tells us that $h = g^*g \in M$. It is then easily checked that p is dual to $dL_{h^{-1}}(\dot{h}) \in T_{[e]}M$ via the natural metric on M . Here $dL_{h^{-1}}: T_{[h]}M \rightarrow T_{[e]}M$ is the derivative of the left translation of M by $h = g^*g \in G$. Let \mathbf{H} be the restriction of Hitchin's map (4) to $T^*\mathcal{M}_{D^d}^{\mathbf{M}}$. The Ad-invariance of \mathbf{H} allows us to define the map

$$\mathbf{H}_{\mathbf{M}}: T^*M \longrightarrow \bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$$

by the formula

$$\mathbf{H}_{\mathbf{M}}(h, p) = \mathbf{H}(\text{Ad}_h(\alpha_0) + \frac{1}{2}z p - z^2\alpha_0^*) . \quad (26)$$

The coefficient of z^2 in $Q_1(x, y) = \mathcal{K}(x, y)$ yields the Hamiltonian H_M given by the expression (25) as expected.

We now prove (i). Observe that $\Phi = (T_2 + iT_3) - 2ziT_1 + z^2(T_2 + iT_3)$ and suppose that $\Phi \in T^*\mathcal{M}_{D^d}^{\mathbf{C}}$. From $(T_2 + iT_3), (T_2 - iT_3) \in \mathfrak{g}$ we get $T_2 \in \mathfrak{g}$ and $T_3 \in i\mathfrak{g}$. Obviously, $T_1 \in i\mathfrak{g}$. Now let Φ also be also an element of $T^*\mathcal{M}_{D^d}^{\mathbf{A}}$. This gives $d\tilde{\tau}(T_2 + iT_3) = (T_2 - iT_3)$. The real structure $d\tilde{\tau}: \mathfrak{g}^{\mathbf{C}} \rightarrow \mathfrak{g}^{\mathbf{C}}$ restricted to $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ is the involution having \mathfrak{u} as its +1 eigenspace and \mathfrak{p} as its -1 eigenspace. From this we see that $T_2(I) \subset \mathfrak{u}$ and $T_3(I) \subset i\mathfrak{p}$. From $d\tilde{\tau}(iT_1) = -iT_1$ we finally get $T_1(I) \subset i\mathfrak{p}$.

To prove (ii) we have to find $\dim(M)$ integrals of the system (T^*M, ω, H_M) which Poisson-commute. From the construction of the map \mathbf{H}_M given by (26) and from the discussion in Section 3 it is clear that the components of \mathbf{H}_M are $n = \dim(G)$ real Poisson-commuting functions, one of them being H_M . Since $\dim(M) < n$, they are not functionally independent therefore the only thing we have to prove is that the reduction to T^*M imposes only $\dim(G) - \dim(M) = \dim(U)$ relations among our functions and not more. This is equivalent to the map

$$d(\mathbf{H})_{(h,p)} : T_{(h,p)}(T^*M) \longrightarrow \bigoplus_{i=1}^r H^0(\mathbb{C}\mathbb{P}^1; \mathcal{O}(2d_i))$$

having rank $\text{rk}(d(\mathbf{H})_{\Phi})$ equal to $\dim(M)$ for a generic $\Phi \in T^*\mathcal{M}_{D^d}^{\mathbf{M}}$. But this in turn is a direct corollary of Proposition 5, where we showed that the map $d(\mathbf{H})_{G^{\mathbf{C}}}$ restricted to the vertical subspace has no kernel. \square

5 Examples

5.1 C. Neumann system

First we will describe the example that justifies the title of this paper. The variational problem describing the harmonic motion on \mathbb{S}^n is known as C. Neumann's problem since it was first described by Carl Neumann in [23]. More recently, this system was studied by many authors, see e.g. [19], [20], [4], [5], [3] and [21]. In Hamiltonian terms, the C. Neumann problem is given by the system $(T^*\mathbb{R}^{(n+1)}, \omega, H_N)$, where

$$H_N(q, p) = \frac{1}{2}\|p\|^2 - \langle Aq, q \rangle,$$

and by the constraints $\|q\| = 1$, $\langle q, p \rangle = 0$. Here A is a symmetric matrix of dimension $(n+1) \times (n+1)$ which we can assume to be diagonal without loss of generality.

