University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 36 (1998), 618

WEAKLY FLAG-TRANSITIVE CONFIGURATIONS AND HALF-ARC-TRANSITIVE GRAPHS

Dragan Marušič Tomaž Pisanski

ISSN 1318-4865

July 8, 1998

Ljubljana, July 8, 1998

WEAKLY FLAG-TRANSITIVE CONFIGURATIONS AND HALF-ARC-TRANSITIVE GRAPHS

Dragan Marušič[1]
IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1000 Ljubljana
Slovenija
dragan.marusic@uni-lj.si

Tomaž Pisanski[2]
IMFM, Oddelek za teoretično računalništvo
Univerza v Ljubljani
Jadranska 19, 1111 Ljubljana
Slovenija
tomaz.pisanski@fmf.uni-lj.si

Abstract

A configuration is *weakly flag-transitive* if its group of automorphisms acts intransitively on flags but the group of all automorphisms and anti-automorphisms acts transitively on flags. It is shown that weakly flag-transitive configurations are in one-to-one correspondence with bipartite $\frac{1}{2}$-arc-transitive graphs of girth not less than 6. Several infinite families of weakly flag-transitive configurations are given via their Levi graphs. Among others an infinite family of non-self-polar weakly flag-transitive configurations is constructed. The smallest known weakly flag-transitive configuration has 27 points and the smallest known non-self-polar weakly flag-transitive configuration has 34 points.

1 Introduction

The topic studied in this paper touches theories of graphs, groups and configurations and the reader is thus referred to [1, 5, 8, 10, 14, 16] for the terms not defined here. Unless specified otherwise, all objects considered in this paper are assumed to be finite. Moreover, all graphs are simple and undirected.

By a *configuration* we shall always mean a symmetric configuration. More precisely, an n_k-configuration is an ordered triple $\mathcal{C} = (P, \mathcal{B}, I)$ of mutually

[1]Supported in part by "Ministrstvo za znanost in tehnologijo Slovenije", proj.no. J1-7035-0101-97.

[1]Supported in part by "Ministrstvo za znanost in tehnologijo Slovenije", proj. no. J2-6193-0101-97 and J1-6161-0101-97.

disjoint sets P, $\mathcal{B}$ and I (whose elements are called, respectively, *points*, *blocks (lines)* and *flags*) with $|P| = n = |\mathcal{B}|$ and $I \subseteq P \times \mathcal{B}$, and (with a point p and a block B called *incident* if $(p, B) \in I$) such that each block (point) is incident with the same number k of points (blocks), and two distinct points (blocks) are incident with at most one common block (point). To each n_k-configuration $\mathcal{C} = (P, \mathcal{B}, I)$ the *dual* n_k-configuration $\mathcal{C}^* = (\mathcal{B}, P, I^*)$ may be associated in the usual way by reversing the roles of points and blocks in $\mathcal{C}$.

An *automorphism* of a configuration $\mathcal{C}$ is an incidence preserving permutation on the (disjoint) union $P \cup \mathcal{B}$ which maps P to P. Similarly, an *anti-automorphism* of a configuration $\mathcal{C}$ is an incidence preserving permutation on the (disjoint) union $P \cup \mathcal{B}$ which interchanges P and $\mathcal{B}$. The configuration $\mathcal{C}$ is said to be *self-dual* if it admits an *anti-automorphism*, that is, if it is isomorphic with its dual $\mathcal{C}^*$. An anti-automorphism of order 2 is called a *polarity.* We say that $\mathcal{C}$ is *self-polar* if it admits a polarity. We let $\mathrm{Aut}_0\,\mathcal{C}$ denote the group of all automorphisms of $\mathcal{C}$, and we let $\mathrm{Aut}\,\mathcal{C}$ denote the group of all automorphisms and anti-automorphisms of $\mathcal{C}$. Then $\mathrm{Aut}_0\,\mathcal{C}$ is a proper subgroup of $\mathrm{Aut}\,\mathcal{C}$ (of index 2) if $\mathcal{C}$ is self-dual, and coincides with $\mathrm{Aut}\,\mathcal{C}$ otherwise.

