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**Preprint series, Vol. 36 (1998), 617**

LIFTING GRAPH  
AUTOMORPHISMS BY  
VOLTAGE ASSIGNMENTS

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ISSN 1318-4865

June 22, 1998

Ljubljana, June 22, 1998

# Lifting Graph Automorphisms by Voltage Assignments

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## Abstract

The problem of lifting graph automorphisms along covering projections is considered in a purely combinatorial setting. Because of certain natural applications and greater generality, graphs are allowed to have semiedges. This requires careful reexamination of the whole subject, which leads to simplification and generalization of several known results. For example, Cayley graphs, including those with involutory or repeated generators, are characterized as regular coverings over one-vertex graphs. The ordinary and the permutation voltage constructions are unified into the concept of a voltage space, constituting the crucial tool for combinatorialization of the lifting problem. Particular attention is paid to the structure of lifted groups, with focus on split extensions having a nice geometrical and combinatorial description. Some applications of these results to regular maps on surfaces are given.

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<sup>1</sup>Supported in part by “Ministrstvo za znanost in tehnologijo Slovenije”, proj. no. P1-0208-1010-94.

<sup>2</sup>Supported in part by VEGA, grant no. 1/331/96.

<sup>3</sup>Supported in part by VEGA, grant no. 1/331/96.

# 1 Introduction

Structural properties of various mathematical objects are to a large extent reflected by their automorphisms. It is therefore natural to compare related objects through their automorphism groups. Unfortunately, the automorphism group of an object is usually not a functorial invariant. This excludes relatively easy comparison of the respective automorphism groups via morphisms. Much better chances to get relevant results occur when the general problem is relaxed. Typically, one considers a morphism  $m : Y \rightarrow X$  satisfying some additional properties and asks whether to an automorphism  $f$  of  $X$  there exists an automorphism  $\tilde{f}$  of  $Y$  such that the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{f}} & Y \\ m \downarrow & & \downarrow m \\ X & \xrightarrow{f} & X \end{array}$$

commutes. This is the essence of the *problem of lifting automorphisms* which has recently received considerable attention in the context of covering projections, and which is considered in this paper.

In topology, the problem was extensively studied several decades ago. In recent years, however, it has reappeared in a discrete rather than in a continuous setting, most notably in the theory of graph coverings. By now classical results of Alpert, Biggs, Gross, Tucker and others, graph coverings and consequently, branched coverings of maps on surfaces can be defined and studied from a purely combinatorial point of view. It is therefore natural to study the lifting problem in this combinatorial setting as well.

Our main objective is to present a topology-free, self-contained approach to the above lifting problem in the context of graphs. Because of certain natural applications in the theory of Cayley graphs as well as in the theory of maps on surfaces, and for the sake of convenience, we extend our definition of a graph to allow semiedges. This slight modification has some far-reaching consequences. For instance, the fundamental group of such a graph may *not* be a free group(!). Therefore, it has been necessary to carefully reexamine the whole basic theory of coverings as it is known in topology and, so far, in topological graph theory. On the other hand, our revision has an additional pleasant consequence in that it enables a smooth and unified transition to the existing theory of voltage assignments. As a result, a genuine combinatorial theory of coverings with no direct topological counterpart is obtained.

The paper is organized as follows. The next section gives a brief historical account of ideas relevant to the topic of this paper. In Sections 3 and 4 we introduce graphs with semiedges and develop the combinatorial counterpart of homotopy the-

ory and of theory of coverings.

Section 5 treats the lifting problem much in the spirit of the classical theory. However, the natural action of the fundamental groupoid on the covering graph, rather than merely the action of the fundamental group on the fibre, is emphasized. Loosely speaking, an automorphism has a lift if and only if this automorphism is consistent with the action of the fundamental groupoid. The action of the fundamental groupoid is used in Section 6 to define a voltage space, a concept which unifies and extends the notions of ordinary, relative and permutation voltages.

In Section 7 we transfer the lifting problem to the abstract voltage level. In order to do this it is necessary that the associated voltage space satisfies a certain condition which we call local invariance. Briefly speaking, this condition requires that a closed walk carrying the trivial voltage is mapped (by the automorphisms in question) to closed walks with trivial voltage. It further transpires that this condition is necessary and sufficient for the automorphism to lift whenever both the covering projection and the voltage action are regular. Our final stage of combinatorialization of the lifting problem is to derive a system of equations in the symmetric group  $S_n$ ,  $n$  being the size of the fibre, which has a solution if and only if the considered automorphism lifts. This establishes a bridge between the solution of the lifting problem in classical terms and within the combinatorial setting.

In Section 8 we illustrate our approach in characterizing all coverings over the Petersen graph such that the whole automorphism group lifts.

The starting point of Section 9 is the observation that any abstract group extension can be treated as a lifting problem on the corresponding Cayley graphs. This brings us to consider the structure of lifted groups with particular focus on split extensions.

In the final section we apply our results to coverings of regular maps on surfaces. It is proved that valency preserving homomorphisms of orientable regular maps can be grasped by means of locally invariant Cayley voltage spaces.

## 2 History

Combinatorial treatment of graph coverings had its primary incentive in the solution of Heawood's Map Colour Problem due to Ringel, Youngs and others [32]. That coverings underlie the techniques which led to the eventual solution of the problem was recognized by Alpert and Gross [11]. These ideas further crystalized in 1974 in the work of Gross [10] where voltage graphs were introduced as a means of a purely combinatorial description of regular graph coverings. In parallel, the very same idea appeared in Biggs' monograph [4]. Much of the theory of combinatorial graph coverings in its own due was subsequently developed by Gross and Tucker in

the seventies. We refer the reader to [13, 38] and the references therein. The theory was extended to combinatorial graph bundles introduced by Pisanski and Vrabec [31] in the eighties.

The study of the lifting problem in the context of regular coverings of graphs had its primary motivation in constructing infinite families of highly transitive graphs. The first notable contribution along these lines appeared, incidentally, in 1974 in Biggs' monograph [4] and in a paper of Djoković [7]. While Biggs gave a combinatorial sufficient condition for a lifted group to be a split extension, Djoković found a criterion, in terms of the fundamental group, for a group of automorphisms to lift at all.

A decade later, several different sources added further motivation for studying the lifting problem. These include: counting isomorphism classes of coverings and, more generally, graph bundles, as considered by Hofmeister [15] and Kwak and Lee [17, 18]; constructions of regular maps on surfaces based on covering space techniques due to Archdeacon, Gvozđjak, Nedela, Richter, Širáň, Škovič and Surowski [1, 2, 14, 28, 29, 35]; and construction of transitive graphs with prescribed degree of symmetry, for instance by Du, Malnič, Nedela, Marušič, Scapellato, Seifert, Trofimov and Waller [8, 22, 23, 25, 26, 34]. Lifting and/or projecting techniques play a prominent role also in the study of imprimitive graphs, cf. Gardiner and Praeger [9] among others. The lifting problem in terms of voltages in the context of general topological spaces was considered by Malnič [20]. Very recently, Venkatesh [37] obtained structural results which refine the work of Biggs [4, 5] and Djoković [7] on lifting groups.

The study of the lifting problem within the combinatorial framework was of course preceded by that in the topological setting. We refer the reader to the work of Armstrong, Birman, Hilden, MacBeath and Zieschang [3, 6, 19, 39] and many others.

### 3 Graphs

A *graph* is an ordered quadruple  $X = (D, V; I, \lambda)$  where  $D$  is a set of *darts*,  $V$  is a nonempty set of *vertices*, which is required to be disjoint from  $D$ ,  $I$  is a mapping of  $D$  onto  $V$  and  $\lambda$  is an involutory permutation of  $D$ . For convenience or if  $\lambda$  is not explicitly specified we sometimes write  $x^{-1}$  instead of  $\lambda x$ . Intuitively, the mapping  $I$  assigns to each dart its *initial vertex*, and the permutation  $\lambda$  interchanges a dart and its *reverse*. The *terminal vertex* of a dart  $x$  is the initial vertex of  $\lambda x$ . The set of all darts having a given vertex  $u$  as their common initial vertex is denoted by  $D_u$ . The cardinality of  $D_u$  is the *valency* of the vertex  $u$ . The orbits of  $\lambda$  are called *edges*; thus each dart determines uniquely its *underlying edge*. An edge is called a *semi-edge* if  $\lambda x = x$ , a *loop* if  $\lambda x \neq x$  and  $I\lambda x = Ix$ , and it is called a *link* otherwise.

We represent a graph, as defined above, by a topological space in the usual way as a 1-dimensional CW-complex. Note that from a topological point of view a semiedge is identical with a pendant edge except that its free endpoint is not listed as a vertex.

A *morphism of graphs*  $f : (D, V; I, \lambda) \rightarrow (D', V'; I', \lambda')$  is a function  $f : D \cup V \rightarrow D' \cup V'$  such that  $fD \subseteq D'$ ,  $fV \subseteq V'$ ,  $fI = I'f$  and  $f\lambda = \lambda'f$ . Thus, a morphism is an incidence-preserving mapping which takes vertices to vertices and edges to edges. Note that the image of a link can be a link, a loop or a semiedge, the image of a loop can be a loop or a semiedge, and the image of a semiedge can be just a semiedge. Composition of morphisms is defined as composition of functions from right to left. This defines the category **Grph**. Note that the condition  $fI = I'f$  implies that  $f$  is completely determined by its restriction  $f|_D$  on the set of darts. Moreover, if  $X$  is a *simple graph* (one without loops, semiedges or multiple adjacencies) then  $f|_V$  coincides with the usual notion of a graph morphism used by graph theorists.

