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Square-edge graphs, partial cubes and their subclasses

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Abstract

An edge of a graph is called a square-edge if it lies in exactly one 4-cycle. A graph G is a square-edge graph if it contains a sequence of square-edges whose removal produces a spanning tree of G. Cube-free median graphs can be characterized as square-edge graphs which contain no Q_3^- as a subgraph. Among square-edge graphs, the class of partial cubes coincides with the class of semi-median graphs. A recognition algorithm for square-edge graphs of complexity $O(a(G)|E(G)|)$ is also presented.

Proposed **running title**: Square-edge graphs

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1 Introduction

An edge of a graph G is a *square-edge* if it lies in exactly one 4-cycle of G. Square-edges were introduced in [8], at least in the context of median graphs and partial cubes. From [8] we recall the following result to be used in the sequel.

Theorem 1.1 *Let e be a square-edge of a median graph G. Then $G - e$ is a median graph. Conversely, let e be a square-edge of a graph G and let $G-e$ be a median graph. If e does not lie in a subgraph of G isomorphic to Q_3^- then G is a median graph.*

Here we continue the investigation of the concept of square-edges and proceed as follows. In the rest of this section we recall the necessary concepts and notations. In Section 2 we introduce square-edge graphs which, for instance, include cube-free median graphs. We prove that square-edge graphs are bipartite and contain $m - n + 1$ 4-cycles, where n is the number of vertices of a given graph and m the number of its edges. In Section 3 we first show that cube-free median graphs can be characterized as square-edge graphs which contain no Q_3^- as a subgraph. We also give a short argument for a result from [7] and prove that among square-edge graphs, partial cubes and semi-median graphs coincide. In Section 4 we present a recognition algorithm for square-edge graphs of complexity $O(a(G)m)$, G and $a(G)$ its arboricity. We conclude the paper with an open problem.

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \Box H)$ whenever $ab \in E(G)$ and $x = y$, or, if $a = b$ and $xy \in E(H)$. *n-cube* Q_n is the Cartesian product of n copies of the complete graph on two vertices K_2. Q_3 is also shortly called the *cube*, and a graph is *cube-free* if it contains no Q_3 as a subgraph. Q_3^- denotes the graph obtained from the 3-cube by removing one of its vertices.

The *distance* $d_G(u, v)$ between vertices u and v of a graph G will be the usual shortest path distance. A subgraph H of a graph G is an *isometric* subgraph, if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$ and H is *convex* if for all $u, v \in V(H)$ all the shortest u, v-paths lie in H. *Partial cubes* are isometric subgraphs of n-cubes. A *median graph* is a connected graph such that, for every triple of its vertices, there is a unique vertex lying on a geodesic (i.e. shortest path) between each pair of the triple.

For an edge ab of a graph G let

$$\begin{aligned}
W_{ab} &= \{w \mid w \in V(G),\ d_G(w, a) < d_G(w, b)\}\,, \\
U_{ab} &= \{u \in W_{ab} \mid u \text{ is adjacent to a vertex in } W_{ba}\}\,, \\
F_{ab} &= \{uv \mid u \in U_{ab}, v \in U_{ba}\}\,.
\end{aligned}$$

A partial cube G is a *semi-median* graph if the sets U_{ab} induce connected subgraphs. Median graphs form a proper subclass of semi-median graphs, and, by the definition, semi-median graphs form a proper subclass of partial cubes.

Perhaps the most important and useful tool for studying median graphs, partial cubes and related classes of graphs is the expansion procedure, first studied by Mulder [9, 10]. Let G' be a graph and let G'_1 and G'_2 be subgraphs of G' such that $V(G'_1) \cap V(G'_2) \neq \emptyset$ and $V(G'_1) \cup V(G'_2) = V(G')$. Assume in addition that G'_1 and G'_2 are isometric subgraphs of G' and that there is no edge between a vertex of $G'_1 \setminus G'_2$ and a vertex of $G'_2 \setminus G'_1$. An *expansion* of a graph G' (with respect to G'_1 and G'_2) is a graph G, obtained from G' in the following way.

(i) Replace each vertex $v \in V(G'_1) \cap V(G'_2)$ by adjacent vertices v_1 and v_2.