If X is a graph we let $V(X)$ and $E(X)$ denote the respective sets of vertices and edges. For $v_1, \ldots, v_k \in V(X)$ and a positive integer d we let $N^d(v_1, \ldots, v_k)$ denote the set of all vertices in X at distance d from the set $\{v_1, \ldots, v_k\}$. In particular, $N(v_1, \ldots, v_k) = N^1(v_1, \ldots, v_k)$ is the set of neighbors of $\{v_1, \ldots, v_k\}$. Furthermore, by $\mathrm{Aut}\,X$ we denote the automorphism group of X. For a bipartite graph X we let $\mathrm{Aut}_0\,X$ be the subroup of $\mathrm{Aut}\,X$ fixing the bipartition.

The concept of a *Levi graph* of a configuration was introduced by Coxeter in 1950 (see [7]). Given a configuration $\mathcal{C} = (P, \mathcal{B}, I)$ we let $L(\mathcal{C}) = L(P, \mathcal{B}, I)$ be the bipartite graph with "black" vertices P and "white" vertices $\mathcal{B}$ and with an edge joining some $p \in P$ and some $B \in \mathcal{B}$ if and only if $(p, B) \in I$. Note that dual configurations have the same Levi graph with the roles of black and white vertices interchanged. Clearly, a complete information about the configuration can be recovered from its Levi graph with a given black and white coloring of vertices. The following proposition from [7] characterizes n_k-configurations in terms of their Levi graphs.

Proposition 1.1 *A graph X is a Levi graph of some n_k-configuration if and only if is a regular bipartite graph of valency k with girth at least 6.*

In view of Proposition 1.1 we may call a configuration *connected* if and only if its Levi graph is connected. In this paper we consider, unless explicitely stated otherwise, only connected graphs and configurations.

A graph X is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive*, respectively, if its automorphism group $\mathrm{Aut}\,X$ transitively on the sets of its vertices, edges and arcs. Further, we shall say that X is $\frac{1}{2}$-*arc-transitive* provided it is vertex- and edge- but not arc-transitive. More generally, by a $\frac{1}{2}$-*arc-transitive* action of a subgroup $G \leq \mathrm{Aut}\,X$ on X we shall mean a vertex- and edge- but not arc-transitive action of G on X. In this case we shall say that the graph X is $(G, \frac{1}{2})$-*transitive*. For further references on $\frac{1}{2}$-arc-transitive graphs (also $\frac{1}{2}$-transitive graphs) see [6, 12, 13] and the survey paper [11].

Given a configuration $\mathcal{C} = (P, \mathcal{B}, I)$, we shall, for the purpose of this paper, make no distinction between the two groups $\mathrm{Aut}\,\mathcal{C}$ and $\mathrm{Aut}\,L(\mathcal{C})$, as well as the two groups $\mathrm{Aut}_0\,\mathcal{C}$ and $\mathrm{Aut}_0\,L(\mathcal{C})$. We say that $\mathcal{C}$ is *point-*, *block-*, and *flag-transitive* provided $\mathrm{Aut}\,\mathcal{C}$ acts transitively on the sets of its points, blocks and flags, respectively. Moreover, a flag-transitive configuration $\mathcal{C}$ is *weakly flag-transitive* if $\mathrm{Aut}\,\mathcal{C}$ acts intransitively on the sets of its flags, and is *strongly flag-transitive* otherwise.

For each n there is only one n_2-configuration, called the *n-gon*; in particular, the configuration 3_2 is called the *triangle*. The n-gon is a self-dual, point-transitive, block-transitive and strongly flag-transitive configuration whose Levi graph is the cycle C_{2n}. As for n_3-configurations, the situation is much more complex since their number grows very fast with growing n (see [3]). Many historically important configurations are of this kind and have received considerable attention over the years. Among others, an important problem is the question of realizability of such configurations (see [4]). However, no n_3-configuration is weakly flag-transitive, making n_4-configurations the simplest candidates for weak flag-transitivity. This is the primary motivation for this work. We explore the connection between weakly flag-transitive configurations and $\frac{1}{2}$-arc-transitive graphs. Using this connection we give several new families of weakly flag-transitive configurations with certain prescribed properties.

In Section 2 we show that weakly flag-transitive configurations are in one-to-one correspondence with bipartite $\frac{1}{2}$-arc-transitive graphs of girth at least 6 (Theorem 2.6). As a consequence, there are no weakly flag-transitive n_k-configurations, for k odd (Corollary 2.7). Some structural properties of weakly flag-transitive configurations, such as the concept of a kernel, are also discussed there.