A *walk of length*  $n \geq 1$  is a sequence of  $n$  darts  $W = x_1x_2 \dots x_n$  such that, for each index  $1 \leq k \leq n - 1$ , the terminal vertex of  $x_k$  coincides with the initial vertex of  $x_{k+1}$ . Moreover, we define each vertex to be a *trivial walk* of length 0. The *initial vertex* of  $W$  is the initial vertex of  $x_1$ , and the *terminal vertex* of  $W$  is the terminal vertex of  $x_n$ . The walk is *closed* if the initial and the terminal vertex coincide. In this case we say that the walk is *based* at that vertex. If  $W$  has initial vertex  $u$  and terminal vertex  $v$ , then we usually write  $W : u \rightarrow v$ . The walk  $W^{-1} = x_n^{-1}x_{n-1}^{-1} \dots x_1^{-1}$  is the *inverse* of  $W$ . Let  $W_1$  and  $W_2$  be two walks such that the terminal vertex of  $W_1$  coincides with the initial vertex of  $W_2$ . We define the *product*  $W_1W_2$  as the juxtaposition of the two sequences. The product of a walk  $W$  and a trivial walk is  $W$  itself. A graph is *connected* if for every pair of vertices  $u$  and  $v$  there exists a walk  $u \rightarrow v$ . A walk  $W$  is *reduced* if it contains no subsequence of the form  $xx^{-1}$ . Clearly, each walk gives rise to a unique reduced walk by a repeated cancellation of all occurrences of the form  $xx^{-1}$ . For instance, if  $x$  is a dart underlying a semiedge, then  $xx$  is a closed walk which reduces to the trivial walk at the initial vertex of  $x$ . Declaring two walks to be (*combinatorially*) *homotopic* provided that they give rise to the same reduced walk, we obtain an equivalence relation similar to the usual homotopy relation known in topology. The essential difference from the homotopy on the associated 1-CW complex is that a walk of length 1 traversing a semiedge is not homotopically trivial.

By  $\pi(X)$  we denote the *fundamental groupoid* of a graph  $X$ , that is, the set of all reduced walks equipped with the product  $W_1 \cdot W_2$  (often denoted simply by  $W_1W_2$ ) being the reduction of  $W_1W_2$ , whenever defined. Note that, for every vertex  $u$ , the trivial walk  $1_u$  behaves as the local identity in the groupoid. Clearly, the subset  $\pi(X, u) \subseteq \pi(X)$  of all reduced closed walks based at a vertex  $u \in X$  forms a group, the *fundamental group* of  $X$  at  $u$ . Note that if  $X$  is connected, then the fundamental

groups at any two vertices are isomorphic. Moreover, if  $X$  is finite, then  $\pi(X, u)$  is finitely generated. We would like to warn the reader that, in contrast to the classical case of graphs without semiedges, the fundamental group need not be a free group. Nevertheless, it is a free product of cyclic groups, each being isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}_2$ . A minimal generating set of  $\pi(X, u)$  can be constructed by the standard algorithm employing a *spanning tree* of  $X$  (an inclusion-minimal connected spanning subgraph) as in the classical case. This algorithm yields a bijective correspondence between the set of semiedges of  $X$  and the set of free factors in  $\pi(X, u)$  isomorphic to  $\mathbb{Z}_2$ .

The rest of the terminology not explicitly defined here (but freely used in the sequel) is tacitly adopted from graph theory as well as group theory and topology [27, 33, 13]. We only briefly mention the following. A (right) *action* of a group  $G$  on a set  $Z$  is defined by a function  $\sigma : Z \times G \rightarrow Z$ , with the common notation  $\sigma(z, g) = z \cdot g$ , such that  $z \cdot 1_G = z$  and  $z \cdot (gh) = (z \cdot g) \cdot h$ . We stick to the standard notation  $G_z = \{g \in G \mid z \cdot g = z\}$  to denote the *stabilizer* of  $z \in Z$  under the action of  $G$ . The left action is defined in a similar manner. We shall later need the concept of a group action extended to an action of a fundamental groupoid which can be introduced in the obvious way, see Proposition 4.3. By  $\text{Sym}_L Z$  and  $\text{Sym}_R Z$  we denote the *left* and the *right symmetric group* on a (nonempty) set  $Z$ , respectively. The right symmetric group on the set  $\{0, 1, \dots, n-1\}$  is commonly denoted by  $S_n$ . Suppose that an action of  $\mathcal{A}$  – a group or a fundamental groupoid – on some set  $Z$  is “represented” by a subgroup of the symmetric group  $\text{Sym}_R Z$ . We then consistently use the “bar notation”  $\bar{a}$  to denote the permutation which corresponds to  $a \in \mathcal{A}$ . Finally, we call an isomorphism  $f : \mathcal{A} \rightarrow \mathcal{A}'$  of two such algebraic structures *consistent* with the respective actions if there exists an isomorphism of these actions which has the form  $(-, f)$ ; more precisely, there exists a bijection  $\phi$  of the corresponding sets such that  $\phi(x \cdot a) = \phi x \cdot f a$ .

## 4 Action of the fundamental groupoid

Let  $X = (D, V; I, \lambda)$  and  $\tilde{X} = (\tilde{D}, \tilde{V}, \tilde{I}, \tilde{\lambda})$  be graphs. A graph epimorphism  $p : \tilde{X} \rightarrow X$  is called a *covering projection* if, for every vertex  $\tilde{u} \in \tilde{X}$ , the set  $\tilde{D}_{\tilde{u}}$  is bijectively mapped onto  $D_{p\tilde{u}}$ . The graph  $X$  is usually referred to as the *base graph* and  $\tilde{X}$  as the *covering graph*. By  $\text{fib}_u = p^{-1}u$  and  $\text{fib}_x = p^{-1}x$  we denote the *fibre* over  $u \in V$  and  $x \in D$ , respectively.

**Example 4.1** Figure 1 shows a covering  $Q_3 \rightarrow st_3$ , where  $Q_n$  denotes the  $n$ -cube and  $st_n$  the  $n$ -semistar, that is a graph on one vertex and  $n$  semiedges.

Consider an arbitrary dart  $x \in X$  with its initial vertex  $u$ . By definition, for each vertex  $\tilde{u} \in \text{fib}_u$  there exists a unique dart  $\tilde{x}$  which is the lift of  $x$ , that is,

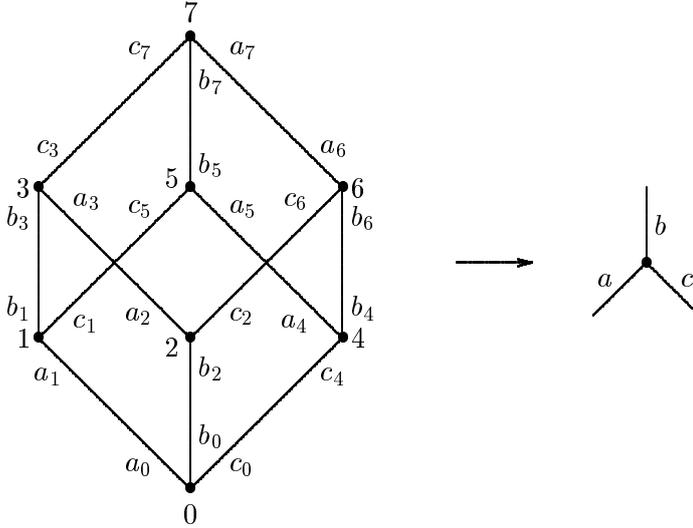


Figure 1: A covering projection  $Q_3 \rightarrow st_3$ , given by  $x_i \rightarrow x$ .

$p\tilde{x} = x$ . This means that every walk of length 1 starting at  $u$  lifts uniquely to a walk starting at  $\tilde{u}$ . Consequently, by induction we obtain the following “Unique walk lifting theorem”. (For the classical topological variant, see [27, pp. 151].)

**Proposition 4.2** *Let  $p : \tilde{X} \rightarrow X$  be a covering projection of graphs and let  $W : u \rightarrow v$  be an arbitrary walk in  $X$ . Then:*

- (a) *For every vertex  $\tilde{u} \in \text{fib}_u$  there is a unique walk  $\tilde{W}$  which projects to  $W$  and has  $\tilde{u}$  as the initial vertex.*
- (b) *Homotopic walks lift to homotopic walks.*
- (c) *If  $X$  is connected, then the cardinality of  $\text{fib}_u$  does not depend on  $u \in X$ . ■*

Note that this theorem may fail when considering a path starting at the free endpoint of a semiedge in the 1-CW complex associated to a graph. This difficulty does not occur with combinatorial graphs since we only consider walks which start and end in vertices. It follows from Proposition 4.2 that the cardinality  $|\text{fib}_u|$  of a covering projection  $p : \tilde{X} \rightarrow X$ , where  $X$  is connected, does not depend on the vertex  $u$ . If  $|\text{fib}_u| = n$  is an integer, we say that the covering projection  $p$  is  $n$ -fold.

Let  $p : \tilde{X} \rightarrow X$  be a covering projection. To simplify the notation we write  $\pi = \pi(X)$  and  $\pi^u = \pi(X, u)$ . As an immediate consequence of the unique walk lifting we have the following simple but useful observation.

**Proposition 4.3** *Let  $p : \tilde{X} \rightarrow X$  be a covering projection of graphs. Then there exists a right action of  $\pi$  on the vertex set of  $\tilde{X}$  defined by*

$$\tilde{u} \cdot W = \tilde{v},$$

where  $W : p\tilde{u} \rightarrow p\tilde{v}$  and  $\tilde{v}$  is the endvertex of the unique walk  $\tilde{W}$  over  $W$  starting at  $\tilde{u}$ . Moreover,

$$\hat{W} : \tilde{u} \mapsto \tilde{u} \cdot W$$

is a bijection  $\text{fib}_{p\tilde{u}} \rightarrow \text{fib}_{p\tilde{v}}$ . ■

In view of Proposition 4.2(b), the action of the fundamental groupoid can, in fact, be extended to an action of all walks. In particular, the fundamental group  $\pi^u$  acts on  $\text{fib}_u$ . If  $\tilde{X}$  (and hence  $X$  as well) is connected, then this action is transitive. To simplify the notation we denote by  $\pi_{\tilde{u}} = \pi_{\tilde{u}}^u$  the stabilizer of  $\tilde{u} \in \text{fib}_u$  under the action of  $\pi^u$ . Clearly, the actions of the fundamental groups at two distinct vertices are isomorphic. Indeed, if  $\hat{W}$  is the bijection as in Proposition 4.3 and if  $W_*S = W^{-1}SW$ ,  $S \in \pi^u$ , then  $(\hat{W}, W_*) : (\text{fib}_u, \pi^u) \rightarrow (\text{fib}_v, \pi^v)$  is an isomorphism of actions.