For every n we have $\mathbb{S}^n = SO(n+1)/SO(n)$. The complexification of $SO(n+1)$ is the group $SO(n+1; \mathbb{C})$. According to Proposition 6 the sphere \mathbb{S}^n is the simultaneous fixed point set of the involutions

$$\tau, \tilde{\sigma}: SO(n+1; \mathbb{C}) \rightarrow SO(n+1; \mathbb{C})$$

given by

$$\tau(h) = \bar{h} \quad , \quad \tilde{\sigma}(h) = J\bar{h}^\top J \quad ,$$

where $J = \text{diag}(-1, 1, \dots, 1)$. Thus $\mathbb{S}^n \cong \mathbf{S} = \{h \in SO(n+1); JhJ = h^\top\}$.

Proposition 7 *Let the Riemannian symmetric space M from Theorem 3 be the sphere \mathbf{S} . Then the Hamiltonian system $(T^*\mathbf{S}, \omega, H_{\mathbf{S}})$ is the classical C. Neumann system.*

Proof: Each row h_i of the matrix $h \in \mathbf{S}$ is a vector in $\mathbb{R}^{(n+1)}$ with unit norm. Moreover, for each i the map $\pi_i: \mathbf{S} \rightarrow \mathbb{S}^n$ given by $\pi_i(h) = h_i$ is an isometry, provided we have multiplied the metric on the target \mathbb{S}^n by $\sqrt{n+1}$ which is the radius of \mathbf{S} . (For every $h \in SO(n+1)$ we have $\text{Tr}(hh^\top) = \text{Tr}(hh^{-1}) \equiv n+1$.) We give an explicit formula for $\pi_1: \mathbf{S} \rightarrow \mathbb{S}^n$. Let $\mathfrak{p} \subset \mathfrak{so}(n+1)$ be the subspace whose elements are matrices of the form

$$\alpha = \begin{pmatrix} 0 & a \\ -a^\top & 0 \end{pmatrix} \quad , \quad (27)$$

and $a = (a_1, \dots, a_n)$. Then $\mathbf{S} = \exp(\mathfrak{p})$. For $h = \exp(\alpha)$ we have

$$\pi_1(h) = \left(\cos \|a\|, \frac{a_1}{\|a\|} \sin \|a\|, \dots, \frac{a_n}{\|a\|} \sin \|a\| \right) \quad .$$

The motion $h(t): I \rightarrow \mathbf{S}$ is therefore uniquely determined by the motion $h_i(t): I \rightarrow \mathbb{S}^n$ for arbitrary $i = 1, \dots, n$. The opposite is of course also true. We shall show that $h_i(t): I \rightarrow \mathbb{S}^n$ is the harmonic motion on the sphere if and only if $h(t): I \rightarrow \mathbf{S}$ is a solution of the system $(T^*\mathbf{S}, \omega, H_{\mathbf{S}})$. Since the Killing form on $SO(n+1; \mathbb{C})$ is given by $\mathcal{K}(x, y) = \text{Tr}(x \cdot y^\top)$, we have

$$H_{\mathbf{S}} = \frac{1}{2} \|p\|^2 - \text{Tr}(\text{Ad}_h(\beta) \cdot J\beta^\top J) \quad .$$

Suppose β is a diagonal matrix. Then for the potential function $V(h)$ we have

$$V(h) = \text{Tr}(\text{Ad}_h(\beta) \cdot J\beta^\top J) = \text{Tr}\left((\beta h\beta) \cdot h^\top\right),$$

since $h^{-1} = h^\top = JhJ$. Denoting $\beta = \text{diag}(b_0, \dots, b_n)$ and $h = (h_{ij})$, we get

$$V(h) = \sum_{i=0}^n b_i \langle \beta \cdot h_i, h_i \rangle,$$

where $\langle -, - \rangle$ is the usual Euclidean scalar product on $\mathbb{R}^{(n+1)}$. From this we see that each h_i moves as a C. Neumann system and, since the motion of h on \mathbf{S} is completely determined by the motion of any h_i on \mathbb{S}^n , the proposition is proved. \square

The above construction differs slightly from that in Theorem 3. For the sake of simplicity the matrix β was taken to be diagonal and is not an element of the appropriate \mathfrak{g} . Nevertheless the theorem and its proof apply to the above example with a slight modification. The involution $\mathbf{A} = A \otimes \tilde{\tau}: T^*\mathcal{M}_{D^d} \rightarrow T^*\mathcal{M}_{D^d}$ has to be replaced by $\mathbf{A} = (-A) \otimes \tilde{\tau}$. As a result we get that in this case $T_1(I), T_3(I) \subset \mathfrak{p}$ and $T_2(I) \subset \mathfrak{u}$. A longer calculation (see [25]) shows that a faithful application of the construction of Theorem 3 also yields the C. Neumann system. The same remark applies to the next two examples.