Section 3 is devoted to specific constructions of weakly flag-transitive configurations. In view of Bouwer's construction of a family of bipartite $\frac{1}{2}$-arc-transitive graph of girth 6 for every even valency greater than 2 (see [6]), it follows that there are weakly flag-transitive n_k-configurations for every even $k > 2$ (see Proposition 3.1). Several constructions of weakly flag-transitive n_4-configuration whose kernels are odd length polygons are known [12]. In Theorem 3.3 we give a construction of weakly flag-transitive n_4-configurations whose kernels are even length polygons. Note that all weakly flag-transitive configurations are self-dual. An infinite family of non-self-polar, weakly flag-transitive n_4-configurations is given in Theorem 3.4.

2 Weakly flag-transitive configurations and $\frac{1}{2}$-arc-transitive graphs

To a graph X of valency $2k$, and a subgroup G of $\mathrm{Aut}\, X$, acting $\frac{1}{2}$-arc-transitively on X, two oppositely oriented graphs may be associated in a natural way. Let $D_G(X)$ be one of these two oriented graphs, fixed from now on. We perform the following operation on $D_G(X)$. Each vertex v splits into two vertices: v^+ which keeps all incoming arcs and v^- which keeps all outgoing arcs. A possibly disconnected $k-$valent oriented graph of order $2|V(X)|$ is thus obtained. Let X^*_G denote its underlying undirected graph. (Note that X^*_G depends solely on X and G.) Let $V^+_G = \{v^+ : v \in V(X)\}$ and $V^-_G = \{v^- : v \in V(X)\}$. For the speacial case $G = \mathrm{Aut}\, X$ the symbol G is omitted in all of the above notations.

Lemma 2.1 *Let X be a $(G, \frac{1}{2})$-transitive graph for a subgroup G of $\mathrm{Aut}\, X$. Then the graph X^*_G is bipartite with bipartition $\{V^+_G, V^-_G\}$, edge-transitive and has isomorphic connected components. Moreover, $\mathrm{Aut}\, X^*_G$ contains an isomorphic copy of G, acting transitively on $E(X^*_G)$ and intransitively on $V(X^*_G)$ with orbits V^+_G and V^-_G.*

Proof. Since $V^+_G = \{v^+ : v \in V(X_G)\}$ and $V^-_G = \{v^- : v \in V(X_G)\}$ are independent sets, X^*_G is bipartite. The action of G extends to X^*_G in a natural way by letting $\alpha(u^+) = \alpha(u)^+$ and $\alpha(u^-) = \alpha(u)^-$ for each $u \in V(X)$ and each $\alpha \in \mathrm{Aut}\, X$. Hence V^+_G and V^-_G are the orbits of the copy of G in $\mathrm{Aut}\, X^*_G$, while edge-transitivity of its action follows from the edge-transitivity of the action of G on X. Clearly, all components of X_G^* are isomorphic graphs. ■

Observe that for any component C of X_G^* the constituent $G^{V(C)}$ acts transitively on $V(C)$. Furthermore, in view of the nature of the operation on $D_G(X)$ which produces X_G^*, there are isomorphic copies of the above components of X_G^* inside X. Any such component (or its isomorphic counterpart in X) will be called a *G-kernel of X* and will be denoted by $\mathrm{Ker}_G X$. As above, for the special case $G = \mathrm{Aut}\, X$ the symbol G is omitted. The proof of the proposition below, which summarizes some basic properties of $\mathrm{Ker}_G X$, is left to the reader.

Proposition 2.2 *Let X be $(G, \frac{1}{2})$-transitive graph of valency $2k$ with $G \leq \mathrm{Aut}\, X$. Then the following statements hold.*

(i) $\mathrm{Ker}_G X$ *is a k-valent, bipartite graph admitting an edge-transitive action of the corresponding constituent of G.*

(ii) Either X has two G-kernels, both spanning X, or X has at least three G-kernels, which are all induced subgraphs; in both cases the collection of G-kernels gives rise to a decomposition of $E(X)$.

(iii) The edge sets of G-kernels are blocks of imprimitivity of G in its action on $E(X)$.

Proof. The proof of (i) is an immediate consequence of Lemma 2.1.