## 5 Lifting automorphisms

Let  $p : \tilde{X} \rightarrow X$  be a covering projection of graphs and let  $f$  be an automorphism of  $X$ . We say that  $f$  *lifts* if there exists an automorphism  $\tilde{f}$  of  $\tilde{X}$ , a *lift* of  $f$ , such that the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ p \downarrow & & \downarrow p \\ X & \xrightarrow{f} & X \end{array}$$

is commutative. Note that if  $f$  lifts, then so does  $f^{-1}$ . More generally, let  $A \leq \text{Aut } X$ . Then the lifts of all automorphisms in  $A$  form a group, the *lift* of  $A$ , often denoted by  $\tilde{A} \leq \text{Aut } \tilde{X}$ . In particular, the lift of the trivial group is known to be the group of *covering transformations* and is denoted by  $\text{CT}(p)$ . There is an associated group epimorphism  $p_A : \tilde{A} \rightarrow A$  with  $\text{CT}(p)$  as the kernel. The set of all lifts  $\text{Lft}(f)$  of a given  $f \in \text{Aut } X$  is a coset of  $\text{CT}(p)$  in  $\tilde{A}$ .

As an immediate consequence of the unique walk lifting we have the following observation.

**Proposition 5.1** *Let  $p : \tilde{X} \rightarrow X$  be a covering projection of connected graphs. Then  $\text{CT}(p)$  acts semiregularly on  $\tilde{X}$ ; that is,  $\text{CT}(p)$  acts without fixed points both on vertices and on darts of  $\tilde{X}$ . ■*

Again, the corresponding topological version of Proposition 5.1 may fail to be true. The next proposition is a trivial but important consequence of Proposition 5.1.

**Proposition 5.2** *Let  $p : \tilde{X} \rightarrow X$  be a covering projection of connected graphs and let  $f \in \text{Aut } X$  have a lift. Then each  $\tilde{f} \in \text{Lft}(f)$  is uniquely determined by the mapping of a single vertex (as well as by the mapping of a single dart) of  $\tilde{X}$ . ■*

The fact that  $\text{CT}(p)$  acts semiregularly if  $\tilde{X}$  is connected is particularly nice. Therefore, for the rest of this section as well as for most of the paper we shall assume the graphs  $X$  and  $\tilde{X}$  to be connected. By semiregularity of  $\text{CT}(p)$ , we have  $|\text{CT}(p)| \leq |\text{fib}_u|$ , with equality occurring if and only if  $\text{CT}(p)$  acts regularly on each fibre. We therefore define the covering projection to be *regular* if the action of  $\text{CT}(p)$  is regular on each fibre.

The next three results, much in the spirit of classical theory [27], are aimed at a smooth transition to a combinatorial treatment of lifting automorphisms. The first one states that, briefly speaking, an automorphism of the base graph has a lift if and only if this automorphism is consistent with the actions of the fundamental groups.

**Theorem 5.3** *Let  $p : \tilde{X} \rightarrow X$  be a covering projection of connected graphs and let  $f$  be an automorphism of  $X$ . Then  $f$  lifts to some  $\tilde{f} \in \text{Aut } \tilde{X}$  if and only if there exists, for an arbitrarily chosen base vertex  $b \in X$ , an isomorphism of actions*

$$(\phi, f) : (\text{fib}_b, \pi^b) \rightarrow (\text{fib}_{fb}, \pi^{fb})$$

*of the fundamental groups such that  $\tilde{f}|_{\text{fib}_b} = \phi$ . Moreover, there is a bijective correspondence between  $\text{Lft}(f)$  and all functions  $\phi$  for which  $(\phi, f)$  is such an isomorphism, and we have*

$$\tilde{f}\tilde{u} = \phi(\tilde{u} \cdot W) \cdot fW^{-1}, \quad W : p\tilde{u} \rightarrow b.$$

**PROOF.** Let  $\tilde{f}$  be a lift of  $f$ . We first show that for an arbitrary walk  $W : u \rightarrow v$  in  $X$  and  $\tilde{u} \in \text{fib}_u$  we have

$$\tilde{f}(\tilde{u} \cdot W) = \tilde{f}\tilde{u} \cdot fW. \tag{1}$$

Indeed, let  $\tilde{W} : \tilde{u} \rightarrow \tilde{u} \cdot W$ . Then  $\tilde{f}\tilde{W} : \tilde{f}\tilde{u} \rightarrow \tilde{f}(\tilde{u} \cdot W)$  projects to  $fW$ , implying (1). In particular, (1) holds true for each  $\tilde{u} \in \text{fib}_b$  and  $S \in \pi^b$ . Since  $\phi := \tilde{f}|_{\text{fib}_b}$  and  $f$  are bijections, we have the required isomorphism of actions.

Conversely, let  $(\phi, f)$  be such an isomorphism. Define the required lift  $\tilde{f}$  on vertices of  $\tilde{X}$  as follows. Let  $\tilde{u}$  be an arbitrary vertex of  $\tilde{X}$  and let  $u = p\tilde{u}$ . Choose an arbitrary walk  $W : u \rightarrow b$  and set

$$\tilde{f}\tilde{u} = \phi(\tilde{u} \cdot W) \cdot fW^{-1}.$$

We have to show several things. First of all, this mapping is well defined, that is to say, it does not depend on the choice of  $W$ . Indeed, let  $W_1, W_2 : u \rightarrow b$ . Then  $\tilde{u} \cdot W_1 = (\tilde{u} \cdot W_2) \cdot W_2^{-1}W_1$ . Therefore,  $\phi(\tilde{u} \cdot W_1) = \phi(\tilde{u} \cdot W_2) \cdot f(W_2^{-1}W_1) = \phi(\tilde{u} \cdot W_2) \cdot fW_2^{-1} \cdot fW_1$ , and hence  $\phi(\tilde{u} \cdot W_1) \cdot fW_1^{-1} = \phi(\tilde{u} \cdot W_2) \cdot fW_2^{-1}$ . Next, from the definition of  $f$  it is easily seen that  $pf\tilde{u} = fp\tilde{u}$ . We verify that it is a bijection. To show that it is onto, let  $\tilde{v}$  be an arbitrary vertex of  $\tilde{X}$  and choose  $W : p\tilde{v} \rightarrow fb$  arbitrarily. It is easily checked that the vertex  $\phi^{-1}(\tilde{v} \cdot W) \cdot f^{-1}W^{-1}$  is mapped to  $\tilde{v}$ . To show that it is 1-to-1, let  $\phi(\tilde{u}_1 \cdot W_1) \cdot fW_1^{-1} = \tilde{f}\tilde{u}_1 = \tilde{f}\tilde{u}_2 = \phi(\tilde{u}_2 \cdot W_2) \cdot fW_2^{-1}$ . Then  $fW_1$  and  $fW_2$  have the same initial vertex and hence  $W_1$  and  $W_2$  as well. Therefore,  $\tilde{u}_1$  and  $\tilde{u}_2$  belong to the same fibre. Moreover, it follows easily that  $\phi(\tilde{u}_1 \cdot W_1) \cdot f(W_1^{-1}W_2) = \phi(\tilde{u}_2 \cdot W_2)$ . This implies  $\phi(\tilde{u}_1 \cdot W_1 \cdot W_1^{-1}W_2) = \phi(\tilde{u}_2 \cdot W_2)$  and hence  $\phi(\tilde{u}_1 \cdot W_2) = \phi(\tilde{u}_2 \cdot W_2)$ . Thus  $\tilde{u}_1 \cdot W_2 = \tilde{u}_2 \cdot W_2$  and so  $\tilde{u}_1 = \tilde{u}_2$ .

Having  $f$  defined on vertices it is immediate that the mapping extends to darts. We conclude that  $\tilde{f}$  as above is a lift of  $f$ . This shows that taking the restriction  $\text{Lft}(f) \rightarrow \text{Lft}(f)|_{\text{fib}_b}$  defines a function onto the set of all such  $\phi$  for which  $(\phi, f)$  is an isomorphism of fundamental groups. That this function is 1-to-1 follows from Proposition 5.2. ■

It is a consequence of Theorem 5.3 that the problem of whether some automorphism has a lift can be reduced to the problem of whether this automorphism gives rise to an isomorphism of actions of fundamental groups. This in turn can be tested just by comparing how the automorphism maps a stabilizer under the action of the fundamental group. In addition, it enables us to give a more explicit formula expressing how an arbitrary lifted automorphism acts on the vertex fibres.

**Theorem 5.4** *Let  $p : \tilde{X} \rightarrow X$  be a covering projection of connected graphs, and let  $f$  be an automorphism of  $X$ . Then:*

(a) *There exists an isomorphism of actions*

$$(\phi, f) : (\text{fib}_b, \pi^b) \rightarrow (\text{fib}_{fb}, \pi^{fb})$$

*if and only if  $f$  maps the stabilizer  $\pi_{\tilde{b}}$  of an arbitrarily chosen base point  $\tilde{b} \in \text{fib}_b$  isomorphically onto some stabilizer  $\pi_{\tilde{v}} \leq \pi^{fb}$ . In this case we have that  $\tilde{v} = \phi\tilde{b}$ , and there is a bijective correspondence between all choices of such a vertex  $\tilde{v}$  and all such isomorphisms.*

(b) Choosing a base point  $\tilde{d} \in \text{fib}_{fb}$  and  $Q \in \pi^{fb}$  such that  $Q^{-1}\pi_{\tilde{d}}Q = f\pi_{\tilde{b}}$ , we have that all such bijections  $\phi = \phi_P$  are given by

$$\phi_P(\tilde{b} \cdot S) = \tilde{d} \cdot Pf(S), \quad (S \in \pi^b), \quad (2)$$

where  $P$  belongs to the coset  $N(\pi_{\tilde{d}})Q$  of the normalizer of  $\pi_{\tilde{d}}$  within  $\pi^{fb}$ . Moreover,  $\phi_{P'} = \phi_P$  if and only if  $P' \in \pi_{\tilde{d}}P$ .

