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Relations between median graphs, semi-median graphs and partial cubes

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Abstract

A rich structure theory has been developed in the last three decades for graphs embeddable into hypercubes, in particular for isometric subgraphs of hypercubes, also known as partial cubes, and for median graphs. Median graphs, which constitute a proper subclass of partial cubes, have attracted considerable attention. In order to better understand them semi-median graphs have been introduced but have turned out to form a rather interesting class of graphs by themselves. This class lies strictly between median graphs and partial cubes and satisfies an expansion theorem for which we give a self-contained proof. Furthermore, we show that these graphs can be characterized as tiled partial cubes and we prove that for a semi-median graph G with n vertices, m edges and k equivalence classes of Djoković's relation Θ, we have $2n - m - k \leq 2$. Moreover, equality holds if and only if G contains no $K_2 \Box C_{2t}$ with $t \geq 2$ as a subgraph. For median graphs we show that they can be characterized as semi-median graphs which contain no convex Q_3^-, i.e. the 3-cube minus a vertex, as a convex subgraph. Also, we introduce the concept of weak 2-convexity and use it, among other things, to prove that median graphs are bipartite, meshed graphs with weakly 2-convex intervals.

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1 Introduction

A *partial cube* is a connected graph that admits an isometric embedding into a hypercube. The structure of partial cubes is relatively well understood. Let us just mention here three fundamental results. First, Djoković [7] characterized these graphs via convexity of certain vertex partitions. He also introduced a relation Θ, which was later used by Winkler [24] to characterize partial cubes as those bipartite graphs for which Θ is transitive. As the third result we mention an expansion theorem of Chepoi [3]. Partial cubes have found several application, see, for instance, early references [11, 12] related to communication theory and recent applications [6, 15] to chemical graph theory. (For efficient computation of Θ see [1, 8].)

A *median graph* is a connected graph such that, for every triple of its vertices, there is a unique vertex lying on a geodesic (i.e. shortest path) between each pair of the triple. By now, the class of median graphs has been well investigated and a rich structure theory is available, see e.g. [2, 5, 17, 23]. Although median graphs form a proper subclass of partial cubes, no nontrivial characterization of median graphs as partial cubes fulfilling some additional conditions is known.

In order to better understand median graphs and their recognition complexity, semi-median graphs were introduced in [13] as partial cubes satisfying a certain additional condition to be explained in Section 2. In the same paper semi-median graphs are characterized in a similar way as Winkler characterized partial cubes in [24], and an expansion theorem for semi-median graphs was established. This theorem is similar to Mulder's expansion theorem for median graphs [18, 19] and to the one of Chepoi for partial cubes [3]. For further results on isometric embedding and in particular on partial cubes, we refer to [9, 10, 22].

In this paper we study interrelations between the above mentioned classes of graphs. In the next section we introduce semi-median graphs and present results concerning Djoković's relation Θ which will be used in the sequel. In Section 3 we study expansion procedures for graphs and give a direct, self-contained proof of the expansion theorem for semi-median graphs. This result was first established in [13] with a proof based on Chepoi's expansion theorem for partial cubes from [3]. Moreover, as several results of this paper rely on the expansion for semi-median graphs, we believe that the new detailed proof is justified. In Section 4 we introduce the concept of weak 2-convexity and prove that median graphs are precisely semi-median graphs without convex Q_3^-. Then we establish a connection between partial cubes and semi-median graphs by characterizing semi-median graphs as tiled partial cubes. These two results are then combined to describe median graphs as those tiled partial cubes which contain no convex Q_3^-. In Section 6 we prove an Euler-type inequality for semi-median graphs and show that the equality occurs if and only if the semi-median graph considered contains no $K_2 \Box C_{2t}$ with $t \geq 2$ as a subgraph. This result is then used to study the number of faces of a planar median graph. We conclude the paper with two open problems.

For an arbitrary relation τ we will write τ^* for its transitive closure. Q_3^- stands for the graph obtained from the 3-cube by removing one of its vertices. For a graph $G = (V, E)$ and $X \subseteq V$ let $\langle X \rangle$ denote the subgraph induced by X.

For $u, v \in V(G)$ let $d_G(u, v)$ denote the length of a shortest path in G from u to v. A subgraph H of a graph G is an *isometric* subgraph, if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$.

The *interval* $I(u, v)$ between vertices u and v consists of all vertices on shortest paths between u and v. A subgraph H of G is *convex*, if for any $u, v \in V(H)$, we have $I(u, v) \subseteq V(H)$. Clearly, a convex subgraph is connected.

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \Box H)$ whenever $ab \in E(G)$ and $x = y$, or, if $a = b$ and $xy \in E(H)$. The n-cube Q_n is the Cartesian product of n copies of the complete graph on two vertices K_2.

2 The relation Θ and semi-median graphs

The Djoković's relation Θ introduced in [7] is defined on the edge-set of a graph in the following way. Two edges $e = xy$ and $f = uv$ of a graph G are in relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)\,.$$

Clearly, Θ is reflexive and symmetric. If G is bipartite, then Θ can be defined as follows: $e = xy$ and $f = uv$ are in relation Θ if

$$d(x, u) = d(y, v) \;\text{ and }\; d(x, v) = d(y, u)\,.$$

Among bipartite graphs, Θ is transitive precisely for partial cubes (i.e. isometric subgraphs of hypercubes), as has been proved by Winkler in [24]. Since median graphs form a subclass of partial cubes, Θ is transitive in particular for median graphs. In other words, Θ is a congruence on median graphs.

Let $G = (V, E)$ be a connected, bipartite graph. For any edge ab of G we write

$$\begin{aligned} W_{ab} &= \{w \in V \mid d_G(a, w) < d_G(b, w)\}, \\ U_{ab} &= \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\}, \\ F_{ab} &= \{e \in E \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}\,. \end{aligned}$$

Note that, G being bipartite, we have $V = W_{ab} \cup W_{ba}$.