(ii) Join v_1 and v_2 to all neighbors of v in $V(G'_1) \setminus V(G'_2)$ and $V(G'_2) \setminus V(G'_1)$, respectively.

(iii) If $v, u \in V(G'_1) \cap V(G'_2)$ are adjacent in G', then join v_1 to u_1 and v_2 to u_2.

Let $G'_0 = G'_1 \cap G'_2$. Then the expansion is *connected* (*convex*) if G'_0 is connected (convex). In 1978 Mulder [9, 10] proved his convex expansion theorem: A graph is a median graph if and only if it can be obtained from the one vertex graph by a sequence of convex expansions. Later Chepoi [2] proved the analogous result for partial cubes. They can be obtained by a sequence of expansions from the one vertex graph. Recently it was demonstrated that semi-median graphs are the graphs that can be obtained by a sequence of connected expansions from the one vertex graph [5, 6].

The Djoković-Winkler's relation Θ introduced implicitly in [4] and explicitly in [11] is defined on the edge-set of a graph in the following way. Two edges $e = xy$ and $f = uv$ of a graph G are in relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u) .$$

Θ need not be transitive, in fact Winkler [11] proved, that a bipartite graph is a partial cube if and only if Θ is transitive. Hence is partial cubes and in particular in semi-median graphs and median graphs, Θ is an equivalence relation.

Finally, for a vertex set $X \subseteq V(G)$ we write $\langle X \rangle$ for the induced subgraph of G on the vertex set X.

2 Square-edge graphs

Let G be a graph and suppose that there exists a sequence of connected graphs $G = G_j$, G_{j-1}, ..., $G_0 = T$, and a sequence of edges $e_j, e_{j-1}, \ldots, e_1$, where

(i) $G_i = G_{i+1} - e_{i+1}$, for $i = j-1, j-2, \ldots, 0$,

(ii) e_i is a square-edge of G_i, for $i = j, j-1, \ldots, 1$, and

(iii) T is a spanning tree of G.

Then we say that G is a *square-edge graph* and the sequence of edges $e_j, e_{j-1}, \ldots, e_1$ is a *square-edge sequence*. Example of a square-edge graph is given in Fig. 1.

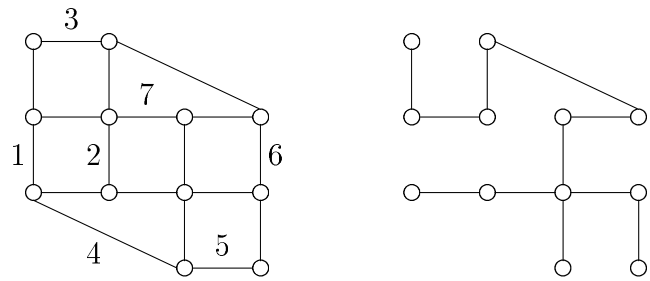

Figure 1: A square-edge graph with its square-edge sequence

Proposition 2.1 *Square-edge graphs are bipartite.*

Proof. Let G be a square-edge graph with a square-edge sequence $e_j, e_{j-1}, \ldots, e_1$. Let $G = G_j$, $G_i = G_{i+1} - e_{i+1}$ for $j - 1 \geq i \geq 0$ and set $G_0 = T$. Clearly, T is bipartite. We need to show that G_{i+1} is bipartite provided that G_i is such. Let $e_{i+1} = uv$. As G_i is bipartite, $d_{G_i}(u, v) = 3$. This means that u and v belong to different bipartition sets of G_i an so G_{i+1} is bipartite as well. $\square$

Note that square-edge graphs contain no $K_{2,3}$ as a subgraph. Indeed, every edge of $K_{2,3}$ is contained in two 4-cycles, thus no edge of $K_{2,3}$ will be removed in an eliminating process. Analogously we see that a square-edge graph contains no Q_3 as a subgraph.