As for part (ii), the fact that the edge sets of the G-kernels decompose $E(X)$, is obvious. Furthermore, note that there are two G-kernels containing a given vertex $v \in V(X)$: call them $\mathrm{Ker}_G^+(v, X)$ and $\mathrm{Ker}_G^-(v, X)$, the superscripts $+$ and $-$ reflecting the relative orientation of the arcs at v. More precisely $\mathrm{Ker}_G^+(v, X)$ is determined by all G-alternating paths in $D_G(X)$ originating from v while $\mathrm{Ker}_G^-(v, X)$ is determined by all G-alternating paths in $D_G(X)$ terminating at v. Assume that the G-kernels are not induced subgraphs. Call an edge of X *major* if it belongs to the edge set of some G-kernel, that is, to $E(\mathrm{Ker}_G^+(v, X))$ for some $v \in V(X)$ and *minor* if it belongs to the edge set of the subgraph induced by the set of vertices of some G-kernel. Note that there is precisely one G-kernel in which a given edge is major. But, in view of the fact that there are precisely two G-kernels containing a given vertex, it follows that there is also precisely one G-kernel in which this same edge is minor. For transitivity reasons each induced subgraph on a given G-kernel contains the same number of minor edges (and by definition also the same number of major edges). But then these two numbers must be the same and equal to $|E(X)|/E(\mathrm{Ker}_G X)$. In this case the graph consists of an edge disjoint union of two G-kernels, both a spanning subgraph of X.

Part (iii) is clear. ■

The proof of the first lemma below is straightforward. The second lemma is an extension of a classical result from [9, Theorem 1].

Lemma 2.3 *Let G be a transitive permutation group on a finite set V and let G_0 be an intransitive subgroup of G of index 2. Then G_0 has two orbits on V of equal size and each element of $G \setminus G_0$ interchanges these two orbits.*

Lemma 2.4 *let X be a connected graph and G a subgroup of $\mathrm{Aut}\, X$ acting transitively on $E(X)$ and intransitively on $V(X)$. Then X is bipartite and the two parts of bipartition are orbits of the action of G on $V(X)$.*

The following lemma is a straightforward consequence of Lemmas 2.1 and 2.3.

Lemma 2.5 *Let X be a $(G, \frac{1}{2})$-transitive and bipartite graph with bipartition (U, W) and let $D_G(X)$ be one of the two oriented graphs associated with the action of G. Color each arc with tail in U green and each arc with tail in W red. Then the red and green subgraphs of X consist of components, isomorphic to $\mathrm{Ker}_G X$.*

The next result links weakly flag-transitive configurations to bipartite $\frac{1}{2}$-arc-transitive graphs of girth at least 6.

Theorem 2.6 *A configuration $\mathcal{C} = (P, \mathcal{B})$ is weakly flag-transitive if and only if its Levi graph $L(\mathcal{C})$ is a bipartite $\frac{1}{2}$-arc-transitive graph of girth at least 6.*

Proof. Observe that $X = L(\mathcal{C})$ is by Proposition 1.1 a bipartite graph with girth at least 6. Assume that X is $\frac{1}{2}$-arc-transitive. Then $\mathrm{Aut}\, X$ acts transitively on $V(X)$ and $E(X)$ and moreover $\mathrm{Aut}_0\, X$ is a subgroup of index 2 in $\mathrm{Aut}\, X$ acting intransitively on $V(X)$ and $E(X)$. In the language of configurations, $\operatorname{Aut}\mathcal{C}$ acts transitively on flags while $\operatorname{Aut}_0\mathcal{C}$ acts intransitively on flags. Hence $\mathcal{C}$ is weakly flag-transitive.