For any two edges uv and xy that are in relation Θ, we write $uv // xy$ if $d(u, x) = d(v, y) = d(u, y) - 1 = d(v, x) - 1$. A set of edges is called Θ-*transitive* if any two edges in F are in relation Θ.

Lemma 2.1 *Let G be a connected, bipartite graph, and let ab be any edge of G. Then F_{ab} is the set of all edges in relation Θ with ab.*

Proof. Let uv be any edge in F_{ab} with u in U_{ab} and v in U_{ba}. Then, by the definitions of W_{ab} and U_{ab}, we have

$$d(a, u) = d(b, u) - 1 \leq d(b, v) = d(a, v) - 1 \leq d(a, u)\,.$$

So uv and ab are in relation Θ. Since both ends of any edge in W_{ab} are closer to a than to b, no edge in W_{ab} can be in relation Θ with ab. □

Theorem 2.2 *Let G be a connected, bipartite graph, and let ab be an edge of G. Then the set F_{ab} is Θ-transitive if and only if W_{ab} and W_{ba} are convex in G.*

Proof. First, let both W_{ab} and W_{ba} be convex in G. Take any two edges uv and xy in F_{ab} with u, x in U_{ab} and v, y in U_{ba}. Then neither v nor y is on a u, x-geodesic, whence $d(u, x) = d(u, y) - 1 = d(x, v) - 1$. Similarly, neither u nor x is on a v, y-geodesic, whence $d(v, y) = d(u, y) - 1 = d(x, v) - 1$. This implies that $uv // xy$. So F_{ab} is Θ-transitive.

Conversely, let F_{ab} be Θ-transitive. First we claim that, for any edges uv and xy in F_{ab} with $ab // uv$ and $ab // xy$, we have $uv // xy$. Assume the contrary, and let uv and xy be such that $d(u, y)$ is as small as possible. Note that now we must have $uv // yx$. Let P be a u, v-geodesic. Since u is in W_{ab} and y is in W_{ba}, there must be an edge pq on P, going from u to y, such that p is in W_{ab} and q is in W_{ba}. So pq is in F_{ab}. By the minimality of $d(u, y)$, we have $pq // xy$, so that $d(p, x) = d(q, y) = d(p, y) - 1$. But this is impossible, since q is between p and y on P, and the subpath of P between p and y is a p, y-geodesic. This proves the claim.

Next, choose any two vertices u and v in W_{ab}. Let P be any u, v-path that passes through W_{ba}. Then P must contain at least two edges of F_{ab}. Going from u to v along P, let xy be the first and qp be the last edge in F_{ab}, say, with $ab // xy$ and $ab // pq$. Then we have $xy // pq$, so that $d(x, p) = d(x, q) + 1$. This implies that the subpath of P between x and p is not a geodesic (since it contains q). Hence we conclude that each u, v-geodesic lies completely in W_{ab}. Similarly, W_{ba} is convex. □

The next corollary is an immediate consequence of Theorem 2.2.

Corollary 2.3 *Let G be a connected, bipartite graph, and let ab be an edge of G. If F_{ab} is Θ-transitive, then F_{ab} is a matching and induces an isomorphism between $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$.*

A bipartite graph G is a *semi-median* graph if $\Theta = \Theta^*$ and the subgraph $\langle U_{ab} \rangle$ is connected for any edge ab of G. Equivalently, G is a semi-median graph if G is a partial cube and if all the subgraphs $\langle U_{ab} \rangle$ are connected. Since in median graphs the subgraphs $\langle U_{ab} \rangle$ are convex, median graphs form a proper subclass of semi-median graphs. Two typical semi-median graphs are given in Fig. 1. (Consider two vertices at distance 4 from the outer 8-cycle of the second graph to note that C_8 can be an interval of a semi-median graph.)

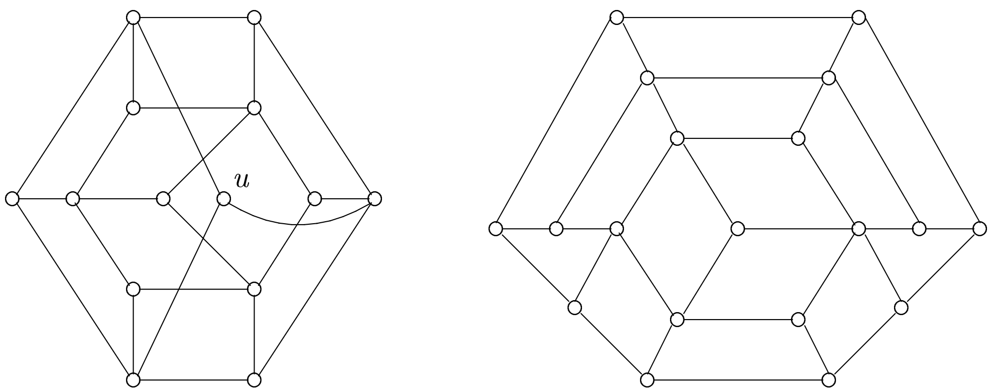

Figure 1: Semi-median graphs

Semi-median graphs are characterized in [13] as bipartite graphs for which $\Theta = \delta^*$ holds. Here an edge e is in relation δ to an edge f if e and f are opposite edges of a square in G or if $e = f$.

3 Expansions, contractions and an expansion theorem

In [21] a broad scheme for expansions and contractions in graphs was presented. Here we focus on one specific instance, the case of semi-median graphs. As the main result of this section we prove an expansion theorem for semi-median graphs. This result has for the first time been given in [13] with a proof which relies on the Chepoi's expansion theorem for partial cubes, see [3, 4]. Here we present a direct, self-contained proof of it. As a by-product we obtain several results which might be of independent interest.