PROOF. It is clear that if  $(\phi, f)$  is an isomorphism of actions, then the condition holds. For the converse, let  $f\pi_b^b = \pi_{\tilde{v}}^{fb}$ . Each  $\tilde{u} \in \text{fib}_b$  can be written as  $\tilde{u} = \tilde{b} \cdot S$  for some  $S \in \pi^b$  because  $X$  is connected. Define  $\phi$  by setting  $\phi\tilde{u} = \tilde{v} \cdot fS$ . It is easy to check that  $(\phi, f)$  is the required isomorphism of actions. The claimed bijective correspondence should also be clear since  $\phi$  is completely determined by the mapping of one point. This proves part (a).

To prove part (b), let  $\tilde{v} = \tilde{d} \cdot P$  be any point satisfying the condition as in part (a). Then  $P^{-1}\pi_{\tilde{d}}P = \pi_{\tilde{d}.P} = Q^{-1}\pi_{\tilde{d}}Q$ , that is  $PQ^{-1} \in N(\pi_{\tilde{d}})$ . The last statement is clear as well. ■

As an immediate corollary of the above two theorems we have the following observation. There exists a covering transformation mapping  $\tilde{u}_1 \in \text{fib}_u$  to  $\tilde{u}_2 \in \text{fib}_u$  if and only if  $\pi_{\tilde{u}_1} = \pi_{\tilde{u}_2}$ . Since a covering projection is regular if and only if  $\text{CT}(p)$  acts transitively, it follows that regular coverings can be characterized by means of stabilizers under the action of the fundamental group.

**Corollary 5.5** *A covering projection of connected graphs  $p : \tilde{X} \rightarrow X$  is regular if and only if, for an arbitrarily chosen vertex  $b$  of  $X$ , all the stabilizers under the action of  $\pi^b$  on  $\text{fib}_b$  coincide, that is to say, the stabilizer is a normal subgroup.* ■

## 6 Voltage spaces

The goal of this section is to explain, unify and generalize the classical concepts of ordinary, permutation and relative voltage assignments introduced in the 70's by Alpert, Biggs, Gross, Tucker and others [13]. In fact, this unified approach works in the general setting of fairly arbitrary topological spaces, see [20].

A *voltage space* on a connected graph  $X$  is a triple  $(F, G; \xi)$  where  $G$  is a group acting on a nonempty set  $F$  and  $\xi : \pi \rightarrow G$  is a homomorphism (where, as usual,  $\pi = \pi(X)$ ). The group  $G$  is called the *voltage group*,  $F$  is the *abstract fibre* and  $\xi_W$  is the *voltage* of a reduced walk  $W$ . Since the product of reduced walks is mapped by the homomorphism  $\xi$  to the product of voltages, it follows that any trivial walk carries the trivial voltage and consequently, inverse walks carry inverse voltages.

In order to define a voltage space on a graph it is obviously enough to specify the voltages on walks of length 1. Even more, we only need to specify the voltages on darts  $\xi : D \rightarrow G$  subject to the condition

$$\xi_{x^{-1}} = (\xi_x)^{-1}.$$

By induction we can then extend the assignment of voltages to all walks, not just to the reduced ones. Clearly, homotopic walks carry the same voltage. Note that it follows from the above condition that a voltage associated to a semiedge is always an involution.

If the voltage group acts faithfully, we speak of a *monodromy voltage space*; by taking the permutation representation of this action we get the *permutation voltage space* as its canonical representative. If the voltage group acts regularly, we speak of a *regular voltage space*. In particular, if  $G$  acts on  $F = G$  by right translation, we speak of the *Cayley voltage space* (ordinary voltages in the terminology of Gross and Tucker [13]).

Obviously,

$$G^u = \xi(\pi^u)$$

is a group, called the *local voltage group* at the vertex  $u$ . Moreover, let  $W : u \rightarrow v$ . Then the inner automorphism

$$W^\#(g) = \xi_W^{-1}g\xi_W$$

of  $G$  takes  $G^u$  onto  $G^v$ . A voltage space is called *locally transitive* whenever some (and hence any) local voltage group acts transitively on the abstract fibre.

We say that a voltage space  $(F, G; \xi)$  on a graph  $X$  is *associated to a covering projection*  $p : \tilde{X} \rightarrow X$  if there exists a labelling  $\ell : \tilde{V} \rightarrow F$  of the vertex-set  $\tilde{V}$  of  $\tilde{X}$  such that, for each vertex  $u$ , we have that the restrictions

$$\ell_u = \ell|_{\text{fib}_u} : \text{fib}_u \rightarrow F$$

are bijections and moreover,

$$(\ell, \xi) : (\tilde{V}, \pi) \rightarrow (F, G)$$

is a morphism of actions. In other words, for every (reduced) walk  $W : u \rightarrow v$  of  $X$  and for each  $\tilde{u} \in \text{fib}_u$  we have

$$\ell(\tilde{u} \cdot W) = \ell\tilde{u} \cdot \xi_W.$$

(For simplicity, we use the same dot sign for the action of  $\pi$  and the action of  $G$ .) Voltage spaces associated to isomorphic covering projections are called *equivalent*.

Note that the action of  $\xi_W$  “represents” the bijection  $\hat{W} : \tilde{u} \mapsto \tilde{u} \cdot W$  introduced in Proposition 4.3: indeed, its permutation representation  $\overline{W} \in \text{Sym}_L F$  is precisely the permutation  $\overline{\xi}_W \in \text{Sym}_R F$ . Note further that  $(\ell_u, \xi) : (\text{fib}_u, \pi^u) \rightarrow (F, G^u)$  is an epimorphism of actions with  $\ell_u$  being a bijection. Hence  $G^u$  acts on  $F$  “in the same way modulo relabelling” as  $\pi^u$  does on  $\text{fib}_u$ , and thus both actions have “the same local monodromy”. In particular, a voltage space associated to a covering projection is necessarily locally transitive, and as in the classical case [36], a covering projection is regular if and only if  $G^u$  has a normal stabilizer.

Each covering projection of graphs gives rise to an associated voltage space, for instance to the permutation voltage space (cf. [12]) obtained as follows: choose the labelling  $\ell$  of each vertex fibre arbitrarily and set, for each  $W \in \pi$ , the voltage  $\xi_W$  to be the permutation in  $\overline{W} \in \text{Sym}_R F$  representing the bijection  $\hat{W}$ . In particular, if the covering projection  $p$  is  $n$ -fold we may assume that  $\text{fib}_u = \{u_1, \dots, u_n\}$  for every vertex  $u$  of  $X$  and that the label of the vertex  $u_i$  is  $i \in \{1, \dots, n\}$ . Then the permutation voltage  $\xi_x \in S_n$  of a dart  $x$  joining its initial vertex  $u$  to its terminal vertex  $v = Ix^{-1}$  is defined by the following rule:  $\xi_x(i) = j$  if and only if there is a dart  $\tilde{x} \in \text{fib}_x$  joining  $u_i$  to  $v_j$ .

To each regular covering projection we can associate a Cayley voltage space  $(G, G; \xi)$  as follows. First label the elements of every vertex fibre by elements of  $G \cong \text{CT}(p)$  such that the left action of  $\text{CT}(p)$  is viewed as left translation on itself. Then the voltage action of  $G$  is the right translation on itself, that is,  $\overline{W}(g) = g\xi_W$  (cf. [10]).

**Example 6.1** We revisit Example 4.1. The regular voltage space  $(\{0, 1, \dots, 7\}, \mathbb{Z}_2^3; \xi)$  on the semistar  $st_3$ , where the homomorphism  $\xi : \pi \rightarrow \mathbb{Z}_2^3$  is defined by setting  $\xi_a = 001$ ,  $\xi_b = 010$  and  $\xi_c = 100$ , is associated to the covering  $Q_3 \rightarrow st_3$ . Replacing each element of the abstract fibre  $\{0, 1, \dots, 7\}$  by its 3-digit binary code, the Cayley voltage space, associated to the same covering, is obtained. Another equivalent regular voltage space is  $(\{0, 1, \dots, 7\}, S_8; \psi)$ , where  $\psi_a = (01)(23)(45)(67)$ ,  $\psi_b = (02)(13)(46)(57)$  and  $\psi_c = (04)(15)(26)(37)$ .

Conversely, every locally transitive voltage space  $(F, G; \xi)$  on a graph  $X$  gives rise to a covering projection, namely to the one to which this voltage space is associated. (If the voltage space is not locally transitive, then the resulting graph is disconnected.) This *derived* covering projection is determined up to isomorphism. In particular, regular (locally transitive) voltage spaces give rise to regular covering projections. The proof of these facts is analogous to the proof in the classical case of graphs without semiedges [13, pp. 57–71]. Details are left to the reader.

**Example 6.2** Let us call a graph with one vertex and arbitrary number of darts a *monopole*. Given a group  $G$  and a symmetric generating (multi)set  $S = S^{-1}$  of

$G$  we define the *Cayley graph*  $\text{Cay}(G, S)$  to be the graph  $(G \times S, G; I_S, \lambda_S)$ , where  $I_S = \text{pr}_1$  is the projection onto the first factor and the involution on darts is given by  $\lambda_S(g, s) = (gs, s^{-1})$ . (Note that  $S$  being possibly a multiset allows Cayley graphs to have repeated generators.) Consider the monopole  $X = (S, v; I, \lambda)$  where  $I$  is the constant function and  $\lambda(s) = s^{-1}$ . It is easy to see that the projection onto the second factor  $\text{pr}_2 : G \times S \rightarrow S$  extends to a regular covering projection  $\text{Cay}(G, S) \rightarrow X$ . Conversely, let  $(G, G; \xi)$  be an arbitrary Cayley voltage space on a monopole  $X = (D, v; I, \lambda)$ . Then the derived covering graph is the Cayley graph  $\text{Cay}(G, S)$  where  $S = \{\xi_x \mid x \in D\}$  is the corresponding symmetric (multi)set of generators for the group  $G$ .