Suppose now that $\mathcal{C}$ is a weakly flag-transitive configuration. By Proposition 1.1 its Levi graph $X = L(\mathcal{C})$ is bipartite of girth at least 6. Since $\operatorname{Aut} X$ acts transitively on $E(X)$, whereas $\operatorname{Aut}_0 X$ acts intransitively on $E(X)$, we see that $\operatorname{Aut}_0 X$ is a proper subgroup of $\operatorname{Aut} X$. But X is connected and so $[\operatorname{Aut} X : \operatorname{Aut}_0 X] = 2$. Furthermore $\operatorname{Aut} X$ acts transitively on $V(X)$, for otherwise the two parts P and $\mathcal{B}$ of the bipartition would, by Lemma 2.4, coincide with the two orbits of $\operatorname{Aut} X$, forcing $\operatorname{Aut} X = \operatorname{Aut}_0 X$. We conclude that X is vertex- and edge-transitive. Also, $\operatorname{Aut}_0 X$ is a subgroup of index 2 in $\operatorname{Aut} X$ acting intransitively on $V(X)$ as well as on $E(X)$. Applying Lemma 2.3 for the action of $\operatorname{Aut}_0 X$ on $V(X)$, we see that $\operatorname{Aut}_0 X$ has two orbits on $V(X)$, namely the sets P and $\mathcal{B}$. Similarly, applying Lemma 2.3 for the action of $\operatorname{Aut}_0 X$ on $E(X)$ we conclude that $\operatorname{Aut}_0 X$ has two equal-size orbits on $E(X)$ (corresponding to the "red" and "green" edges of Lemma 2.5). But since P and $\mathcal{B}$ are orbits of $\operatorname{Aut}_0 X$ in its action on $V(X)$ it follows that every vertex of X is incident with the same number of red and green edges. Orienting the red edges from P to $\mathcal{B}$ and the green edges from $\mathcal{B}$ to P, we see, by Lemma 2.3, that $\operatorname{Aut} X$ preserves this orientation. We conclude that X is $\frac{1}{2}$-arc-transitive. ■

The proof of the result below follows from the well known fact that $\frac{1}{2}$-arc-transitive graphs have even valency (see [15]), but it may also be deduced from Lemma 2.5.

Corollary 2.7 *There is no weakly flag-transitive n_k-configuration for k odd.*

We may extend the concept of a kernel to weakly flag-transitive configurations in the following way. Given a weakly flag-transitive n_{2k}-configuration $\mathcal{C}$ it follows from Proposition 2.2 that its Levi graph $L(\mathcal{C})$ is an edge-disjoint union of isomorphic copies of $\operatorname{Ker} L(\mathcal{C})$, the latter being connected, bipartite, of girth at least 6, and such that $\operatorname{Aut}_0 \operatorname{Ker} L(\mathcal{C})$ acts transitively on the set of its edges. We let the *kernel* $\operatorname{Ker}\mathcal{C}$ of $\mathcal{C}$ be, up to duality, the strongly flag-transitive r_k-configuration associated with $\operatorname{Ker} L(\mathcal{C})$. In the particular case of an n_4-configuration $\mathcal{C}$ the kernel is always an r-gon for some $r \geq 3$. The corresponding $2r$-cycles in the $\frac{1}{2}$-arc-transitive graph $L(\mathcal{C})$ are referred to as *alternating cycles* and the parameter r is referred to as the *radius* of $L(\mathcal{C})$ (see [12]).

A natural question arises. Which strongly flag-transitive configurations can be kernels of weakly flag-transitive configurations? The known constructions of $\frac{1}{2}$-arc-transitive graphs of valency 4 for odd radius greater than or equal to 7 (see [12]) together with the constructions of such graphs for certain even radii given in Section 3, suggest that, for every $r \geq 3$, a weakly flag-transitive n_4-configuration whose kernel is the r-gon may exist. As for kernels which are strongly flag-transitive r_k-configurations, where $k \geq 3$, practically nothing seems to be known. In particular, it would be interesting to know if there exists a weakly-transitive configuration with a non-self-dual kernel. (Note that a weakly flag-transitive configuration is by definition self-dual. But this may not necessarily be the case for its kernel.)

The next section is devoted to the study of weakly flag-transitive n_4-configurations. We give a construction of an infinite family of weakly flag-transitive configurations whose kernels are even length polygons and a construction of non-self-polar weakly flag-transitive configurations.

3 Constructions of weakly flag-transitive n_4-configurations with prescribed properties

In 1970 Bouwer [6, Proposition 2] gave a construction of a bipartite $\frac{1}{2}$-arc-transitive graph of girth 6 and valency $2k$ for every $k \geq 2$. As a consequence we have the following result.

Proposition 3.1 (Bouwer) *There exist weakly flag-transitive n_{2k}-configurations for every even $k \geq 2$.*

The smallest graph in Bouwer's family has 54 vertices and it gives rise to a weakly flag-transitive 27_4-configuration which is the smallest known weakly flag-transitive configuration. As mentioned in Section 2, the kernel of an n_4-configuration is either an even length or an odd length polygon. For example, the kernel of the above 27_4-configuration is the 9-gon. In fact this configuration belongs to an infinite family of weakly flag-transitive $(2m+1)_4$-configurations which have the property that any two adjacent kernels, that is, $(2m+1)$-gons with either a common point or a common line, have respectively all points or all lines in common. The corresponding Levi graphs belong to a family of graphs defined below.