The notations we develop in this section will be used in the sequel without further explanation.

Let $G = (V, E)$ be a connected graph. For two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G, the *intersection* $G_1 \cap G_2$ is the subgraph of G with vertex-set $V_1 \cap V_2$ and edge-set $E_1 \cap E_2$, and the *union* $G_1 \cup G_2$ is the subgraph of G with vertex-set $V_1 \cup V_2$ and edge-set $E_1 \cup E_2$. A *cover* G_1, G_2 of G consists of two subgraphs G_1 and G_2 of G such that $G_0 = G_1 \cap G_2$ is non-empty and $G = G_1 \cup G_2$. Note that there are no edges between $G_1 - G_2$ and $G_2 - G_1$. We call G_0 the *intersection of the cover*. If both G_1 and G_2 are isometric, then we call G_1, G_2 an *isometric cover*, and if both G_1 and G_2 are convex, then we call G_1, G_2 a *convex cover*.

Let G_1, G_2 be a cover of a connected graph G with $G_0 = G_1 \cap G_2$. Clearly, G_0 is convex if and only if both G_1 and G_2 are convex. Note that G_1 and G_2 may both be isometric, but not G_0 which may even be disconnected.

Let G' be a connected graph, and let G'_1, G'_2 be a cover of G' with $G'_0 = G'_1 \cap G'_2$. The *expansion* of G' with respect to G'_1, G'_2 is the graph G constructed as follows. Let G_i be an isomorphic copy of G'_i, for $i = 1, 2$, and, for any vertex u' in G'_0, let u_i be the corresponding vertex in G_i, for $i = 1, 2$. Then G is obtained from the disjoint union $G_1 \cup G_2$, where for each u' in G'_0 the vertices u_1 and u_2 are joined by an edge; see Fig. 2.

For example, if $G' = G'_1 = G'_2 = Q_n$, then $G = Q_{n+1}$. If G' is a tree, and if $G'_1 = G'$ and G'_2 consist of a single vertex u' of G', then G is the tree obtained from G' by adding a new vertex pending at u'. Let G_{0i} be the subgraph of G_i in G corresponding to the subgraph $G'_0 = G'_1 \cap G'_2$ in G', for $i = 1, 2$, and let $F = F_{12}$ be the set of edges between G_1 and G_2 in G, cf. Fig. 2. Then F is a matching, which induces an isomorphism ϕ between G_{01} and G_{02} defined by $\phi(u_1) = u_2$, for any edge u_1u_2 in F. Furthermore, it follows that $G_{01} \cup G_{02}$ together with F form a subgraph of G isomorphic to $G'_0 \Box K_2$. Note that this notion of expansion is called binary Cartesian expansion in the much more general setting of [21]. The following lemma follows straightforward from the definitions.

Lemma 3.1 *Let G be the expansion of a connected graph G' with respect to the cover G'_1, G'_2. Then G'_1 and G'_2 are both isometric in G' if and only if G_1 and G_2 are both convex in G.*

The converse notions to cover and expansion are, respectively, split and contraction. Let G be a connected graph. Let G_1 and G_2 be disjoint subgraphs of G, and let $F = F_{12}$ be

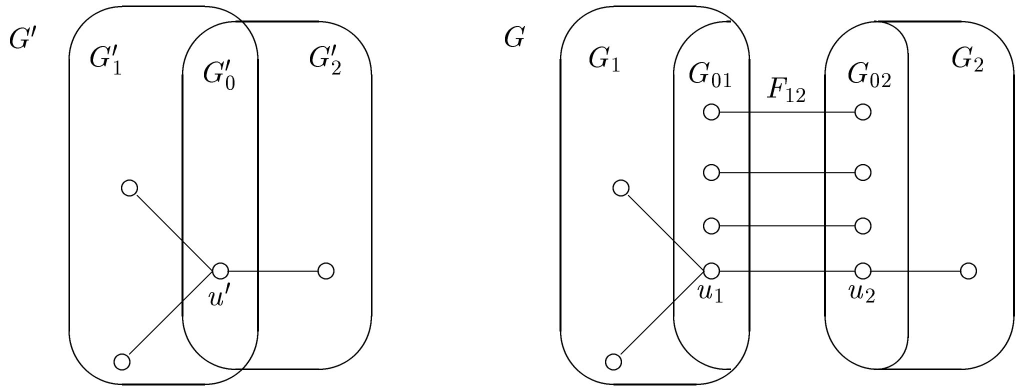

Figure 2: Graph G as the expansion of G' with respect to G'_1, G'_2

the set of edges between G_1 and G_2, where G_{0i} is the subgraph of G_i consisting of the end vertices of the edges in F, for $i = 1, 2$. Then G_1, G_2 is a *split* in G if G_1 and G_2 together with F form the whole of G, and F is a matching inducing an isomorphism along its edges between G_{01} and G_{02}. Note that G_{01} and G_{02}, together with F, form a graph isomorphic with $G_{01} \square K_2$. The subgraphs G_1 and G_2 are the *sides* of the split. In Fig. 2 we have a graph G with split G_1, G_2 and a contraction G' with cover G'_1, G'_2.

Note that both sides of the split are convex if and only if both sides are isometric. We call G_1, G_2 a *convex split* if both sides of the split are convex. Now Lemma 3.1 can be alternatively formulated as follows: if G is the expansion of a connected graph G' with respect to the cover G'_1, G'_2, then G'_1, G'_2 is an isometric cover of G' if and only if G_1, G_2 is a convex split in G.

The next lemma follows immediately from the definitions.