5.2 C. Neumann systems on hyperbolic spaces

We have mentioned in Section 4.1 that symmetric spaces appear in dual pairs. The non-compact symmetric space dual to the sphere \mathbb{S}^n is the hyperbolic space $\mathbb{H}^n = SO_0(1, n)/SO(n)$. Here $SO_0(1, n)$ denotes the unit component of the group $SO(1, n)$ and $SO(1, n)$ consists of those elements $h \in SL(n; \mathbb{R})$ which preserve the quadratic form $-x_0^2 + x_1^2 + \dots + x_n^2$ in \mathbb{R}^{n+1} . It is therefore reasonable to expect that the C. Neuman system will have a non-compact analogue on \mathbb{H}^n .

We proceed following the scheme described in Section 4.1. First we define the Lie algebra $\tilde{\mathfrak{so}}(n+1) = \mathfrak{so}(n) \oplus i\mathfrak{p}$, where the subspace \mathfrak{p} consists of the elements of the form (27). Clearly we have $\tilde{\mathfrak{so}}(n+1)^\mathbb{C} = \mathfrak{so}(n+1; \mathbb{C})$, and $\tilde{\mathfrak{so}}(n+1)$ is the fixed-point set of the involution $d\tau: \alpha \mapsto J\bar{\alpha}J$ with $J = \text{diag}(-1, 1, \dots, 1)$.

The subspace $i\mathfrak{p}$ is the fixed-point set of the involution $d\tilde{\sigma}: \tilde{\mathfrak{so}}(n+1) \rightarrow \tilde{\mathfrak{so}}(n+1)$ given by $d\tilde{\sigma}(\alpha) = -\bar{\alpha} = \bar{\alpha}^\top$. The Lie algebra $\tilde{\mathfrak{so}}(n+1)$ is isomorphic to $\mathfrak{so}(1, n)$ via the map $\kappa: \tilde{\mathfrak{so}}(n+1) \rightarrow \mathfrak{so}(1, n)$ given by $\kappa(\alpha) = K \cdot \alpha \cdot \bar{K}$, where $K = \text{diag}(i, 1, \dots, 1)$. On the subalgebra $\mathfrak{so}(n) \subset \tilde{\mathfrak{so}}(n+1)$ the map κ is equal to the identity. From this we see, that the hyperbolic space $\mathbb{H}^n = SO_0(1, n)/SO(n)$ is isometric to the simultaneous fixed-point set $\mathbf{Hyp} \subset SO(n+1; \mathbb{C})$ of the involutions $\tau, \tilde{\sigma}: SO(n+1; \mathbb{C}) \rightarrow SO(n+1, \mathbb{C})$ given by $\tau(h) = J\bar{h}J$ and $\tilde{\sigma}(h) = \bar{h}^\top$.

Denote by $\mathbb{R}^{1,n}$ the $(n+1)$ -dimensional real space equipped with the Minkowski metric $\langle x, x \rangle_{1,n} = -x_0^2 + x_1^2 + \dots + x_n^2$ for $x = (x_0, x_1, \dots, x_n)$. It is well known that the space \mathbb{H}^n can be thought of as the unit sphere $\mathbb{H}^n = \{x \in \mathbb{R}^{1,n}; \langle x, x \rangle_{1,n} = 1\}$ in $\mathbb{R}^{1,n}$.

Proposition 8 *Let the space M from Theorem 3 be the hyperbolic space \mathbf{Hyp} . Then the Hamiltonian system $(T^*\mathbf{Hyp}, \omega, H_{\mathbf{Hyp}})$ describes the motion of a particle in $\mathbb{R}^{1,n}$ confined to the Minkowski unit sphere $\mathbb{H}^n \subset \mathbb{R}^{1,n}$ and under the influence of the force potential*

$$V_{hyp}(x) = \langle B(x), x \rangle_{1,n},$$

where B is a symmetric $(n+1) \times (n+1)$ matrix. In other words, the system $(T^*\mathbf{Hyp}, \omega, H_{\mathbf{Hyp}})$ is the Minkowskian version of C. Neumann's system.