Let $t \geq 3$ be an integer, $r \geq 3$ be an odd integer and let $s \in \mathbb{Z}_r^*$ satisfy $s^t = \pm 1$. The graph $X(s;t,r)$ is defined to have the vertex set $\{v_i^j : i \in \mathbb{Z}_t, j \in \mathbb{Z}_r\}$ and edges of the form $v_i^j v_{i+1}^{j+s^i}, v_i^j v_{i+1}^{j-s^i}$, $(i \in \mathbb{Z}_t, j \in \mathbb{Z}_r)$. It is easily checked that the permutations ρ, σ and τ mapping according to the rules

$$v_i^j \rho = v_i^{j+1}, \qquad i \in \mathbb{Z}_t,\ j \in \mathbb{Z}_r. \tag{1}$$

$$v_i^j \sigma = v_{i+1}^{sj}, \qquad i \in \mathbb{Z}_t,\ j \in \mathbb{Z}_r, \tag{2}$$

$$v_i^j \tau = v_i^{-j}, \qquad i \in \mathbb{Z}_t,\ j \in \mathbb{Z}_r. \tag{3}$$

are automorphisms of $X(s;t,r)$.

The following result from [12] classifies $\frac{1}{2}$-arc-transitive graphs among the graphs $X(s;t,r)$.

Theorem 3.2 [12, Theorem 3.4] *The graph $X \cong X(s;t,r)$, where $r \geq 3$ is odd, $t \geq 3$ and $s \in \mathbb{Z}_r^*$ satisfies $s^t = \pm 1$, is $\frac{1}{2}$-arc-transitive if and only if none of the following conditions is fulfilled.*

(i) $s^2 = \pm 1$;

(ii) $(s;t,r) = (2;3,7)$;

(iii) $(s;t,r) = (s;6,7k)$, *where $k \geq 1$ is odd, $(7,k) = 1$, $s^6 = 1$, and there exists a unique solution $q \in \{s, -s, 1/s, -1/s\}$ of the equation $x^2 + x - 2 = 0$ such that $7(q-1) = 0$ and $q \equiv 5 \pmod 7$.*

In particular, when X is $\frac{1}{2}$-arc-transitive then $\mathrm{Aut}\, X = \langle \rho, \sigma, \tau \rangle$.

Clearly, if $t = 2u$ even, the graph $X(s;t,r)$ is bipartite and moreover, if $s \neq 1, -1$, the girth of $X(s;t,r)$ is either 6 or 8. Let $\mathcal{C}(s;t,r)$ denote the $(ur)_4$-configuration corresponding to the graph $X(s;t,r)$ under these assumptions. Note that the kernel of $\mathcal{C}(s;t,r)$ is the r-gon. It then follows by Theorem 3.2 above that for each odd $r \geq 7$ there exist a pair (s,t) such that the configuration $\mathcal{C}(s;t,r)$ is weakly flag-transitive, giving us weakly flag-transitive configurations whose kernels are polygons of length greater than or equal to 7. In particular, $X(2;6,9)$ is isomorphic to the above mentioned smallest graph of Bouwer [6], and so $\mathcal{C}(2;6,9)$ is the above mentioned smallest known weakly flag-transitive configuration. This leaves 3-gons and 5-gons as kernels as the only open question in the case of weakly flag-transitive n_4-configurations whose kernels are odd length polygons. This brings us to the existence of weakly flag-transitive configurations whose kernels are even length polygons. We give below an infinite family of such configurations by constructing the corresponding Levi graphs.

Let $r, t \geq 4$, be even integers and let $s \in \mathbb{Z}_r^*$ satisfy $s^t = 1$. We define the graph $Y(s;t,r)$ to have vertex set $\{v_i^j : i \in \mathbb{Z}_t, j \in \mathbb{Z}_r\}$ and edges of the form $v_i^j\, v_{i+1}^j, v_i^j\, v_{i+1}^{j+s^i}$, $(i \in \mathbb{Z}_t, j \in \mathbb{Z}_r)$. (These graphs belong to a more general family of 4-valent graphs admitting a $\frac{1}{2}$-transitive group action studied in [13]. It is easily checked that the permutations ρ, σ mapping according to the rules (1) and (2), respectively, and the permutation τ mapping according to the rule

$$v_i^j\, \tau = v_i^{-j+s^{i-1}}, \qquad i \in Z_t,\ j \in Z_r. \tag{4}$$

are automorphisms of $Y(s;t,r)$. (Note that $Y(s;t,r)$ is the Cayley graph of the group $\langle \rho, \sigma \rangle \cong \mathbb{Z}_r \rtimes \mathbb{Z}_t$ with respect to the set of generators $\{\sigma, \sigma\rho^s\}$.) Moreover, the group $\langle \rho, \sigma, \tau \rangle$ acts $\frac{1}{2}$-arc-transitiviley on $Y(s;t,r)$.