Trees are trivial examples of square-edge graphs. It is also easy to see that complete grids, i.e. Cartesian products of two paths are square-edge graphs. In order to find more interesting examples of square-edge graphs, we recall the following result from [6]:

Proposition 2.2 *A graph is a $(K_2 \Box C_{2t})$-free semi-median graph, $t \geq 2$, if and only if it can be obtained from the one vertex graph by a connected expansion procedure, in which every expansion step G'_0 is a tree.*

to prove the following lemma:

Lemma 2.3 *Let G be a graph and let F be its Θ-equivalence class with at least two edges. Then F contains at least two square-edges in each of the following two cases:*

(i) G is a $(K_2 \Box C_{2t})$-free semi-median graph, $t \geq 2$;

(ii) G is a cube-free median graph.

Proof. (i) Let F be a Θ-equivalence class of G with at least two edges. Let $ab \in F$. It follows from Proposition 2.2 that $\langle U_{ab} \rangle$ is a tree, hence it contains at least two vertices of degree 1, say x and y. If xx' and yy' are the edges from F then by the expansion theorem we conclude that xx' and yy' are square-edges.

(ii) If G is a median graph, then G is also a semi-median graph. Moreover, if G is cube-free then it can be obtained from the one vertex graph by a (convex) expansion procedure, in which every expansion step is done with respect to a convex cover with a convex tree as intersection, cf. [7]. Thus G is a $(K_2 \Box C_{2t})$-free semi-median graph, and we are in the first case. □

Corollary 2.4 *Cube-free median graphs are square-edge graphs. In particular, the Cartesian product $T_1 \Box T_2$ of trees T_1 and T_2 is a square-edge graph.*

Proof. Let G be a cube-free median graph. Then by Lemma 2.3 (ii) it has a square-edge e and by Theorem 1.1 the graph $G - e$ is a cube-free median graph. Induction completes the proof that cube-free median graphs are square-edge graphs.

Cartesian products of median graphs are median graphs. As trees are median graphs and $T_1 \Box T_2$ is clearly cube-free, $T_1 \Box T_2$ is a cube-free median graph. □

Another interesting property of square-edge graphs is the following.

Proposition 2.5 *Let G be a square-edge graph with n vertices and m edges. Then the number of 4-cycles in G is equal to $m - n + 1$.*

Proof. Let $G = G_j$, G_{j-1}, …, $G_0 = T$ and $e_j, e_{j-1}, \ldots, e_1$, be a square-edge sequences of G.

If $j = 0$ then G is a tree. Hence $m = n - 1$ and the number of 4-cycles is zero.

Suppose now that $j > 0$ and that the lemma holds for any square-edge graph with smaller number of 4-cycles. Clearly, G_{j-1} is a square-edge graph and the number of 4-cycles of G_{j-1} is one less that the number of 4-cycles of G. By the induction hypothesis the number of 4-cycles of G_{j-1} is equal $(m - 1) - n + 1$. We conclude that the number of 4-cycles of G is $m - n + 1$. □

To show that the converse of Proposition 2.5 does not hold in general, consider the graph G which we obtain from $C_6 \Box C_6$ by adding a path of length two between diagonal vertices of one of its squares. Clearly, G has 37 vertices, 74 edges and 36+2=38 squares and so $m - n + 1 = 38$. (We note that one can find easier examples of this type, but the one presented here is also tiled, cf. the next section.)

We conclude this section with the following result, which is in particular useful from the algorithmic point of view.

Theorem 2.6 *Let G be a square-edge graph and let e^* be a square-edge of G. Then $G^* = G - e^*$ is a square-edge graph.*

Proof. Suppose that the theorem is false and let G be a counterexample with $|E(G)|$ as small as possible. Let $S = e_j, e_{j-1}, \ldots, e_1$ be an arbitrary square-edge sequence of G and let $G_j = G$ and $G_i = G_{i+1} - e_{i+1}$ for $0 \leq i \leq j-1$.

Suppose first that $e^* = e_k$ for some $1 \leq k \leq j$. Clearly, as G^* does not have a square-edge sequence, $k < j$. We claim that $S^* = e_j, e_{j-1}, \ldots, e_{k+1}, e_{k-1}, \ldots, e_1$ is a square-edge sequence of G^*. Since $e^* = e_k$, it is enough to see that $e_j, \ldots, e_{k+1}$ form a part of a square-edge sequence in G^*. Assume thus that e_t cannot be removed from $G_t - e^*$ for some t with $j > t \geq k+1$. Thus, e_t must lie in a common square with e_k. However, since S is a square-edge sequence of G, this implies that $e_k = e^*$ lies on two different 4-cycles of G, which is not possible. It follows that S^* is a square-edge sequence of G^*, a contradiction.