Proof: Obviously, we only have to adapt the proof of Proposition 7. Let the map $\pi: \mathbf{Hyp} \rightarrow \mathbb{R}^{n+1}$ assign to the element $h \in \mathbf{Hyp}$ its first row. Due to $\mathbf{Hyp} = \exp(i\mathfrak{p})$ we have

$$\pi(h) = \left(\cosh \|a\|, \frac{ia_1}{\|a\|} \sinh \|a\|, \dots, \frac{ia_n}{\|a\|} \sinh \|a\| \right)$$

for some $a = (a_1, \dots, a_n)$. Let $L = \text{diag}(1, -i, \dots, -i)$ and denote by $\tilde{\pi}$ the composition $L \circ \pi$. Dividing the metric $\text{Tr}(hh^\top)$ on \mathbf{Hyp} by $\sqrt{n+1}$ turns the map $\tilde{\pi}: \mathbf{Hyp} \rightarrow \mathbb{H}^n \subset \mathbb{R}^{1,n}$ into an isometry. The rest of the proof is the same as that of Proposition 7, the only difference being that we have to replace the scalar product $\langle -, - \rangle$ by the product $\langle -, - \rangle_{1,n}$. \square

5.3 A many-body C. Neumann system

The Grassmannian manifold $\text{Gr}_{p,q}(\mathbb{R})$ is a Riemannian symmetric space and it can be obtained as the quotient $\text{Gr}_{p,q}(\mathbb{R}) = SO(n)/SO(p) \times SO(q)$, where $p+q = n$. The space $\mathbf{Gr} = \text{Gr}_{p,q}(\mathbb{R})$ is the fixed-point set of the real involutions $\tau, \tilde{\sigma}: SO(n; \mathbb{C}) \rightarrow SO(n; \mathbb{C})$ given by $\tau(h) = \bar{h}$ and $\tilde{\sigma}(h) = J_p \bar{h}^\top J_p$, where $J_p = \text{diag}(-1, \dots, -1, 1, \dots, 1)$ contains p negative entries. The rows h_i of a $h \in \mathbf{Gr}$ are orthonormal. Suppose $p < \lfloor \frac{n}{2} \rfloor$. Then the element h is determined by the choice of the first p rows h_1, \dots, h_p , while the choice of these is arbitrary as long as they are orthonormal. Let $\beta = (b_1, \dots, b_n)$ again be diagonal. Then, as before, we have

$$V(h) = \sum_{i=0}^n b_i \langle \beta(h_i), h_i \rangle. \quad (28)$$

Let $h(t): I \rightarrow T^*\mathbf{Gr}$ be a solution of the Hamiltonian system $(T^*\mathbf{Gr}, \omega, H_{\mathbf{Gr}})$. From (28) we see that each component $h_i(t)$ of $h(t)$ moves as a C. Neumann system. Suppose that the particle on \mathbb{S}^n represented by h_i has charge b_i . Then all these particles move as harmonic oscillators on \mathbb{S}^n under the influence of the potential $V(x) = \langle \beta(x), x \rangle$. This proves the following proposition.

Proposition 9 *The Hamiltonian system $(T^*\mathbf{Gr}, \omega, H_{\mathbf{Gr}})$ describes the motion of p particles h_1, \dots, h_p on the unit sphere \mathbb{S}^n under the influence of the harmonic potential*

$$V(x) = \langle \beta(x), x \rangle,$$

where $\beta = \text{diag}(b_1, \dots, b_n)$. The particle h_i has charge b_i . In addition, the system obeys the constraints given by $h_i \in \left(\text{span}(h_1, \dots, h_{i-1}) \right)^\perp$ for $i = 1, \dots, p$. \square

The system $(T^*\mathbf{Gr}, \omega, H_{\mathbf{Gr}})$ has its Minkowskian counterpart as well. Instead of the Grassmannian we have to take the symmetric space $SO_0(p, q)/SO(p) \times SO(q)$. Then we get a system of p particles h_1, \dots, h_p of different charges moving on the hyperbolic space \mathbb{H}^n , where $n = p + q$. They are constrained by the relations $h_i \in \left(\text{span}(h_1, \dots, h_{i-1}) \right)^\perp$. Here $h \perp k$ means that the two vectors are perpendicular with respect to the Minkowskian scalar product $\langle -, - \rangle_{1,n}$.

5.4 Magnetic terms, coadjoint orbits and the spherical pendulum

Let G be a real semi-simple Lie group and let $\tau: G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ be the real structure of $G^{\mathbb{C}}$ whose fixed point set is G . Then G is obviously the simultaneous fixed-point set of τ and $\tilde{\sigma}$, where $\tilde{\sigma}: G^{\mathbb{C}} \times G^{\mathbb{C}} \rightarrow G^{\mathbb{C}} \times G^{\mathbb{C}}$ is given by $\tilde{\sigma}(g_1, g_2) = (g_2, g_1)$. If we put $M = G$, then the system (T^*G, ω, H_G) describes the motion of a particle on G under the influence of the force potential $V(g) = \mathcal{K}(\text{Ad}_g^*(\beta), \tau(\beta))$