We are now ready to prove the existence of weakly flag-transitive configurations whose kernels are even length polygons by identifying certain $\frac{1}{2}$-arc-arc-transitive graphs in the family of graphs $Y(s;t,r)$.

Theorem 3.3 *Let $p \equiv 1 \,(mod\, 3)$ be a prime and let $s \in \mathbb{Z}_{2p}^*$ satisfy $s \neq \pm 1$ and $s^3 = -1$. Then $Y(s;6,2p)$ is a bipartite $\frac{1}{2}$-arc-transitive graph with girth 6 and radius $2p$. The corresponding configuration $\mathcal{D} = \mathcal{D}(s;6,2p)$ is thus a weakly flag-transitive $(12p)_4$-configuration and its kernel is the $2p$-gon.*

Proof. For each $i \in \mathbb{Z}_6$ let $W_i = \{v_i^j : j \in \mathbb{Z}_{2p}\}$. The graph $Y = Y(s;6,2p)$ is clearly bipartite with the sets $W_0 \cup W_2 \cup W_4$ and $W_1 \cup W_3 \cup W_5$ being the two parts of bipartition.

Observe first that there are no 4-cycles in Y. (In other words, there is no relation in σ and $\sigma\rho^s$ of length 4 in the group $\langle \rho, \sigma \rangle$.) Namely, a 4-cycle in Y would necesarily have to contain a 2-path of the form $v_{i+1}^j v_i^j v_{i+1}^{j+s^i}$ for some $i \in \mathbb{Z}_6$ and $j \in \mathbb{Z}_{2p}$. But since $s \neq \pm 1$, we see that v_{i+1}^j and $v_{i+1}^{j+s^i}$ have no common neighbor other than v_i^j. Since $Y \cong Cay(\langle \rho, \sigma \rangle, \{\sigma, \sigma\rho^s\})$ and σ has order 6, it therefore follows that Y has 6. (For example $C = v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_0^0$ is a 6-cycle.) In fact, every 6-cycle in $Y(s;6,2p)$ contains precisely one vertex from each of W_i, $i \in \mathbb{Z}_6$. Namely, a 6-cycle not of this kind would have to contain three vertices from W_i, and either one vertex from W_{i+1} and two vertices from W_{i-1} or one vertex from W_{i-1} and two vertices from W_{i+1}. For transitivity reasons we may let $i = 0$, giving us a 6-cycle of the form $v_1^0 v_0^0 v_1^1 v_0^1 v_1^2 v_2^j v_1^0$ or of the form $v_0^0 v_1^1 v_0^1 v_1^1 v_0^2 v_5^j v_0^0$ for some $j \in \mathbb{Z}_{2p}$. But then $j \in \{2, 2p-2\}$ in the first case and $j \in \{2^{-1}, 2p - 2^{-1}\}$ in the second case. Both are impossible as $2 \notin \mathbb{Z}_{2p}^*$. Consequently, no 2-path with a central vertex in W_i and either both endvertices in W_{i+1} or both endvertices in W_{i-1} is contained in a 6-cycle. On the other hand, every 2-path connecting three neighboring sets W_i, W_{i+1}, W_{i+2} is contained in a 6-cycle. For transitivity reasons it suffices to verify this statement for all such 2-paths with central vertex, say v_1^1. For example, the 6-cycles $v_0^1 v_1^1 v_2^1 v_3^1 v_4^1 v_5^1 v_0^1$, $v_0^1 v_1^1 v_2^{1+s} v_3^{1+s} v_4^{1+s} v_5^1 v_0^1$, $v_0^0 v_1^1 v_2^1 v_3^1 v_4^0 v_5^0 v_0^0$ and $v_0^0 v_1^1 v_2^{1+s} v_3^{1+s} v_4^s v_5^0 v_0^0$, respectively, contain the corresponding four 2-paths $v_0^1 v_1^1 v_2^1$, $v_0^1 v_1^1 v_2^{1+s}$, $v_0^0 v_1^1 v_2^1$ and $v_0^0 v_1^1 v_2^{1+s}$ of that kind.