Lemma 3.2 *Let G be a connected graph, and let G_1, G_2 be a split in G. Then G_1, G_2 is a convex split in G if and only if $G_1 = \langle W_{uv} \rangle$ and $G_2 = \langle W_{vu} \rangle$ for each edge uv with u in G_1 and v in G_2.*

The *contraction* of G with respect to the split G_1, G_2 is the graph G' obtained from G by contracting each edge of F to a single vertex. Clearly, if G is the expansion of G' with respect to the cover G'_1, G'_2 of G', then, with the above notation, G_1, G_2 is a split in G, and G' is the contraction of G with respect to the split G_1, G_2, and vice versa.

Next we establish some basic facts about the behaviour of subgraphs with respect to expansions or contractions.

Let G be the expansion of a connected graph G' with respect to the cover G'_1, G'_2 with $G'_0 = G'_1 \cap G'_2$. Let H' be an induced subgraph of G' such that $H' \cap G'_0$ is nonempty. Then $H' \cap G'_1, H' \cap G'_2$ is a cover of H'. Let H be the expansion of H' with respect to $H' \cap G'_1, H' \cap G'_2$. Then H is an induced subgraph of G such that $H \cap G_1, H \cap G_2$ is a split in H and the union of $H \cap G_{01}$ and $H \cap G_{02}$ induces a subgraph of G isomorphic to $[H' \cap G'_0] \square K_2$. Conversely, H' is the contraction of H with respect to the split $H \cap G_1, H \cap G_2$. Clearly, H is connected in G if and only if H' is connected in G'. The next lemma follows straightforwardly from the definitions.

Lemma 3.3 *Let G' be a connected graph, let G'_1, G'_2 be an isometric cover of G', and let G be the expansion of G' with respect to this cover. Let H' be a subgraph of G', and let H be the expansion in G with respect to the cover $H' \cap G'_1$, $H' \cap G'_2$. Then H is isometric, resp. convex, in G if and only if H' is isometric, resp. convex, in G'.*

Let G be a connected graph, and let G_1, G_2 be a split in G where F the set of edges between the sides of the split. Let H be an induced subgraph of G with both $H \cap G_1$ and $H \cap G_2$ nonempty. Then the subgraphs $H_1 = H \cap G_1$ and $H_2 = H \cap G_2$ form a split in H. Let F_H be the set of edges between the sides of this split in H, and let H_{0i} denote the ends of F_H in H_i, for $i = 1, 2$. Then we have $H_{0i} \subseteq H \cap G_{0i}$, but we do not necessarily have $H_{0i} = H \cap G_{0i}$, for $i = 1, 2$; cf. Fig. 3. Let G' be the contraction of G with respect to the split G_1, G_2. Then the subgraph H' of G' produced by this contraction is precisely the contraction of H with respect to the split $H_1 = H \cap G_1$, $H_2 = H \cap G_2$ of H. Note, however, that H is not necessarily the expansion of H' with respect to $H' \cap G'_1$ and $H' \cap G'_2$. This is so because H is the expansion of H' with respect to $H' \cap G'_1$, $H' \cap G'_2$ if and only if $H_{0i} = H \cap G_{0i}$, for $i = 1, 2$. A simple example is given in Figure 3, where H'' is the corresponding expansion of H'.

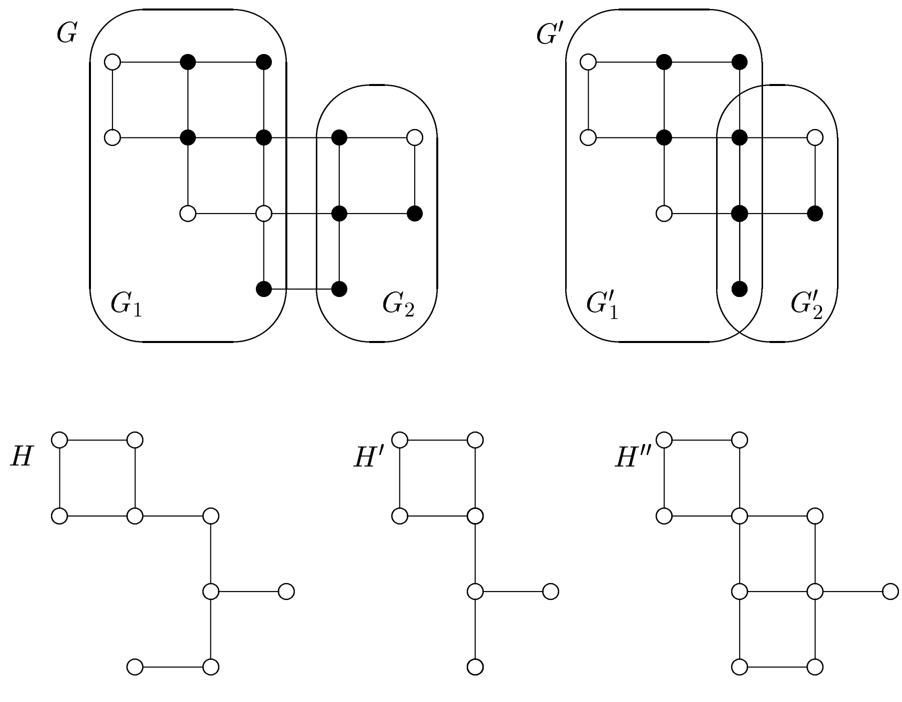

Figure 3: Expansions and contractions of subgraphs

Again we see that H is connected in G if and only is H' is connected in G'.

Lemma 3.4 *Let G be a connected graph, and let G_1, G_2 be a convex split in G. Let H be a convex subgraph of G with nonempty $H_i = H \cap G_i$, for $i = 1, 2$. If uv is an edge with u in G_1 and v in G_2, then u is in H if and only if v is in H.*

Proof. Note that, since H is convex, there is an edge xy in H with x in G_1 and y in G_2. Assume that u lies in H. By Lemma 3.2, we have that $\langle W_{vu} \rangle = \langle W_{yx} \rangle = G_2$, so that v is on a geodesic between u and y. By the convexity of H, we see that v is in H as well. Similarly, if v lies in H, we deduce that u lies in H too. □

The next corollary combines some of the above lemmata and observations.