Denote by $T \subset G$ the stabilizer of $\beta \in \mathfrak{g}$ with respect to the coadjoint action of G on \mathfrak{g} . Then T acts on (T^*G, ω, H_G) . Let $\mu: T^*G \rightarrow \mathfrak{t}^*$ be the moment map of this action. The T -action enables us to take two different symplectic quotients of T^*G , namely $\mu^{-1}(0)/T$ and $\mu^{-1}(\alpha)/T$ for some nonzero $\alpha \in \mathfrak{t}^*$. Both quotients are diffeomorphic to $T^*\mathcal{O}$, where $\mathcal{O} \cong G/T$ is the coadjoint orbit containing β . They differ in their symplectic structures. The symplectic structure of $\mu^{-1}(0)/T$ is the usual cotangent structure ω_{ctg} on $T^*\mathcal{O}$, while the structure ω_α on $\mu^{-1}(\alpha)/T$ contains an additional magnetic term. More about this can be found in [17]. Application of the symplectic quotient construction of the system (T^*G, ω, H_G) yields the systems $(T^*\mathcal{O}, \omega_{ctg}, H_{\mathcal{O}})$ and $(T^*\mathcal{O}, \omega_{ctg} + \omega_{mg}, \tilde{H}_{\mathcal{O}})$, where $\mu^{-1}(0)/T \cong (T^*\mathcal{O}, \omega_{ctg})$ and $\mu^{-1}(\alpha)/T \cong (T^*\mathcal{O}, \omega_{ctg} + \omega_{mg})$ symplectically. By $H_{\mathcal{O}}$ and $\tilde{H}_{\mathcal{O}}$ we denoted the induced Hamiltonians. From Theorem (3) and the discussion in [17] we get the following result.

Proposition 10 *The system $(T^*\mathcal{O}, \omega_{ctg}, H_{\mathcal{O}})$ describes the motion of a particle on the coadjoint orbit under the influence of a potential force. The particle governed by the system $(T^*\mathcal{O}, \omega_{ctg} + \omega_{mg}, \tilde{H}_{\mathcal{O}})$ is additionally influenced by a magnetic force. Both systems are integrable in the Liouville-Arnold sense.*

We now take a closer look at the simplest case of the above proposition. The spherical pendulum is a classical mechanical system which describes the motion of a particle confined to the sphere $S^2 \in \mathbb{R}^3$ under the influence of the gravitational force. This system was studied already by Huyghens and more recently by Duistermaat in [9]. The spherical pendulum is given by the Hamiltonian

$$H(q, p) = \|p\|^2 + q_3$$

on the phase space $T^*\mathbb{R}^3 = \{(q_1, q_2, q_3, p_1, p_2, p_3)\}$ and by the constraints $\|q\| = 1$ and $\langle q, p \rangle = 0$. Let the group G be the group of rotations $SO(3)$. There is

an isometry $\mathcal{I} : (\mathfrak{so}(3), \mathcal{K}) \rightarrow (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, where \mathcal{K} is the Killing form on the Lie algebra $\mathfrak{so}(3)$ and $\langle \cdot, \cdot \rangle$ the standard Euclidean structure on \mathbb{R}^3 . Let $\beta \in \mathfrak{so}(3)$ correspond to the point $e_3 = (0, 0, 1)$ under this isometry.

Proposition 11 *Let $U(1)_\beta \subset SO(3)$ be the stabiliser of $\beta \in \mathfrak{so}(3)^*$. Then the action of $U(1)_\beta$ on $T^*SO(3)$ preserves the Hamiltonian H . Denote by $\mu : T^*SO(3) \rightarrow i\mathbb{R}$ be the corresponding moment map. The reduced system $(\mu^{-1}(0)/U(1)_\beta, \tilde{\omega}, \tilde{H})$ is the spherical pendulum. Let $\alpha \in i\mathbb{R}$ be a nonzero element. The reduced system $(\mu^{-1}(\alpha)/U(1)_\beta, \tilde{\omega}_\alpha, \tilde{H}_\alpha)$ describes the motion of a spherical pendulum in the field of a magnetic monopole lying in the centre of the sphere.*