We may thus assume that e^* does not belong to any square-edge sequence of G. Let a, b and c be the edges which, together with e^*, form the unique square containing e^*. Clearly, at least one of the edges a, b and c belongs to S. We may assume that $a = e_k$ is the first among these three edges that appears in S. If $k = j$, then it is easy to see that $e^*, e_{j-1}, \ldots, e_1$ is a square-edge sequence of G containing e^*, which is not possible. Therefore $k < j$. Note that e^* is a square-edge in G_k. Now, by the minimality assumption, we may assume that $G_k - e^*$ is a square-edge graph and that there is a square-edge sequence of G_k of the form $e^*, e'_{k-1}, \ldots, e'_1$. But then $e_t, \ldots, e_{k+1}, e^*, e'_{k-1}, \ldots, e'_1$ is a square-edge sequence of G containing e^*. This final contradiction completes the proof. $\square$

3 Square-edge graphs and subclasses of partial cubes

In this section we present some results which demonstrate that square-edge graphs are not only interesting as such, but can also be used in studying (subclasses of) partial cubes. We begin with the following characterization of cube-free median graphs, which follows from Corollary 2.4. (See [1] for two very interesting characterizations of this class of graphs.)

Proposition 3.1 *A graph G is a cube-free median graph if and only if G is a square-edge graph and contains no Q_3^- as a subgraph.*

Proof. Let G be a cube-free median graph. Then G is also Q_3^--free, for otherwise a Q_3^- would give rise to a 3-cube. G is a square-edge graph by Corollary 2.4.

Conversely, if G is Q_3^--free, then it is clearly cube-free. Moreover, by Theorem 1.1 G is also a median graph and so the induction on $|E(G)|$ completes the proof. $\square$

Let G be a cube-free median graph with n vertices, m edges and k equivalence classes of the relation Θ. Then by Corollary 2.4 G has a square-edge sequence. Note that whenever we remove a square-edge, the number of Θ-classes increases by one. We end up with a spanning tree, thus finally we have $n-1$ Θ-classes. Since we have removed $m-n+1$ edges, we conclude that $k+(m-n+1)=n-1$, i.e. $2n-m-k=2$. Thus, in a cube-free median graph we have $2n-m-k=2$, a result established in [7].

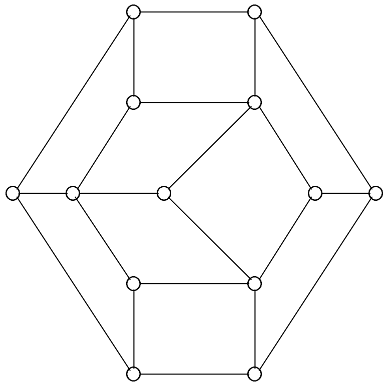

Figure 2: A semi-median graph

We can not apply the above argument to semi-median graphs, since semi-median graphs are not hereditary for removing square-edges. Consider, for instance, the semi-median graph G from Fig. 2. Any edge e of its outer 6-cycle is a square-edge, but $G-e$ is not a semi-median graph. (In fact, $G-e$ is not even a partial cube.)

For the sake of simplicity we will next consider a cycle also as the set of its edges. An *even* graph is a graph whose every vertex has even degree. Let H be a even subgraph of a graph G. Then a set of 4-cycles $\mathcal{F}=\{C_1, C_2, \ldots, C_p\}$ in G is a *tiling* of H if

$$H = C_1 \oplus C_2 \oplus \cdots \oplus C_p \,.$$

Call a graph G *tiled* if every cycle of G has a tiling.

Theorem 3.2 *Square-edge graphs are tiled.*

Proof. Suppose that the theorem is false and let G be a counterexample with $|E(G)|$ as small as possible. Then there exits a cycle C of G which is not tiled.

By the minimality there exists a square-edge $e=ab$ such that the graph $G^* = G\backslash\{e\}$ is a tiled square-edge graph.