Corollary 3.5 *Let G be a connected graph, let G_1, G_2 be a convex split in G with contraction G', and let H be a subgraph of G with the corresponding contraction H'. If H is convex in G then H' is convex in G'.*

Proof. Let H be a convex subgraph of G, and assume that the subgraphs $H \cap G_{01}$ and $H \cap G_{02}$ are nonempty. It suffices to show that, for each vertex in $H \cap G_{01}$, its neighbor in G_{02} lies in H.

Because of the convexity of H, there exists an edge xy with x in $H \cap G_1$ and y in $H \cap G_2$.

Choose any vertex u in $H \cap G_{01}$, and let v be its neighbor in G_{02}. Because G_2 is convex, every geodesic between v and y lies in G_2. So v is on a geodesic between u and y, whence v is in H. Similarly, we deduce that, if v is a vertex in $H \cap G_2$ with neighbor u in G_1, then u is also in H. By the previous observations we infer that H' is also convex. □

Theorem 3.6 *A graph is a semi-median graph if and only if it can be obtained from the one-vertex graph K_1 by expansions with respect to isometric covers with connected intersection.*

Proof. First assume that G is a semi-median graph. We will show that G can be obtained from a semi-median graph with fewer vertices by an expansion with respect to an isometric cover with a connected intersection.

Let ab be any edge of G, and let $G_1 = \langle W_{ab} \rangle$, and $G_2 = \langle W_{ba} \rangle$, and $G_{01} = \langle U_{ab} \rangle$, and $G_{02} = \langle U_{ba} \rangle$. From Lemma 3.2, Theorem 2.2 and Corollary 3.5 it follows that G_1, G_2 is a convex split with connected G_{01} and G_{02}. Let G' be the contraction of G with respect to this split. Then G'_1, G'_2 is an isometric cover of G' with nonempty intersection G'_0, and G is the expansion of G' with respect to this cover. It suffices to show that G' is a semi-median graph.

Let $u'v'$ be any edge of G', and let uv be an edge in G corresponding to $u'v'$, and let $H_1 = \langle W_{uv} \rangle$ and $H_2 = \langle W_{vu} \rangle$ and $H_{01} = \langle U_{uv} \rangle$ and $H_{02} = \langle U_{vu} \rangle$. Let H'_i be the contraction of H_i, and let H'_{0i} be the contraction of H_{0i}, for $i = 1, 2$. Then H_1, H_2 is a convex split in G, hence, by Corollary 3.5, H'_1 and H'_2 are convex in G'. This implies that H'_1 consists of all vertices nearer to u' than to v' in G', and that H'_2 consists of all vertices nearer to v' than to u'. Moreover, we have in G' that $\langle U_{uv} \rangle = H'_{01}$, which is connected, because H_{01} is. Finally, we may conclude that H'_1, H'_2 is a convex split in G', so that the edges between the sides of this split form a Θ-transitive Θ-class containing the edge $u'v'$. From these facts we conclude that G' is a semi-median graph.

Conversely, let G be the expansion of a semi-median graph G' with respect to an isometric cover G'_1, G'_2 of G' with connected intersection G'_0. Then G_1 and G_2 are convex in G, so that F_{12} is Θ-transitive. Furthermore, for any edge ab in F_{12}, the subgraph $\langle U_{ab}\rangle$ is connected, because it is isomorphic to G'_0. Let uv be any edge in G not in F_{12}, and let $u'v'$ be the edge in G' corresponding to uv. Let $H'_{u'} = \langle W_{u'v'}\rangle$ and $H'_{v'} = \langle W_{v'u'}\rangle$ in G'. Then, by Theorem (ii), $H'_{u'}$ and $H'_{v'}$ are convex subgraphs of G'. Let H_u be the expansion of $H'_{u'}$ with respect to the cover $H'_{u'} \cap G'_1$, $H'_{u'} \cap G'_2$, and, similarly, let H_v be the expansion of $H'_{v'}$ with respect to the cover $H'_{v'} \cap G'_1$, $H'_{v'} \cap G'_2$. Then H_u and H_v are disjoint subgraphs of G, the union of which contains all vertices of G. By Lemma 3.3, both H_u and H_v are convex in G. This implies that $H_u = \langle W_{uv}\rangle$ and $H_v = \langle W_{vu}\rangle$. Moreover, $H_{0u} = \langle U_{uv}\rangle$ and $H_{0v} = \langle U_{vu}\rangle$ are connected and matched by the set of edges F_{uv} between H_u and H_v. Hence F_{uv} is a Θ-transitive Θ-class. From this we deduce that G is a semi-median graph as well. □

4 Median graphs versus semi-median graphs

In this section we characterize median graphs in terms of semi-median graphs. For this purpose we study convexity concepts in bipartite graphs, and, in particular we introduce the notion of weakly 2-convex subgraphs. This notion also allows us to slightly strengthen one of the known characterizations of median graphs.

Let H be a subgraph of a graph G. Then the *boundary* ∂H of H in G is the set of all edges xy of G with $x \in H$ and $y \notin H$. The following result is from [13].

Lemma 4.1 (Convexity Lemma) *An induced connected subgraph H of a bipartite graph G is convex if and only if no edge of ∂H is in relation Θ to an edge in H.*

Let H be an induced connected subgraph of a graph G. Then H is 2-*convex* if for any two vertices u and v of H with $d_G(u,v) = 2$, every common neighbor of u and v belongs to H. The subgraph H is *weakly 2-convex*, if for any vertices u and v of H with $d_H(u,v) = 2$ every common neighbor of u and v belongs to H. Clearly, a 2-convex subgraph is a weakly 2-convex subgraph. The converse is not true, consider for instance the path on five vertices as a subgraph of the 6-cycle C_6.