This shows that *Cayley graphs are nothing but regular covers over monopoles*. Summing up, by allowing semiedges we have obtained a characterization of Cayley graphs which overcomes the trouble encountered in the classical approach that not all Cayley graphs were regular coverings over bouquets of circles [13, pp. 68–69].

Observe that the voltage group can be larger than really necessary. In fact, it is the local group which is responsible for the structure of the covering. The following proposition shows that any given voltage space can be replaced by an equivalent voltage space where the local groups do not depend on the vertex and coincide with the whole voltage group.

**Proposition 6.3** *Let  $(F, G; \xi)$  be a voltage space associated to a covering projection  $p : \tilde{X} \rightarrow X$  of connected graphs, and let  $b \in X$  be a base vertex. Then there exists an equivalent voltage space  $(F, G; \xi')$  which has all local voltage groups equal and the local group at  $b$  has not changed.*

PROOF. For each vertex  $u \in X$  choose a preferred walk  $W^u : u \rightarrow b$ , where  $W^b$  is the trivial walk at  $b$ . (This can easily be done, for instance, by specifying a spanning tree for  $X$ .) Let  $W : u \rightarrow v$  be an arbitrary walk in  $X$ . Set  $\xi'_W = (\xi_{W^u})^{-1} \xi_W \xi_{W^v}$ . As a result, the new voltage of each preferred walk is trivial and  $\xi'_S = \xi_S$  for every closed walk  $S \in \pi^b$ . Now, modify the labelling of fibres according to the rule  $\ell'_t = \ell_t \cdot \xi_{W^u}$ ,  $t \in X$ . It is easy to see that the new voltage space has all the claimed properties. ■

Since any two vertices can be joined by a walk carrying the trivial voltage, no walk can have voltage outside  $G^b$ . For if  $W : u \rightarrow v$  had a voltage not in  $G^b$ , then the closed walk  $S = (W^u)^{-1} W W^v$  at  $b$  would have its voltage  $\xi'_S = \xi'_W \notin G^b$ , which is absurd. Hence, the local group  $G^b$  can be chosen as the new voltage group without affecting the covering. Assuming that a voltage space  $(F, G; \xi)$  satisfies the condition  $G^b = G$  therefore does not mean any loss of generality, but can sometimes considerably simplify the notation.

## 7 Combinatorialization by voltages

Let  $p : \tilde{X} \rightarrow X$  be a covering projection of connected graphs. The question whether an automorphism  $f$  of  $X$  has a lift is equivalent, by Theorem 5.3, to the problem whether  $f$  is consistent with the actions of fundamental groups  $\pi^b$  and  $\pi^{fb}$ .

Let  $(F, G; \xi)$  be a voltage space associated to the above covering. As remarked in Section 6, the action of the fundamental group  $\pi^u$  on  $\text{fib}_u$  is “the same, modulo relabelling” as the action of the local group  $G^u$  on  $F$ . Can we transfer the isomorphism problem involving fundamental groups to a similar one by just considering the local voltage groups? The answer is *yes*, provided that certain additional requirements are imposed on the voltage space. This is especially relevant in the context of lifting groups rather than individual automorphisms.

Let  $A$  be a group of automorphisms of  $X$ . We say that the voltage space is *locally  $A$ -invariant* at a vertex  $v$  if, for every  $f \in A$  and for every walk  $W \in \pi^v$ , we have

$$\xi_W = 1 \Rightarrow \xi_{fW} = 1. \quad (3)$$

We remark that we can speak of local invariance without specifying the vertex, as local invariance at some vertex implies the same at all vertices. A voltage space is *locally  $f$ -invariant* for an individual automorphism  $f$  of  $X$  if it is locally invariant with respect to the group  $\langle f \rangle$  generated by  $f$ . This is equivalent to requiring that the condition (3) be satisfied by both  $f$  and  $f^{-1}$ . If  $f$  has finite order it is even sufficient to check (3) for  $f$  alone.

Assume that the voltage space is locally  $A$ -invariant. Since with each  $f \in A$  its inverse  $f^{-1}$  must also satisfy the condition (3), local  $A$ -invariance is equivalent to the requirement that for each  $f \in A$  there exists an induced isomorphism  $f^{\#u} : G^u \rightarrow G^{fu}$  of local voltage groups such that the following diagram

$$\begin{array}{ccc} \pi^u & \xrightarrow{f} & \pi^{fu} \\ \xi \downarrow & & \downarrow \xi \\ G^u & \xrightarrow{f^{\#u}} & G^{fu} \end{array}$$

is commutative; in other words,  $f^{\#u}(\xi_W) = \xi_{fW}$  where  $W \in \pi^u$ .

It is clear that in order to transfer the isomorphism problem involving fundamental groups to local groups we must require local invariance in the first place. If the voltage space is, say, a monodromy one, then the existence of a lift implies local invariance. However, this need not be true in general.

**Theorem 7.1** *Let  $(F, G; \xi)$  be a voltage space associated to a covering  $p : \tilde{X} \rightarrow X$  of connected graphs, and let  $f$  be an automorphism of  $X$ .*

- (a) *The automorphism  $f$  has a lift if and only if, for an arbitrarily chosen  $i \in F$ , there exists some  $j \in F$  such that, for every walk  $S \in \pi^b$ , we have  $i \cdot \xi_S = i \Leftrightarrow j \cdot \xi_{fS} = j$ .*
- (b) *Let  $(F, G; \xi)$  be locally  $f$ -invariant. Then  $f$  has a lift if and only if the induced isomorphism  $f^{\#b} : G^b \rightarrow G^{fb}$  is consistent with the actions of the local voltage groups.*

PROOF. As already remarked in Section 6,  $(\ell_u, \xi) : (\text{fib}_u, \pi^u) \rightarrow (F, G^u)$  is a morphism of actions with  $\ell_u$  a bijection. Hence, if  $S \in \pi^u$  and  $\ell_u(\tilde{u}) = i$ , then  $S \in (\pi^u)_{\tilde{u}}$  if and only if  $\xi_S \in G_i^u$ . Thus part (a) follows from Theorems 5.3 and 5.4.

Let us prove the statement in (b). By Theorem 5.3, we have that  $f$  has a lift if and only if  $f$  is consistent with the actions of fundamental groups. By the assumption,  $f^{\#b}$  exists. It is easy to see that  $f$  is consistent if and only if  $f^{\#b}$  is consistent. ■

For regular voltage spaces associated to a regular covering we can immediately infer that an automorphism lifts if and only if every closed walk carrying the trivial voltage is mapped to a closed walk with the same property. A special form of this result has been previously observed by Gvozdjak and Širáň [14] and Nedela and Škoviera [29].

**Corollary 7.2** *Let the covering as well as the voltage space be regular. Then  $f$  lifts if and only if the voltage space is locally  $f$ -invariant. ■*

The next corollary constitutes the last stage of combinatorialization of the lifting problem by reducing it to a system of equations in the symmetric group  $S_n$ , where  $n$  is the size of the fibre. This approach has already been used by Malnič and Marušič in [22]. The method is further illustrated in Section 8.

**Corollary 7.3** *Let  $(F, \text{Sym}_R F; \xi)$  be a permutation voltage space on a graph  $X$ . Then, an automorphism  $f$  of  $X$  has a lift if and only if the system of equations in  $\text{Sym}_R F$*

$$\xi_S \cdot \tau = \tau \cdot \xi_{fS},$$

*where  $S$  runs through a generating set for  $\pi^b$ , has a solution. Moreover, there is a bijective correspondence between all solutions and the restrictions  $\text{Lft}(f)|_{\text{fib}_b}$ . ■*

We end this section by giving some explicit formulas for the lifts in terms of voltages. We denote by

$$\tau_{u,\tilde{f}} \in \text{Sym}_L F$$

the permutation representation of  $\tilde{f}|_{\text{fib}_u}$ . Rewriting the formula given in Theorem 5.3 yields

$$\tau_{u,\tilde{f}}(k) = \tau_{b,\tilde{f}}(k \cdot \xi_W) \cdot (\xi_{fW})^{-1}, \quad (4)$$

where  $W : u \rightarrow b$  is arbitrary. In the case of permutation voltages the formula (4) rewrites as

$$\tau_{u,\tilde{f}} = \xi_W \cdot \tau_{b,\tilde{f}} \cdot (\xi_{fW})^{-1} \quad (5)$$

(note that this multiplication is in  $\text{Sym}_R F$ ), and in the case of Cayley voltages of regular coverings the formula (4) rewrites as

$$\tau_{u,\tilde{f}}(g) = \tau_{b,\tilde{f}}(g\xi_W)(\xi_{fW})^{-1}. \quad (6)$$

It remains to determine  $\tau_{b,\tilde{f}}$ . In the case of permutation voltages we only need to solve the system of equations given in Corollary 7.3. As long as we consider finite graphs, the system is finite because we only need to consider the generators of the fundamental group. However, to solve such a system of permutation equations is algorithmically difficult in general.

An explicit formula which takes into account the mapping of stabilizers can be derived from Theorem 5.4. Let  $i_0, j_0 \in F$  be arbitrary and let  $j \in F$  be as in Theorem 7.1(a). Choose  $Q \in \pi^{fb}$  such that  $j = j_0 \cdot \xi_Q$ . We then have

$$\tau_{b,\tilde{f}}(i_0 \cdot \xi_S) = j_0 \cdot a_{\tilde{f}} \xi_{fS}, \quad S \in \pi^b, \quad (7)$$

where  $a_{\tilde{f}}$  belongs to the coset  $N(G_{j_0})\xi_Q$  and is determined up to a coset of  $G_{j_0}$ ; note that we have here tacitly assumed  $G^b = G$ , cf. the remarks after Proposition 6.3. In the case of Cayley voltages the formula (7) can be simplified. By taking  $i_0 = j_0 = 1$  and using the fact that  $f^{\#b}$  exists, we have

$$\tau_{b,\tilde{f}}(g) = a_{\tilde{f}} f^{\#b}(g). \quad (8)$$

## 8 Coverings of the Petersen graph

Let  $P$  be the Petersen graph, with  $V(P) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  being the vertex-set labelled as in Figure 2. We characterize all covering projections  $\tilde{P} \rightarrow P$  such that the automorphism group  $\text{Aut } P$  lifts.