Proof: Since $SO(3)/U(1)_\beta = S^2$ and since $\mu^{-1}(0)/U(1)_\beta = T^*(SO(3)/U(1)_\beta)$, we see that the phase space of the reduced space is indeed T^*S^2 . What remains to be shown is that the potential part \tilde{V} of the reduced Hamiltonian \tilde{H} is equal to $V_{sp}(q) = q_3$. Under the isometry \mathcal{I} the adjoint action of $SO(3)$ on $\mathfrak{so}(3)$ translates into the usual action of $SO(3)$ as the rotations on \mathbb{R}^3 . From this we see that the 2-sphere $\{q = A(e_3) \in \mathbb{R}^3; A \in SO(3)\}$ is precisely the quotient $SO(3)/U(1)_\beta$. Moreover,

$$\mathcal{K}(Ad_A(\beta), \beta) = \langle A(e_3), e_3 \rangle = \langle q, e_3 \rangle = q_3 ,$$

which proves the proposition. □

Appendix: Bundles with higher order framings

In this Appendix we give a brief description of the moduli spaces with higher order framings. In particular we shall prove Proposition 1 from the page 3.

Let D^d be a divisor of C with $\text{supp}(D^d) = \{p_1, \dots, p_k\}$ and let $\deg(p_i) = d_i$. Denote $\delta = \sum_{i=1}^k d_i$. A holomorphic structure A on P is given by the differential operator

$$\bar{\partial}_A : \Omega^0(C; \text{ad}P) \longrightarrow \Omega^{0,1}(C; \text{ad}P)$$

which satisfies the usual Leibnitz rule. Here $\text{ad}(P) \rightarrow C$ is the vector bundle associated to P via the Ad-representation; its fibre is $\mathfrak{g}^{\mathbb{C}}$. The holomorphic

structure A is by definition δ -stable if for every holomorphic subbundle F of $\text{ad}(P)$ we have

$$\frac{\deg F}{\text{rk} F} < \frac{\deg(\text{ad}(P))}{\text{rk}(\text{ad}(P))} + \delta \left(\frac{1}{\text{rk} F} - \frac{1}{\text{rk}(\text{ad}(P))} \right).$$

Denote by \mathcal{A}^{δ_s} the space of all δ -stable holomorphic structures on P and let $\mathcal{G}_{D^d} \subset \mathcal{G} = \text{Aut}(P)$ be given by

$$\mathcal{G}_{D^d} = \{g \in \mathcal{G}; g(p_i) = \text{id}, (dg)^{(l)}(p_i) = 0, \quad l = 1, \dots, d_i, i = 1, \dots, k\},$$

where $dg^{(l)}$ is the l -th derivative of the map g . The moduli space \mathcal{M}_{D^d} of stable holomorphic structures with d_i -framings at the points $p_i \in \text{supp}(D^d)$ is the quotient

$$\mathcal{M}_{D^d} = \mathcal{A}^{\delta_s} / \mathcal{G}_{D^d}.$$

Proof of Proposition 1: First we prove that for every $P \in \mathcal{M}_{D^d}$ the cotangent space $T^*\mathcal{M}_{D^d}$ can be identified with $H^0(C; \text{ad}(P) \otimes K(D^d))$ whose elements are meromorphic sections of $\text{ad}P \otimes K$ with poles of degree d_i at the points p_i . For every $A \in \mathcal{A}^{\delta_s}$ the tangent space $T_A\mathcal{A}^{\delta_s}$ is isomorphic to $\Omega^{0,1}(C; \text{ad}(P))$. The natural pairing between the spaces $\Omega^{0,1}(C; \text{ad}(P))$ and $\Omega^{1,0}(C; \text{ad}(P) \otimes K) = \Omega^0(C; \text{ad}(P) \otimes K)$ is given by

$$\langle \Phi, \Psi \rangle = \int_C \mathcal{K}(\Phi(z)dz \wedge \Psi(z)d\bar{z}).$$

Therefore $T^*\mathcal{A}^{\delta_s} = \mathcal{D}(C; \text{ad}(P) \otimes K) \times \mathcal{A}^{\delta_s}$, where $\mathcal{D}(C; \text{ad}(P) \otimes K)$ is the space of distributions on C with values in $\text{ad}(P) \otimes K$. Let $\mu : T^*\mathcal{A}^{\delta_s} \rightarrow \text{Lie}(\mathcal{G}_{D^d})^*$ be the moment map of the \mathcal{G}_{D^d} -action on $T^*\mathcal{A}^{\delta_s}$. Then $T^*\mathcal{M}_{D^d} = \mu^{-1}(0) / \mathcal{G}_{D^d}$. Let $\psi \in \text{Lie}(\mathcal{G}_{D^d})$ be arbitrary and let $f_\psi : T^*\mathcal{A}^{\delta_s} \rightarrow \mathbb{C}$ denote the Hamiltonian function belonging to the vector field on $T^*\mathcal{A}^{\delta_s}$ whose infinitesimal generator is ψ . As shown in [12] we have

$$f_\psi(A, \Phi) = \langle \mu(A, \Phi), \psi \rangle = \int_C \mathcal{K}(\bar{\partial}_A \psi \wedge \Phi).$$

Since $\Phi \in T^*\mathcal{M}_{D^d}$ if and only if $f_\psi(A, \Phi) = 0$ for every $\psi \in \text{Lie}(\mathcal{G}_{D^d})$, we see from [12] that Φ is holomorphic with respect to A away from $\text{supp}(D^d)$.