Assume that $e \notin C$. Then there is a tiling $\mathcal{F}$ of C in G^*, which is also a tiling of C in G, a contradiction.

So assume that $e \in C$. Let $C^4 = abcda$ be the 4-cycle in G which contains e. (Since e is a square-edge, this 4-cycle is unique.) Let $C^* = C^4 \oplus C$. Graph C^* is even, hence there exists a decomposition $\mathcal{F} = \{C_1^*, \ldots, C_p^*\}$ of C^* in G^* into pairwise edge disjoint cycles. (Note that in our case p is always ≤ 3, but we will complete the proof without this fact.)

Since C_i^* $(1 \leq i \leq p)$ is a cycle of G^*, there exists a tiling $\mathcal{F}_i^*$ of C_i^* in G^*. Since the cycles of $\mathcal{F}$ are pairwise edge disjoint, we obtain that

$$C^* = C_1^* \oplus C_2^* \oplus \cdots \oplus C_p^*.$$

This implies that

$$\mathcal{F}^* = \mathcal{F}_1^* \oplus \mathcal{F}_2^* \oplus \cdots \oplus \mathcal{F}_p^*$$

is a tiling of C^*. Since $C^* = C^4 \oplus C$, it follows that $\mathcal{F}^* \cup C^4$ is a tiling of C, which is a contradiction. □

Using Theorem 3.2 we can prove the following result which connects partial cubes with semi-median graphs.

Corollary 3.3 *Let G be a square-edge graph. Then G is a partial cube if and only if G is a semi-median graph.*

Proof. As semi-median graphs are partial cubes, we only need to show that a square-edge graph which is a partial cube is also a semi-median graph. It is possible to show this directly by a lengthy proof, but instead we recall a theorem from [6] which states that a graph is a semi-median graph if and only if it is a tiled partial cube. Combining this result with Theorem 3.2 the result follows. □

4 Recognizing square-edge graphs

In this section we present an algorithm of complexity $O(a(G)m)$ which recognizes square-edge graphs, and find a square-edge sequence, if one exists. The algorithm depends on the work of Chiba and Nishizeki [3].

The first part of our algorithm is basically analogous to Algorithm C4 from [3]. It finds all the quadrangles of a given graph and prepare some data structure for the second part. For each vertex v of a graph it finds all the quadrangles containing v. Let w be a vertex with $d_G(v, w) = 2$. Then, as square-edge graphs contain no $K_{2,3}$ as a subgraph, there are at most two potential neighbors of both w and v. They are denoted by f_w and s_w in the algorithm. If on the vertices v, w, f_w, s_w we find an induced triangle, the graph is rejected, since a square-edge graph is bipartite. Every quadrangle, found by the algorithm, is stored in the set Q_c which contains its four edges. The indices of the quadrangles containing the edge e are stored in the set L_e.

In the second part the algorithm constructs a square-edge sequence. The array I represents the list of indices of Q. The indices are sorted such that the quadrangles with a square-edge are at the beginning of I. Applying Theorem 2.6 the algorithm at each step tries to find a quadrangle containing a square-edge. If no such edge is found, the graph is rejected; otherwise the array S representing the square-edge sequence is augmented and the sets L_e are updated for every edge e of the quadrangle involved. If the updated set L_e contains a square-edge, index of its quadrangle is set at the beginning of the list I.

The procedure is repeated until $I = \emptyset$. If we end up with a tree, the graph is accepted.

Procedure SEQUENCE(G);
{ Let $G = (V, E)$ be a connected graph. }
begin

1. Finding quadrangles and computing L-sets

  $c := 0$; { Quadrangles counter }
  Sort the vertices in V in a way that $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$;
  for each vertex $v \in V$ **do begin** $f_v := 0$; $s_v := 0$; **end**;
  for $i := 0$ to n **do begin**
    for each vertex u adjacent to v_i **do**
      for each vertex $w \neq v_i$ adjacent to u **do begin**
        if $v_i w \in E$ **then** REJECT;
        if $f_w = 0$ **then** $f_w := u$
          else if $s_w = 0$ **then** $s_w := u$ **else** REJECT;
      end;
    for each vertex w with $s_w \neq 0$ **do begin**
      if $f_w s_w \in E$ **then** REJECT;
      $c := c + 1$; $Q_c = \{ v_i f_w, v_i s_w, w f_w, w s_w \}$;
      $L_{v_i,f_w} := L_{v_i,f_w} \cup \{c\}$; $L_{v_i,s_w} := L_{v_i,s_w} \cup \{c\}$;
      $L_{w,f_w} := L_{w,f_w} \cup \{c\}$; $L_{w,s_w} := L_{w,s_w} \cup \{c\}$;
    end;
    for each vertex w with $f_w \neq 0$ **do begin** $f_w := 0$; $s_w := 0$; **end**;
    Delete the vertex v_i from G and let G be the new graph;
  end;