Since every covering projection of  $P$  can be described by means of a permutation voltage space, it is sufficient to characterize all the assignments of permutations to

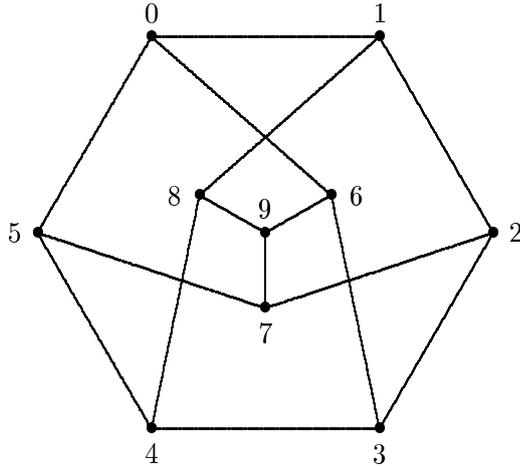


Figure 2: The Petersen graph.

the darts of  $P$  such that the derived covering has the required property. By [36, 13] we may assume that the darts of an arbitrarily chosen spanning tree carry the identity permutation. Consequently, every covering projection  $\tilde{P} \rightarrow P$  is completely described by a 6-tuple of permutations  $(\xi_0, \xi_1, \dots, \xi_5)$ , where  $\xi_i$  is the voltage carried by the dart  $x_i$  joining  $i$  to  $i + 1$ , for  $i = 0, 1, \dots, 5$  with arithmetics mod 6. Since  $\text{Aut } P$  is generated by the automorphisms  $f = (012345)(876)$ ,  $g = (03)(12)(45)(78)$  and  $h = (01845)(29376)$ , Corollary 7.3 gives us three systems of equations  $\xi_S \tau = \tau \xi_{fS}$ ,  $\xi_S \eta = \eta \xi_{gS}$  and  $\xi_S \omega = \omega \xi_{hS}$ , where  $S \in \pi^9$  ranges over all elements of the fundamental group at the vertex 9. The fundamental group  $\pi^9$  is generated by the following six closed walks (for brevity indicated just as cyclic sequences of vertices)  $W_0 = (96018)$ ,  $W_1 = (98127)$ ,  $W_2 = (97236)$ ,  $W_3 = (96348)$ ,  $W_4 = (98457)$  and  $W_5 = (97506)$ . Thus the above systems of equations reduce to the following 18 equations in variables  $\xi_0, \dots, \xi_5$ ,  $\tau$ ,  $\eta$ , and  $\omega$ , where every generator of  $\pi^9$  and every generator of  $\text{Aut } P$  gives rise to one equation:

$$\begin{array}{lll}
\xi_0\tau = \tau\xi_1 & \xi_0\omega = \omega\xi_1^{-1} & \xi_0\eta = \eta\xi_2^{-1}\xi_1^{-1}\xi_3^{-1} \\
\xi_1\tau = \tau\xi_2 & \xi_1\omega = \omega\xi_0^{-1} & \xi_1\eta = \eta\xi_3 \\
\xi_2\tau = \tau\xi_3 & \xi_5\omega = \omega\xi_2^{-1} & \xi_2\eta = \eta\xi_2 \\
\xi_3\tau = \tau\xi_4 & \xi_2\omega = \omega\xi_5^{-1} & \xi_3\eta = \eta\xi_2^{-1}\xi_4^{-1}\xi_3^{-1} \\
\xi_4\tau = \tau\xi_5 & \xi_4\omega = \omega\xi_3^{-1} & \xi_4\eta = \eta\xi_3\xi_4\xi_5 \\
\xi_5\tau = \tau\xi_0 & \xi_3\omega = \omega\xi_4^{-1} & \xi_5\eta = \eta\xi_0\xi_1\xi_2.
\end{array}$$

From the first column it easily follows that the required six voltages  $\xi_i$  can be expressed as  $\xi_i = \tau^{-i}\nu\tau^i$ ,  $i = 0, 1, \dots, 5$ , where  $\nu$  and  $\tau$  are permutations satisfying the identity  $\nu\tau^6 = \tau^6\nu$ . Thus  $f$  lifts if and only if the covering projection is determined by the 6-tuple

$$(\xi_0, \xi_1, \dots, \xi_5) = (\nu, \tau^{-1}\nu\tau, \tau^{-2}\nu\tau^2, \tau^{-3}\nu\tau^3, \tau^{-4}\nu\tau^4, \tau^{-5}\nu\tau^5),$$

where  $\nu$  and  $\tau$  satisfy the identity  $\nu\tau^6 = \tau^6\nu$ . For instance, any choice of  $\nu$  combined with any choice of  $\tau$  of order 2, 3 or 6 is good. By specifying  $\tau$  we actually prescribe how  $f$  should lift. For the automorphism group  $\text{Aut } P = \langle f, g, h \rangle$  we get: the group  $\text{Aut } P$  lifts if and only if the cover over  $P$  is determined by the 6-tuple of voltages

$$(\nu, \tau^{-1}\nu\tau, \tau^{-2}\nu\tau^2, \tau^{-3}\nu\tau^3, \tau^{-4}\nu\tau^4, \tau^{-5}\nu\tau^5),$$

where  $\nu$  and  $\tau$  are two of the four generators of  $G = \langle \nu, \tau, \eta, \omega; (R), \dots \rangle$  and  $(R)$  is the system of 13 identities arising from the above system of equations if we replace the first column by the identity  $\nu\tau^6 = \tau^6\nu$  and substitute all  $\xi_i$  by  $\tau^{-i}\nu\tau^i$  in all the other equations.

**Example 8.1** By taking  $\xi_i = (12) \in S_2$  for  $i = 0, 1, \dots, 5$ , we get the generalized Petersen graph  $\text{GP}(10, 3)$  as a cover over the Petersen graph such that the full automorphism group lifts. The solutions of the above system of equations are  $\tau = \omega = \eta = 1$  in  $S_2$ . If we set  $\xi_0 = \xi_2 = \xi_4 = (12)(34)$ ,  $\xi_1 = \xi_3 = \xi_5 = (13)(24)$  in the group  $S_4$ , another interesting example is obtained. In this case, the solutions of the above system of equations are  $\tau = (14)$ ,  $\omega = (1243)$  and  $\eta = (14)(23)$ . Thus the derived covering graph is a 3-arc-transitive cubic graph and has 40 vertices. In fact, the resulting graph can be viewed as a canonical double covering (see remarks after Corollary 9.6)  $\tilde{D}$  over the dodecahedron  $D$ . Since  $|\text{Aut } \tilde{D}| > |\text{Aut } D \times \mathbb{Z}_2|$  it follows that the dodecahedron is unstable (see [26] for the definition and [28] for further information and results). This is the first symmetrical example of its kind, as  $D$  is sharply 2-arc-transitive.

## 9 Group extensions

Let  $K \rightarrow E \rightarrow Q$  be an arbitrary abstract group extension. Choose an arbitrary symmetric generating set  $S = S^{-1}$  of the group  $E$  and let  $\text{Cay}(E, S)$  be the corresponding Cayley graph. The group  $E$  acts by left multiplication as a group of automorphisms of  $\text{Cay}(E, S)$ . So does the group  $K$  (via its embedding in  $E$ ). Since  $K$  acts freely on  $\text{Cay}(E, S)$ , the projection  $p : \text{Cay}(E, S) \rightarrow \text{Cay}(E, S)/K$  is a regular covering projection with  $\text{CT}(p) \cong K$ . Now  $\text{CT}(p)$  is normal in  $E$ . Therefore  $E$ , as the automorphism group, projects onto an automorphism group of  $\text{Cay}(E, S)/K$ ; the latter is clearly isomorphic to  $Q$ . As a result, any abstract group extension  $K \rightarrow E \rightarrow Q$  can be studied, up to isomorphism of extensions, as a lifting of a group  $Q$  to the group  $E$  by means of a suitable regular covering projection whose group of covering transformations is isomorphic to  $K$ .

This special case of lifting automorphisms of Cayley graphs provides an additional support to our original motivation for studying lifts of automorphisms in terms of voltages. The most interesting case for most applications occurs when the covering projection is regular and the extension is split. For a more general treatment of this problem we refer the reader to [21].

Let  $p : \tilde{X} \rightarrow X$  be an arbitrary covering projection of connected graphs, and let  $A$  be a group of automorphisms of  $X$ . We extend the notion of local  $A$ -invariance of an associated voltage space as follows. Let  $\Omega \subseteq V(X)$  be a nonempty subset of vertices of  $X$ . Denote by  $\pi^\Omega$  the set of all walks  $W \in \pi$  having both endvertices in  $\Omega$ . An associated voltage space  $(F, G; \xi)$  is said to be *A-invariant on  $\Omega$*  whenever the condition

$$\xi_W = 1 \Rightarrow \xi_{fW} = 1. \quad (9)$$

is satisfied for every  $f \in A$  and every walk  $W \in \pi^\Omega$  (compare with (3) from Section 7).

In what follows we shall assume the covering, and hence the voltage space, to be regular. In fact, we may assume that the voltage space is Cayley which brings additional computational convenience. Further, we shall require it to be  $A$ -invariant on some nonempty subset  $\Omega$  of vertices containing the base vertex  $b \in \Omega$ . In addition to the above assumptions on the voltage space we demand, for reasons which will become clear later, that the chosen subset of vertices  $\Omega$  is invariant under the action of  $A$ , that is,  $A(\Omega) = \Omega$ . These assumptions have some very strong implications.