Let z be a local coordinate centered at p and suppose that $\psi(0) = 0$. We can assume that ψ vanishes outside some neighbourhood U of p . Then from Stokes' theorem and the condition $f_\psi(A, \Phi) = 0$ we get

$$f_\psi(A, \Phi) = \int_U \mathcal{K}(\bar{\partial}\psi \wedge \Phi) = \int_U \mathcal{K}(\psi \wedge \bar{\partial}(\Phi)) = \psi(0) = 0$$

for every suitable ψ . This implies that $\bar{\partial}(\Phi) = a\delta(0)$ and therefore $\Phi(z) = \frac{a}{z} + hol(z)$. It now follows easily that in the case where $\deg(p) = d$ the element $\Phi(z)$ is of the form

$$\Phi(z) = \sum_{i=1}^d \frac{a_i}{z^i} + hol(z).$$

(ii) It remains to prove the condition of the Mittag-Leffler type that the elements of the space $T_{[P]}^* \mathcal{M}_{D^d}$ must satisfy. Let $\mathcal{O}(\text{ad}(P) \otimes K)$ be the sheaf of A -holomorphic sections of $\text{ad}(P) \otimes K$ and let $\mathcal{O}_{D^d}(\text{ad}(P) \otimes K)$ be the sheaf of meromorphic sections with poles of degrees d_i at p_i . By \mathcal{PP} we denote the skyscraper sheaf of the principal parts at D^d . Then the exact sequence

$$0 \rightarrow \mathcal{O}(\text{ad}(P) \otimes K) \rightarrow \mathcal{O}_{D^d}(\text{ad}(P) \otimes K) \rightarrow \mathcal{PP} \rightarrow 0$$

gives rise to the following exact sequence on the cohomological level:

$$\dots \xrightarrow{i} H^0(\text{ad}(P) \otimes K(D^d)) \xrightarrow{p} H^0(\mathcal{PP}) \xrightarrow{\delta} H^1(\text{ad}(P) \otimes K) \rightarrow \dots \quad (29)$$

A principal part $\tilde{\lambda} \in H^0(\mathcal{PP})$ is in the image of the map p of the sequence (29) if and only if $\delta(\tilde{\lambda}) = 0$. Let U_i be a neighbourhood of p_i for $i = 1, \dots, k$ and let $\{U_0, U_1, \dots, U_k\}$ be a covering of C . The smooth $(1, 1)$ -form $\delta(\tilde{\lambda})$ is given by

$$\delta(\tilde{\lambda}) = \sum_{i=1}^k \bar{\partial} \left(f_i \cdot \sum_{j=1}^{d_i} \frac{\lambda_i^j}{z_i^j} \right), \quad (30)$$

where $z_i \in U_i$ is a local coordinate centred at p_i . Let f_i be a smooth function equal to 1 on the disc $\Delta_\epsilon \subset U_i$ with radius $\frac{\epsilon}{2}$ and center p_i and to 0 on $C \setminus U_i$. Suppose that the diameters of U_i are equal to ϵ . Since $\bar{\partial}_A(\Psi) = 0$ for every

$\Psi \in H_A^0(C; \text{ad}(P))$ the Leibnitz rule, Stokes' theorem, and the expression (30) give us

$$\begin{aligned} \int_C \mathcal{K}(\delta(\tilde{\lambda}), \Psi) &= \lim_{\epsilon \rightarrow 0} \int_{C \setminus \cup U_i} d\mathcal{K}\left(\sum_{i=1}^k \left(\sum_{j=1}^{\deg(p_i)} \frac{\lambda_i^j}{(z_i)^j}\right), \Psi\right) \\ &= \sum_{i=1}^k \int_{\partial U_i} \mathcal{K}\left(\sum_{j=1}^{\deg(p_i)} \frac{\lambda_i^j}{(z_i)^j}, \Psi\right). \end{aligned}$$