A graph $G = (V,E)$ is called *meshed* if it satisfies the *quadrangle property*: for any vertices u, v, w, z with $d(u,v) = d(u,w) = d(u,z) - 1$ and $d(u,v) = 2$ and z a common neighbor of v and w, there exists a common neighbor x of v and w with $d(u,x) = d(u,v) - 1$.

Let G be a connected, bipartite graph, let u be a vertex of G and let $P = v_1 \to v_2 \to \cdots \to v_k$ be a path in G. We call the edge v_iv_{i+1} on P *upward* with respect to u if $d(u,v_{i+1}) = d(u,v_i) + 1$ and *downward* if $d(u,v_{i+1}) = d(u,v_i) - 1$. The *distance* of u to P is $d(u,P) = \sum_{i=1}^{k} d(u,v_i)$.

Lemma 4.2 *Let G be a connected, bipartite, meshed graph and let H be a subgraph of G. Then the following statements are equivalent:*

(i) H is a convex subgraph of G,

(ii) H is a 2-convex subgraph of G,

(iii) H is a weakly 2-convex subgraph of G.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are trivial. It remains to prove that (iii) implies (i).

Let H be a weakly 2-convex subgraph of G and assume that H is not convex. Let u, v be vertices in H such that $I(u,v)$ is not contained in H with $d(u,v) = k$ as small as possible. Note that $k \geq 3$. By the minimality of k, there is a u,v-geodesic Q of which all internal vertices are outside H. Let x be the neighbor of v on Q. Since H is connected, being weakly 2-convex, there exists a u,v-path in H. Let P be a path from v to u in H that minimizes the distance from u to P. Suppose that P contains an upward edge with respect to u. Then there exists a subpath $p \to q \to r$ of P such that pq is upward and qr is downward. Now, G being meshed, there exists a common neighbor s of p and r with $d(u,s) = d(u,p) - 1$. Since $p \to q \to r$ lies in H, and H is weakly 2-convex, s also lies in H. But now the path P' obtained from P by replacing q by s has smaller distance to u, which creates a conflict with the minimality of P. So P contains only downward edges, that is, P is also a u,v-geodesic. Let y be the neighbor of v on P. Then, as G is meshed, we find a common neighbor w of x and y with $d(u,v) = d(u,y) - 1 = d(u,v) - 2$. Since x is not in H, and since H is weakly 2-convex, w cannot be in H. Hence $I(u,y)$ is not contained in H, which is impossible by the minimality of $k = d(u,v)$. □

Note that a graph is meshed if every triple of vertices has a median. Hence, by Lemma 4.2, we have the following result.

Corollary 4.3 *Let G be a connected, bipartite, graph in which every triple of vertices has a median. Then a subgraph H of G is convex if and only if H is 2-convex.*

Corollary 4.3 is given in [14] in a weaker form, stating that convex is equivalent to isometric and 2-convex.

In [17] it is proved that a graph G is a median graph if and only if G is a bipartite, meshed graph with 2-convex intervals. Thus, from Lemma 4.2 we also infer the following characterization of median graphs:

Corollary 4.4 *Let G be a connected graph. Then G is a median graph if and only if G is a bipartite, meshed graph with weakly 2-convex intervals.*

Almost-median graphs were introduced in [13] as partial cubes for which the subgraph $\langle U_{ab} \rangle$ is isometric for any edge ab of G and it was proved that a graph is a median graph if and only if it is an almost-median graph which contains no convex Q_3^- as a subgraph. We are now able to strengthen this result as follows.

Theorem 4.5 *A graph G is a median graph if and only if G is a semi-median graph and contains no convex Q_3^- as a subgraph.*

Proof. Clearly, a graph which contains a convex Q_3^- cannot be a median graph.

Suppose G is a semi-median graph which contains no convex Q_3^-. Then G can be constructed from the 1-vertex graph by a series of connected expansions. If these expansions are also convex, we obtain a median graph. Thus, there must be a first expansion step which leads to a non-median graph. Since all graphs constructed up to this step are median graphs, this expansion cannot be convex by Mulder's convex expansion theorem. To make notation easier, let us assume for the time being that the last expansion step is the only one not convex in $\langle W_{uv}\rangle$. As $\langle W_{uv}\rangle$ is a median graph, U_{uv} cannot be 2-convex either by Corollary 4.3. Thus, there is a square consisting of the edges ab, bc, cd, da in $\langle W_{uv}\rangle$ where a, b, c are in U_{uv}, but not d. Let a', b', c' be the neighbors of a, b, c in U_{vu}. Then a, b, c, d, a', b', c' form a convex Q_3^- in G, which readily follows from the Convexity Lemma 4.1.

If this is not the last expansion step, we note that all further connected expansions add edges which belong to different Θ-classes from all previous ones, so this Q_3^- remains convex by Lemma 4.1. □

5 Median graphs versus partial cubes

In this section we will answer the question which additional conditions a partial cube must satisfy in order to be a median graph. For this sake we first study the interrelation of partial cubes with semi-median graphs.

An *even* graph is a graph whose every vertex has even degree. Recall that $\oplus$ denotes the symmetric sum of graphs, where we consider graphs as sets of edges. Let C be a cycle of a graph G. Then a set of 4-cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ in G is a *tiling* of H if

$$H = C_1 \oplus C_2 \oplus \cdots \oplus C_p\,.$$

Call a graph G *tiled* if every cycle of G has a tiling. This definition enables us to formulate the following result about partial cubes and semi-median graphs.