Firstly, since  $G^b = G$  there exists a walk with trivial voltage between any pair of vertices of  $X$ . This together with  $A$ -invariance of the voltage space on  $\Omega$  readily implies that, for each  $f \in A$ , there exists an automorphism  $G \rightarrow G$  defined by

$$f^\sharp(\xi_W) = \xi_{fW}, \quad (W \in \pi^\Omega).$$

It is interesting to mention that  $f^{\natural} = f^{\#u}$  for  $u \in \Omega$ , however,  $f^{\#u}$  is computed locally whereas  $f^{\natural}$  can be computed globally on  $\Omega$ . Moreover, taking into account that  $A(\Omega) = \Omega$ , the assignment  $f \mapsto f^{\natural}$  is a homomorphism  $\natural : A \rightarrow \text{Aut } G$ ; details are left to the reader. This homomorphism yields the semidirect product

$$G \rtimes_{\natural} A.$$

Secondly,  $A$ -invariance on  $\Omega$  implies local  $A$ -invariance. Hence  $A$  does lift by Corollary 7.2. Recall the formula (6) due to which  $\tau_{u,\tilde{f}}(g) = \tau_{b,\tilde{f}}(g\xi_W)(\xi_{fW})^{-1}$ , where  $W : u \rightarrow b$ . Because of  $A$ -invariance on  $\Omega$  it is straightforward that

$$\tau_{\tilde{f}} = \tau_{u,\tilde{f}}$$

does not depend on  $u \in \Omega$ . What is more, the following generalization of an old result of Biggs [4] holds.

**Theorem 9.1** *Let  $(G, G; \xi)$  be a Cayley voltage space associated to a regular covering projection  $p : \tilde{X} \rightarrow X$  of connected graphs, and let  $A$  be a group of automorphisms of  $X$ . Suppose  $(G, G; \xi)$  is  $A$ -invariant on some nonempty subset  $\Omega$  of vertices satisfying  $A(\Omega) = \Omega$ . Then the group  $A$  lifts as a split extension of  $\text{CT}(p)$ , and there is an isomorphism  $G \rtimes_{\natural} A \rightarrow \tilde{A}$  given by  $(g, f) \mapsto \tilde{f}_g$ , where  $\tilde{f}_g$  is uniquely determined by the requirement that  $\tau_{\tilde{f}_g}(1) = g$ . This isomorphism takes  $G$  onto  $\text{CT}(p)$  and  $A$  onto the subgroup preserving the set of vertices in fibers over  $\Omega$  which are labelled by 1.*

**PROOF.** We have already proved above that the group  $A$  indeed lifts and that the semidirect product  $G \rtimes_{\natural} A$  exists. Now for each  $g \in G$  and  $f \in A$  there is a unique  $\tilde{f}_g \in \tilde{A}$  with  $\tau_{\tilde{f}_g}(1) = g$  and, clearly, every element of  $\tilde{A}$  can be written in this way. It remains to show that the assignment  $(g, f) \mapsto \tilde{f}_g$  is a homomorphism, that is to say, we need to check that  $\tilde{f}_g \tilde{f}_{g'} = \widetilde{ff'_{gf^{\natural}(g')}}$ . This is done by checking how both sides map the vertex in  $\text{fib}_b$  labelled by 1, which is almost straightforward if we take into account the formula (8) of Section 7, that is,  $\tau_{\tilde{f}_g}(g') = gf^{\natural}(g')$ . ■

**Example 9.2** Note that on every graph  $X$  one can define many nontrivial Cayley voltage spaces  $(G, G, \xi)$  which are  $(\text{Aut } X)$ -invariant on the whole set of vertices. Choose an arbitrary spanning tree of  $X$  and let  $n$  be the number of cotree links and loops, and let  $m$  be the number of semiedges. Set  $G = \mathbb{Z}_k^n \oplus \mathbb{Z}_2^m$  for some positive integer  $k$ . Define the voltages in such a way that the darts of the chosen spanning tree receive the trivial voltage, the voltages on darts of semiedges generate the  $\mathbb{Z}_2$ -part of  $G$ , and the voltages on darts of links or loops of cotree edges generate the  $\mathbb{Z}_k$ -part of  $G$ . That this voltage space is indeed  $(\text{Aut } X)$ -invariant relies on the

following observation. Let  $W$  be a walk in  $X$ . For any dart  $x$  denote by  $|x|_W$  the number of appearances of  $x$  in  $W$ . Now the walk  $W$  has trivial voltage if and only if  $|x|_W \equiv 0 \pmod{2}$  whenever  $x$  belongs to a semiedge, and  $|x|_W - |x^{-1}|_W \equiv 0 \pmod{k}$  whenever  $x$  belongs to a cotree link or a loop. This property is clearly preserved by any graph automorphism. The above construction generalizes to the case involving an arbitrary coefficient ring, thereby extending homological coverings of Biggs [4, 5] and Surowski [35] to graphs with semiedges.

Let  $p : \tilde{X} \rightarrow X$  be a covering projection of connected graphs, and let  $\Omega$  be a subset of vertices of  $X$ . Choosing a vertex in each fibre over  $\Omega$  we get a set of representatives which we call a *geometric transversal* over  $\Omega$ . If some geometric transversal is invariant for a subset of lifted automorphisms, then this subset of automorphisms constitutes a group. This situation has been encountered in Theorem 9.1 where the set of vertices over  $\Omega$  labelled by 1 forms a geometric transversal over  $\Omega$ .

Conversely, if there is a set  $\bar{A}$  of lifted automorphisms  $\bar{f}$ , one for each  $f \in A$ , such that there exists an invariant transversal, then the lifted group  $\tilde{A}$  can be split into the semidirect product of  $\text{CT}(p)$  and  $\bar{A}$  as its complement. Moreover, the following converse to Theorem 9.1 holds.

**Theorem 9.3** *Let  $p : \tilde{X} \rightarrow X$  be a regular covering projection of connected graphs and let  $A$  be a group of automorphisms of  $X$ . Further, let  $\Omega$  be an  $A$ -invariant nonempty subset of vertices of  $X$ . Suppose that  $A$  lifts as an internal semidirect product  $\tilde{A} = \text{CT}(p) \rtimes \bar{A}$  such that there exists an  $\bar{A}$ -invariant geometric transversal. Then there exists a Cayley voltage space associated to  $p$  which is  $A$ -invariant on  $\Omega$ .*

PROOF. In order to define a Cayley voltage space we need to label the vertex-fibres by elements of  $G \cong \text{CT}(p)$ . This is done by first extending the  $\bar{A}$ -invariant geometric transversal over  $\Omega$  to a geometric transversal over the whole vertex set. The vertices of the resulting transversal are then labelled by 1, and the labelling is completed by using the left action of  $\text{CT}(p)$ . Having all vertices of  $\tilde{X}$  labelled we also have determined the voltages of walks, as already described in Section 6.

We now claim that the obtained voltage space is  $A$ -invariant on  $\Omega$ . Let us prove that  $\tau_{u, \bar{f}}$  does not depend on  $u \in \Omega$ . Indeed, we have  $\tau_{u, \bar{f}}(1) = 1$  for every  $u \in \Omega$ . Further note that every covering transformation induces a constant mapping of labels since between any two vertices there exists a walk carrying the identity voltage. Let  $c_g \in \text{CT}(p)$  be the one with  $\tau_{c_g}(1) = g$ . Now we have  $\tau_{u, \bar{f}}(g) = \tau_{u, \bar{f}}\tau_{c_g}(1) = \tau_{u, \bar{f}}\tau_{c_g}\tau_{u, \bar{f}}^{-1}(1)$ . But  $\text{CT}(p)$  is a normal subgroup in  $\tilde{A}$ . Hence  $\bar{f}c_g\bar{f}^{-1} = c_{g'}$ . Therefore  $\tau_{u, \bar{f}}(g) = \tau_{c_{g'}}(1) = g'$ , and the permutation  $\tau_{u, \bar{f}}$  does not depend on  $u \in \Omega$ .

Finally, the fact that  $\tau_{u,\bar{f}}$  does not depend on  $u \in \Omega$  implies that the voltage space is  $A$ -invariant on  $\Omega$ . This is easily seen by using the formula (6) of Section 7. ■

**Corollary 9.4** *Let  $p: \tilde{X} \rightarrow X$  be a regular covering projection of connected graphs and let  $A$  be a group of automorphisms of  $X$  acting semiregularly on a nonempty subset  $\Omega$  of vertices. Suppose that  $A$  lifts as an internal semidirect product  $\tilde{A} = \text{CT}(p) \rtimes \bar{A}$ . Then there exists a Cayley voltage space associated to  $p$  which is  $A$ -invariant on  $\Omega$ .*

PROOF. In view of Theorem 9.3 it is sufficient to prove that there exists an  $\bar{A}$ -invariant geometrical transversal over  $\Omega$ . Such a transversal can be formed as a union of orbits of  $\bar{A}$  intersecting every vertex fibre over  $\Omega$ . We only have to verify that each orbit in question intersects such a fibre in at most one vertex. By way of contradiction, if such an orbit contains two vertices in the same fibre, then there is  $\bar{f} \in \bar{A}$  mapping one vertex to another. Hence  $\bar{f}$  projects to an automorphism stabilizing a vertex in  $\Omega$ . By semiregularity, the latter automorphism must be the identity. Consequently,  $\bar{f} \in \text{CT}(p)$ . But since  $\text{CT}(p) \cap \bar{A}$  is trivial, we have  $\bar{f} = \text{id}$ . ■

Theorems 9.1 and 9.3 show that the existence of a Cayley voltage space which is  $A$ -invariant on  $\Omega$  is a necessary and sufficient condition for the group  $A$  to lift as a split extension of  $\text{CT}(p)$  such that some complement of  $\text{CT}(p)$  has an invariant geometrical transversal over  $\Omega$ . However, a group  $A$  can lift as a split extension of  $\text{CT}(p)$  even if no Cayley voltage space associated to  $p$  is  $A$ -invariant on  $\Omega$ . Here is an example.