From the Cauchy theorem we finally get $\int_C \mathcal{K}(\delta(\tilde{\lambda}), \Psi) = \sum_{i=1}^k \mathcal{K}(\delta(\tilde{\lambda}), \Psi(p_i))$. Since by Serre duality $\delta(\tilde{\lambda}) = 0$ if and only if $\int_C \mathcal{K}(\delta(\tilde{\lambda}), \Psi) = 0$ for every $\Psi \in H_A^0(C; \text{ad}(P))$, the proposition is proved. \square

References

- [1] M. R. ADAMS, J. HARNAD, J. HURTUBISE, Isospectral Hamiltonian flows in finite and infinite dimensions II. Integration of flows, *Comm. Math. Phys.* 134 (1990), pp. 555-585
- [2] M. R. ADAMS, J. HARNAD, J. HURTUBISE, Darboux coordinates and Liouville-Arnold integration in loop algebras, *Comm. Math. Phys.* 155 (1993), pp. 385-413
- [3] M. R. ADAMS, J. HARNAD, E. PREVIATO, Isospectral Hamiltonian flows in finite and infinite dimensions I, *Comm. Math. Phys.* 117 (1988), pp. 451-500
- [4] M. ADLER, P. VAN MOERBEKE, Completely integrable systems, Euclidean Lie algebras and curves, *Advances in Mathematics*, 38 (1980), pp. 267-317
- [5] M. ADLER, P. VAN MOERBEKE, Linearization of Hamiltonian systems, Jacobi varieties, and representation theory, *Advances in Mathematics* 38 (1980), pp. 318-37
- [6] V.I. ARNOLD, Mathematical methods of classical mechanics, Second Edition, *Graduate Texts in Math.* 60 Springer-Verlag (1989)

- [7] A. BEAUVILLE, Jacobiennes des courbes spectrales et systemes hamiltoniens complètement integrables, *Acta Math.* 164 (1990), pp. 211-235
- [8] S.K. DONALDSON, Nahm's Equations and the Classification of Monopoles, *Comm. Math. Phys.* 96 (1984), pp. 387-407
- [9] J.J DUISTERMAAT, On global action-angle coordinates, *Comm. Pure and App. Math.* Vol. XXXIII, (1980) pp. 687-706
- [10] P.A. GRIFFITHS, J. HARRIS, Principles of Algebraic Geometry, John Wiley & Sons (1978)
- [11] S. HELGASON, Differential geometry and symmetric spaces, Academic Press, (1962)
- [12] N.J. HITCHIN, Stable bundles and integrable systems, *Duke Mathematical Journal* 54, (1987) pp. 91-114
- [13] N.J. HITCHIN, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* (3) 55 (1987) pp. 59-126
- [14] J.E. HUMPHREYS, Reflection groups and Coxeter groups, *Cambridge Studies in Advanced mathematics* 29, CUP (1990)
- [15] L. KATZARKOV, T. PANTEV, Stable G_2 bundles and algebraically completely integrable systems, *Compositio-Math.* 92 (1994), pp. 43-60
- [16] E. MARKMAN, Spectral curves and integrable systems, *Compositio-Math.* 93 (1994), pp. 255-290
- [17] J.E. MARSDEN, Lectures on mechanics, *London Mathematical Society Lecture Note Series*, 174, Cambridge University Press (1992)
- [18] A. S. MISHCHENKO, A. T. FOMENKO, Euler equations on finite-dimensional Lie groups, *Izv. Akad. Nauk SSSR*, Ser. Mat 42 (1978) no. 2 pp. 396-414
- [19] J. MOSER, Various aspects of integrable Hamiltonian systems, in Guckenheimer, Moser, Newhouse, Dynamical Systems, C.I.M.E. Lectures, Bressanone, Italy 1978, *Progress in Math.* 8, Birkhäuser(1980)

- [20] J. MOSER, Geometry of quadrics and spectral theory, in The Chern Symposium 1979, Springer-Verlag (1980)
- [21] D. MUMFORD, Tata Lectures on Theta II, *Progress in Mathematics*, Birkhäuser (1984)
- [22] W. NAHM, The construction of all self-dual monopoles by the ADHM method, in Monopoles in quantum field theories, World Scientific, Singapore (1982)
- [23] C. NEUMANN, De problemate quodam mechanico quod ad primam integralium ultraellipticorum revocatur, *J. reine angew. Math.* 56, (1859) pp. 46-63
- [24] A.G. REYMAN, M.A. SEMENOV-TIAN-SHANSKY, Integrable systems II, in *Encyclopedia of Mathematical Sciences*, vol 16, Springer-Verlag (1984)
- [25] P. SAKSIDA, Geometry of integrable systems, *D.Phil. Thesis*, Oxford 1994

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