Theorem 5.1 *A graph is a semi-median graph if and only if it is a tiled partial cube.*

Proof. Let G be a tiled partial cube which is not a semi-median graph. Since G is not a semi-median graph, there exists an edge $ab \in E(G)$ such that $\langle U_{ab}\rangle$ (and $\langle U_{ba}\rangle$) is not connected. Let $\langle U_{ab}^1\rangle$ and $\langle U_{ab}^2\rangle$ be two different connected components of $\langle U_{ab}\rangle$ and let $\langle U_{ba}^1\rangle$ and $\langle U_{ba}^2\rangle$ be the appropriate connected components of $\langle U_{ba}\rangle$. Let u_1v_1 and u_2v_2 be edges from F_{ab} with $u_i \in \langle U_{ab}^i\rangle$ and $v_i \in \langle U_{ba}^i\rangle$ for $i = 1, 2$. We may assume that u_1 and u_2 are as close as possible under the above assumptions. Then there is an induced (in fact even isometric) cycle $C = u_1Pu_2v_2Qv_1u_1$ of G such that the paths P and Q lie in $W_{ab} \setminus U_{ab}$ and $W_{ba} \setminus U_{ba}$, respectively.

Observe first that the length of C is at least 6. Since G is tiled, there exists a set of cycles $\mathcal{C} = \{C_1, \ldots, C_p\}$ $(p \geq 2)$ of G such that $C = C_1 \oplus \cdots \oplus C_p$. Note that if the 4-cycle C_i has an edge in F_{ab}, then it has exactly two edges in F_{ab}. Since u_1v_1 and u_2v_2 are the only two edges of C which are in F_{ab} there exists a subset $\mathcal{C}_r = \{C_{i_1}, \ldots, C_{i_r}\}$ of $\mathcal{C}$, where every C_{i_j} has two edges in F_{ab} and $F_{ab} \cap E(C_{i_1} \oplus \cdots \oplus C_{i_r}) = \{u_1v_1, u_2v_2\}$. But then we observe that in $C_{i_1} \oplus \cdots \oplus C_{i_r}$ there is an u_1, u_2-path with all inner vertices in $\langle U_{ab}\rangle$. This is a contradiction, for we assumed that these two vertices are in different connected components of $\langle U_{ab}\rangle$.

Suppose now that G is a semi-median graph. Then G is a partial cube. We need to show that G is tiled. The proof is by induction on the number of Θ-classes of G. If there is no such class then G is the one vertex graph, which is tiled. Now, let G be a semi-median graph with at least one Θ-class. Then G can be obtained by a connected expansion step from a semi-median graph G', which is tiled by the induction hypothesis. Let F_{ab} be the Θ-class obtained in the expansion step. Suppose that C is an arbitrary cycle of G. We will prove that C is tiled by induction on $|E(C) \cap F_{ab}|$. Let C', W'_{ab}, and W'_{ba} corespond in G' to C, W_{ab}, and W_{ba}, respectively.

Suppose first that $|E(C) \cap F_{ab}| = 0$. Then we may assume that C is a cycle of $\langle W_{ab} \rangle$. By the induction hypothesis there is a tiling $\mathcal{C}' = \{C'_1, \ldots, C'_p\}$ of C' in G'. Let $\mathcal{C}'_b$ be the set of 4-cycles of $\mathcal{C}'$ which have a vertex in $W'_{ba} \setminus W'_{ab}$ and let $\mathcal{C}'_a = \mathcal{C}' \setminus \mathcal{C}'_b$.

Let $\mathcal{C}_b$ be the set of 4-cycles in $\langle W_{ba} \rangle$ which naturally corespond to the 4-cycles of $\mathcal{C}'_b$. Similarly, let $\mathcal{C}_a$ be the set of 4-cycles in $\langle W_{ab} \rangle$ which corespond to the 4-cycles of $\mathcal{C}'_a$. Set $\mathcal{C} = \mathcal{C}_a \cup \mathcal{C}_b$.

Note that the graph $H_b = \oplus_{D \in \mathcal{C}_b} D$ is an even subgraph of U_{ba}. Let H_a be the isomorphic copy in U_{ab} of H_b. Since $C' = C'_1 \oplus \cdots \oplus C'_p$, it follows that

$$H_a = C \oplus \ \oplus_{D \in \mathcal{C}_a} D. \tag{1}$$

Let $\mathcal{C}_{ab}$ be the set of 4-cycles $D = cdefc$ such that $cd \in H_a$, $ef \in H_b$, and $de, fc \in F_{ab}$. Since H_a and H_b are even graphs, by (1), we conclude that $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_{ab}$ is a tiling of C in G.

Assume next that $E(C) \cap F_{ab} = \{u_1v_1, u_2v_2\}$, where $u_1, u_2 \in U_{ab}$ and $v_1, v_2 \in U_{ba}$. Let P_u be an arbitrary path in $\langle U_{ab} \rangle$ between u_1 and u_2. Denote by P_v the isomorphic copy of P_u in $\langle U_{ba} \rangle$. Note that P_u and P_v always exist since $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are connected graphs as G is a semi-median graph. Let $H = P_u \cup P_v \cup \{u_1v_1, u_2v_2\}$. Then H is a cycle and it is easy to see that there exists a tiling $\mathcal{C}_H$ such that each 4-cycle of $\mathcal{C}_H$ has two edges in F_{ab}, one in P_u and one in P_v. Note that we can consider the even graph $C \oplus H$ as a set of pairwise edge disjoint cycles (possibly empty set) such that each one is either in $\langle W_{ab} \rangle$ or $\langle W_{ba} \rangle$. Thus, by the previous case, for each such cycle $D \in C \oplus H$ there exists a tiling $\mathcal{C}_D$. Let

$$\mathcal{C}_{C \oplus H} = \oplus_{D \in C \oplus H} \ \mathcal{C}_D. \tag{2}$$

Now it is easy to see that $\mathcal{C}_{C \oplus H}$ is a tiling of $C \oplus H$. Finally, observe that $C_{C \oplus H} \cup \mathcal{C}_H$ is a tiling of C.