**Example 9.5** Let  $p: \tilde{X} \rightarrow X$  be a 2-fold covering projection of a 4-cycle onto a 2-cycle. Then  $\text{Aut } X \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  lifts to the dihedral group  $\tilde{A} \cong D_8$  which is a split extension of  $\text{CT}(p) \cong \mathbb{Z}_2$ . Take  $\Omega$  to be the vertex set of  $X$  and suppose that there is a  $\mathbb{Z}_2$  voltage space which is  $A$ -invariant on  $\Omega$ . Since  $A$  acts transitively on darts of  $X$ , the assignment of voltages to darts is constant, and in turn yields a covering graph different from a 4-cycle, a contradiction. This example easily generalizes to an infinite family.

**Corollary 9.6** *Let  $p: \tilde{X} \rightarrow X$  be a regular covering projection of connected graphs and let  $A$  be a subgroup of the stabilizer of a vertex. If  $A$  lifts, it lifts as a split extension of  $\text{CT}(p)$ . ■*

In addition to the concept of an  $A$ -invariance of a voltage space on  $\Omega$  we now introduce a stronger notion. Let  $A$  be a group of automorphisms of a graph  $X$  and

$\Omega$  an  $A$ -invariant subset of vertices of  $X$ . We say that an arbitrary voltage space  $(F, G; \xi)$  is *strongly  $A$ -invariant* on  $\Omega$  if, for every walk  $W \in \pi^\Omega$  and every  $f \in A$ , we have that  $\xi_W = \xi_{fW}$ . Note that for every nonbipartite graph  $X$  one can easily define a Cayley voltage space  $(\mathbb{Z}_2, \mathbb{Z}_2, \xi)$  which is strongly  $(\text{Aut } X)$ -invariant on the whole vertex set by setting  $\xi_x = 1$ . The derived covering graph  $\tilde{X}$  is called the *canonical double covering* of  $X$  [28].

**Corollary 9.7** *Let  $p: \tilde{X} \rightarrow X$  be a regular covering projection of connected graphs and let  $A$  be a group of automorphisms of  $X$ . Further, let  $\Omega$  be a nonempty  $A$ -invariant subset of vertices of  $X$ . If some Cayley voltage space associated to  $p$  is strongly  $A$ -invariant on  $\Omega$ , then the group  $A$  lifts as a direct product  $\tilde{A} \cong G \times A$ . ■*

**Corollary 9.8** *Let  $p: \tilde{X} \rightarrow X$  be a regular covering projection of connected graphs, let  $A$  be a group of automorphisms of  $X$ , and let  $\Omega$  be a nonempty  $A$ -invariant subset of vertices of  $X$ . Suppose that  $A$  lifts as an internal direct product  $\tilde{A} = \text{CT}(p) \times \bar{A}$  such that  $\bar{A}$  has an invariant geometrical transversal over  $\Omega$ . Then there exists a Cayley voltage space associated to  $p$  which is strongly  $A$ -invariant on  $\Omega$ .*

PROOF. As in Theorem 9.3 we can construct a Cayley voltage space  $(G, G; \xi)$  associated to  $p$  which is  $A$ -invariant on  $\Omega$ . Note that the  $\bar{A}$ -invariant geometrical transversal over  $\Omega$  is labelled by 1. By Theorem 9.1, the group  $A$  lifts as a split extension  $\tilde{A} \cong G \rtimes_{\mathfrak{h}} A$ . There is an isomorphism which maps  $G \rtimes_{\mathfrak{h}} \text{id}$  onto  $\text{CT}(p)$  and  $\text{id} \rtimes_{\mathfrak{h}} A$  onto the subgroup preserving vertices labelled by 1, that is, onto  $\bar{A}$ . Hence  $\text{id} \rtimes_{\mathfrak{h}} A$  is normal, that is, the semidirect product is in fact the direct product. Therefore, the homomorphism  $\mathfrak{h}: A \rightarrow \text{Aut } G$  is trivial. Consequently, the voltage space is strongly  $A$ -invariant on  $\Omega$ , as required. ■

## 10 Lifting map automorphisms

A *map* is a cell decomposition of an orientable closed surface. A common way to describe a map is endowing its underlying graph  $X$  with a permutation  $R$  (called *rotation*) cyclically permuting darts of  $X$  with the same initial vertex. Thus every map can be identified with a pair  $(X; R)$ , see [16]. A *map homomorphism*  $p: (X; R) \rightarrow (Y; Q)$  is a graph morphism  $X \rightarrow Y$  commuting with the rotations, that is,  $pR = Qp$ . It is well known that if the automorphism group  $\text{Aut } M$  of a map  $M$  acts transitively on the dart set of  $M$ , it does so regularly. Therefore such maps are called *regular maps*. In what follows we prove two statements about homomorphisms of regular maps. The first one is a generalization of a theorem established by Gvozdjak and Širáň [14] which shows how to use covering techniques to construct new regular maps from old ones. The other result is the converse to the first one.

Let  $p : \tilde{X} \rightarrow X$  be a graph covering and let  $M = (X; R)$  be a map with underlying graph  $X$ . Then the *lift* of  $M$  to  $\tilde{X}$  is the map  $\tilde{M} = (\tilde{X}, \tilde{R})$  where  $\tilde{R}$  is defined by the equation  $Rp(x) = p\tilde{R}(x)$ .

**Theorem 10.1** *Let  $M = (X; R)$  be a regular map and let  $(G, G; \xi)$  be a locally  $(\text{Aut } M)$ -invariant Cayley voltage space. Then the lifted map  $\tilde{M}$  is regular and  $\text{Aut } \tilde{M} = (\text{Aut } M)^\sim$ .*

PROOF. By Corollary 7.2,  $\text{Aut } M \leq \text{Aut } X$  lifts. Taking into account the way how  $\tilde{R}$  is defined we see that  $(\text{Aut } M)^\sim \leq \text{Aut } \tilde{M}$ . Since  $\text{Aut } M$  acts regularly on arcs of  $M$ ,  $(\text{Aut } M)^\sim$  acts regularly on the set of arcs of  $\tilde{M}$  as well. Thus  $\tilde{M}$  is a regular map and  $\text{Aut } \tilde{M} = (\text{Aut } M)^\sim$ . ■

Let  $p : \tilde{X} \rightarrow X$ ,  $M$  and  $\tilde{M}$  be as above. Since  $Rp(x) = p\tilde{R}(x)$ , the (graph) covering projection  $p$  is at the same time a homomorphism of maps preserving the valency of vertices. The following theorem shows that every such homomorphism gives rise to a Cayley voltage space which is locally invariant with respect to the automorphism group of the base map.

**Theorem 10.2** *Let  $M = (X; R)$  and  $p : \tilde{M} \rightarrow M$  be a valency preserving homomorphism of regular maps. Then there is an associated locally  $\text{Aut } M$ -invariant Cayley voltage space  $(G, G; \xi)$ .*

PROOF. Let  $\tilde{M} = (\tilde{X}, \tilde{R})$ , and let  $L, \tilde{L}$  be the dart-reversing involutions in  $X$  and  $\tilde{X}$ , respectively. Let us denote by  $\text{Mon } M = \langle R, L \rangle$  the monodromy group of  $M$ . Similarly,  $\text{Mon } \tilde{M} = \langle \tilde{R}, \tilde{L} \rangle$  will stand for the monodromy group of  $\tilde{M}$ . Both  $\text{Mon } M$  and  $\text{Mon } \tilde{M}$  act transitively on darts of the respective maps. Moreover, since  $M$  and  $\tilde{M}$  are regular, the actions of  $\text{Mon } M$  and  $\text{Mon } \tilde{M}$  are regular as well. It is proved in [30, Proposition 2.1] that  $p$  induces a group epimorphism  $p^* : \text{Mon } \tilde{M} \rightarrow \text{Mon } M$  with kernel  $K$ . The orbits of the action of  $K$  on darts of  $\tilde{X}$  determine the fibres of  $p$  over darts of  $X$ . Since the map  $\tilde{M}$  is regular,  $\text{Aut } \tilde{M} \cong \text{Mon } \tilde{M}$ . If we identify the darts with the elements of  $\text{Mon } \tilde{M}$ , then we can describe the action of an element  $g \in \text{Mon } \tilde{M}$  as the right translation by  $g$ , while the map automorphisms act as left translations on  $\text{Mon } \tilde{M}$ . Hence, there is a subgroup  $K^* \leq \text{Aut } \tilde{M}$  corresponding to  $K \leq \text{Mon } \tilde{M}$ . Since  $K^* \cong K$ , and since  $K^*$  is normal in  $\text{Aut } \tilde{M}$  and acts regularly on each dart-fibre of  $p$ , we infer that the covering projection  $p$  is regular with  $CT(p) = K^*$ . Moreover, every automorphism of  $\tilde{M}$  projects onto an automorphism of  $M$ . Consequently,  $\text{Aut } M$  lifts and the existence of the required voltage space follows from 7.2. ■

Note that there are homomorphisms of regular maps which do not preserve the valency of vertices, and thus their restriction to the underlying graphs is not a covering projection. However, there is a way to generalize the above statements to this kind of map homomorphisms as well [24]. We conclude the discussion of coverings of regular maps by giving (without proof) a necessary and sufficient condition for  $\text{Aut } \tilde{M}$  to be a semidirect product of  $CT(p)$  by  $\text{Aut } M$ . For a proof, see [24].

**Corollary 10.3** *Let  $p : \tilde{M} \rightarrow M$  be a valency preserving homomorphism of regular maps. Then  $\text{Aut } \tilde{M} \cong CT(p) \rtimes \text{Aut } M$  if and only if there exists an associated Cayley voltage space which is  $(\text{Aut } M)$ -invariant on the whole vertex-set of  $M$ .*

The above corollary in combination with Theorem 10.2 indicates that not every automorphism group of a regular map lifts as a split extension of  $CT(p)$ , which corrects Theorem 5 of Gvozđjak and Širáň in [14]. Examples of coverings of regular maps with the property that the automorphism group of the lifted map does not split can be found in [24].

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