2. Searching for a square-edge sequence

  Sort the indices of Q such that if exists $e \in Q_i$ with $|L_e| = 1$, then put i in the beginning of list I;
  $k := 1$;

while there is an index in I **do begin**
Let i be the first index of I;
Let e_j, $j \in \{1, 2, 3, 4\}$ be the edges of Q_i such that $|L_{e_1}| = 1$;
if $|L_{e_1}| \neq 1$ **then** REJECT;
$S_k := e_1$; $k := k + 1$;
for $j := 2$ **to** 4 **do** $L_{e_j} := L_{e_j} \setminus \{ i \}$;
for $j := 2$ **to** 4 **do**
if $|L_{e_j}| = 1$ **then** put $\ell \in L_{e_j}$ at the beginning of the list I;
Delete i from I;
end;
if $G \setminus S$ is a tree **then** ACCEPT **else** REJECT;
end.

Recall that the *arboricity* $a(G)$ of a graph G is the minimum number of edge-disjoint spanning forests into which G can be decomposed.

Theorem 4.1 *Let G be a graph with $|E(G)| = m$ edges. Then Algorithm SEQUENCE decides in $O(a(G)m)$ time whether G is a square-edge graph and finds a square-edge sequence if one exists.*

Proof. The correctness of the algorithm follows from the above discussion.

Since the first part of the algorithm is similar to Algorithm C4, it can be shown along the same lines as in [3] that this part of the above algorithm obtains all the quadrangles of G in $O(a(G)m)$ time.

The second part of the algorithm can be divided into three steps. In the first step the indices of the quadrangles are ordered such that the quadrangles with a square-edge are at the beginning of the list. The number of quadrangles is bounded with $O(a(G)m)$, thus this ordering can be done within the same time.

Concerning the while loop, note that the body of it can be executed in a constant time. Therefore, the total running time is again bounded with the number of quadrangles.

In the last step we need to check if a given graph is a tree. This can clearly be done in $O(n)$ time. We conclude that the total running time of the algorithm is bounded by $O(a(G)m)$. □

Corollary 4.2 *Algorithm SEQUENCE obtains a square-edge sequence of a cube-free median graph in $O(n)$ time.*

Proof. We have seen that the first part of SEQUENCE runs in the same time as Algorithm C4. It was shown in [3] that the complexity of C4 is at most:

$$O(m) + O(n) + O(\sum_{uv \in E(G)} min\{d(u), d(v)\}) .$$

Let S be a square-edge sequence of G and let T be a spanning tree obtained after deleting the edges of S. Then the summation from the above can be written as

$$O(\sum_{uv\in S} min\{d(u),d(v)\} + \sum_{uv\in T} min\{d(u),d(v)\}).$$

In Section 2 we have shown that $m = 2n-2-k$ holds in a cube-free median graph. Hence $m = O(n)$. Therefore the complete running time is bounded as follows

$$O(m) + O(n) + O(\sum_{i=1}^{n} d(v_i) + \sum_{i=1}^{n} d(v_i)) \leq O(n).$$

Since the running time of the second part of the SEQUENCE is bounded by the number of quadrangles (that cannot exceed the number of edges) the assertion follows. □

Concluding remark

We conclude the paper with the following two problems which we are not able to solve at the moment.

Problem 1. Let G be a square-edge graph. Is the number of its edges linear, i.e. is it $E(G) = O(|V(G)|)$?

Problem 2. Give a forbidden subgraph characterization of square-edge graphs.

Recall that $K_{2,3}$ and Q_3 would fit into such a characterization.

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