Suppose now that $E(C) \cap F_{ab} = \{u_1v_1, \ldots, u_{2t}v_{2t}\}$ $(t \geq 2)$ and that every cycle D of G with $|F_{ab} \cap D| < 2t$ is tiled. The argument is basically the same as the previous one. We may assume that there is a path P_u between u_1 and u_2 in $C \setminus F_{ab}$. Let P_v be an arbitrary path in $\langle U_{ba} \rangle$ between v_1 and v_2. Let $H = P_u \cup P_v \cup \{u_1v_1, u_2v_2\}$. Since $|E(H) \cap F_{ab}| = 2$, it follows by the above that there exists a tiling $\mathcal{C}_H$ for H. Similarly as above consider the even graph $C \oplus H$ as a set of cycles such that each one has at most $2(t-1)$ edges from F_{ab}. Thus for each cycle $D \in C \oplus H$ there exists a tiling $\mathcal{C}_D$. Let $\mathcal{C}_{C \oplus H}$ be defined as in (2). Then, as above we have that $C_{C \oplus H}$ is a tiling of $C \oplus H$ and hence $C_{C \oplus H} \cup \mathcal{C}_H$ is a tiling of C. $\square$

Combining Theorem 5.1 with Theorem 4.5 we obtain the following characterization of median graphs as partial cubes.

Corollary 5.2 *A graph G is a median graph if and only if G is a tiled partial cube which contains no convex Q_3^-.*

6 An Euler-type formula for semi-median graphs

Let G be a median graph with n vertices, m edges and k equivalence classes of the relation Θ. It was proved in [16] that $2n - m - k \leq 2$, and that equality holds if and only if G is a cube-free median graph. The purpose of this section is to extend this Euler-type result to semi-median graphs. For this sake we need the following corollary to Theorem 3.6.

Corollary 6.1 *A graph is a $(K_2 \Box C_{2t})$-free semi-median graph, $t \geq 2$, if and only if it can be obtained from the one vertex graph by a connected expansion procedure, in which every expansion step is taken with respect to a cover with a tree as intersection.*

Proof. Let G be a $(K_2 \Box C_{2t})$-free semi-median graph which is a connected expansion of G' with respect to G'_1 and G'_2. Then $G'_0 = G'_1 \cap G'_2$ contains no cycle because G is $(K_2 \Box C_{2t})$-free. Moreover, as G is a semi-median graph, G'_0 is connected and hence a tree. Induction completes the argument.

Conversely, assume that G can be obtained from the one vertex graph by a connected expansion procedure, in which every expansion step is taken with respect to a cover with a tree as intersection. By the expansion theorem G is a semi-median graph and we show by induction that G is $(K_2 \Box C_{2t})$-free. Let G be obtained from G' with respect to G'_1 and G'_2 by an expansion step. As G' is a $(K_2 \Box C_{2t})$-free graph, G can only contain an induced $K_2 \Box C_{2t}$ which intersects both G_1 and G_2. Each 4-cycle $K_2 \Box K_2$ as a subgraph of $K_2 \Box C_{2t}$ has two vertices in G_1 and two in G_2. But then we conclude that C_{2t} is a subgraph of G_1 and of G_2, thus $G'_1 \cap G'_2$ is not a tree, a contradiction. $\Box$

We can now state the main result of this section.

Theorem 6.2 *Let G be a semi-median graph with n vertices, m edges and let k be the number of equivalence classes of Θ. Then $2n - m - k \leq 2$. Moreover, $2n - m - k = 2$ if and only if G contains no $K_2 \Box C_{2t}$ with $t \geq 2$ as a subgraph.*

Proof. We prove the inequality by induction on the number of vertices using Theorem 3.6. The inequality reduces to $2 \leq 2$ if $G = K_1$. So assume that G is the connected expansion of the semi-median graph G' with respect to its isometric subgraphs G'_1, G'_2 with $G'_0 = G'_1 \cap G'_2$. By induction we have $2n' - m' - k' \leq 2$ for G', where k', n', m' are the corresponding parameters of G'. Let t be the number of vertices in G'_0, so that G'_0, being connected, has at least $t - 1$ edges. Then we have $n = n' + t$ and $m \geq m' + 2t - 1$. Moreover, an expansion step yields one more Θ-equivalence class and so we have $k = k' + 1$. Thus

$$
\begin{aligned}
2n - m - k &\leq 2(n' + t) - (m' + 2t - 1) - (k' + 1) \\
&= 2n' - m' - k' \\
&\leq 2\,.
\end{aligned}
$$

Clearly, we have equality if and only if, all the expansions have been taken with respect to two isometric subgraphs having a tree as intersection. By Corollary 6.1, this is equivalent with G being a $(K_2 \Box C_{2t})$-free semi-median graph. $\Box$

Corollary 6.3 *Let G be a planar semi-median graph with n vertices, k equivalence classes of the relation Θ and f faces in its planar embedding. Then $f \geq n - k$. Moreover, $f = n - k$ if and only if G is $(K_2 \Box C_{2t})$-free.*

Proof. Combine the Euler's formula $n - m + f = 2$ with Theorem 6.2. $\Box$

7 Two problems

At the end of the paper we wish to state the following two problems.

Problem 7.1 *Is every regular partial cube a Cartesian product of even cycles and K_2's?*

Problem 7.2 *Is every regular semi-median graph a hypercube?*

Recall that Mulder [20] settled the latter question affirmatively for median graphs. Note also that the answer is negative for partial cubes (consider C_6). Finally, observe that by Theorem 5.1, the truth of the first problem implies the truth of the second one.

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