We may now easily deduce that the sets W_i, $i \in \mathbb{Z}_6$, form an imprimitivity block system of $\mathrm{Aut}\, Y$. Let $\alpha \in \mathrm{Aut}\, Y$ and $i \in \mathbb{Z}_6$. Assume that $W_i \cap W_i\alpha \neq \emptyset$ and let $v \in W_i \cap W_i\alpha$. In view of the facts about 2-paths in Y discussed above, it follows that $(W_i \cap N^2(v))\alpha \subseteq W_i$. Continuing this way we see that $(W_i \cap N^{2k}(v))\alpha \subseteq W_i$ for each k and thus $W_i\alpha = W_i$. Hence W_i is a block of $\mathrm{Aut}\, Y$.

Finally, we prove that Y is $\frac{1}{2}$-arc-transitive by showing that $\mathrm{Aut}\, Y = \langle \rho, \sigma, \tau \rangle$. Let α be an arbitrary automorphism of Y fixing v_0^0. We claim that α fixes each of the sets W_i. Assuming the contrary, we must have that α interchanges W_i and W_{6-i} for each $i \in \mathbb{Z}_6$. Consequently, α interchanges the sets $N(v_0^0) \cap W_1 = \{v_1^0, v_1^1\}$ and $N(v_0^0) \cap W_5 = \{v_5^0, v_5^{-s^5}\}$ and so α interchanges the sets $N(v_1^0, v_1^1) \cap W_0 \setminus \{v_0^0\} = \{v_0^1, v_0^{-1}\}$ and $N(v_5^0, v_5^{s^5}) \cap W_0 \setminus \{v_0^0\} = \{v_0^{s^5}, v_0^{-s^5}\}$. Hence α interchanges the sets $N(v_0^{-1}, v_0^1) \cap W_0 = \{v_1^{-1}, v_1^0, v_1^1, v_1^2\}$ and $N(v_0^{-s^5}, v_0^{s^5}) \cap W_0 = \{v_5^{-2s^5}, v_5^{-s^5}, v_5^0, v_5^{s^5}\}$ and hence the sets $\{v_1^{-1}, v_1^2\}$ and $\{v_5^{-2s^5}, v_5^{s^5}\}$. Continuing this way we can see that, for each $j \in \mathbb{Z}_{2p}$, α interchanges the sets $\{v_0^j, v_0^{-j}\}$ and $\{v_0^{js^5}, v_0^{-js^5}\}$, as well as the sets $\{v_1^{j+1}, v_1^{-j}\}$ and $\{v_5^{js^5}, v_5^{-(j+1)s^5}\}$. Consequently, α interchanges the sets $N(v_0^j, v_0^{-j}) \cap W_5 = \{v_5^j, v_5^{j-s^5}, v_5^{-j}, v_5^{-j-s^5}\}$ and $N(v_0^{js^5}, v_0^{-js^5}) \cap W_1 = \{v_1^{js^5}, v_1^{js^5+1}, v_5^{-js^5}, v_5^{-js^5+1}\}$. But then using the formula for α on W_1 we obtain that the former of the two sets must coincide with the set $\{v_5^{js^{10}}, v_5^{-(j+s^5)s^5}, v_5^{(js^5-1)s^5}, v_5^{-js^{10}}\}$. This forces $s^{10} = \pm 1$ and, since $s^3 = -1$, we find that $s = \pm 1$, a contradiction. We have thus shown that α fixes each of the sets W_i, $i \in \mathbb{Z}_6$ and so it is easily seen that $\alpha \in \langle \tau \rangle$ and consequently $\mathrm{Aut}\, Y = \langle \rho, \sigma, \tau \rangle$. This proves that Y is

$\frac{1}{2}$-arc-transitive. We conclude that $\mathcal{D}(s;6,2p)$ is a desired configuration. ■

We remark that the smallest graph in the above family of graphs is $Y(3;6,14)$ and so the corresponding $(84)_4$-configuration $\mathcal{D}(3;6,14)$ is the smallest known weakly flag-transitive configuration whose kernel is an even length polygon, more precisely its kernel is the 14-gon.

We now turn to the existence and construction problem for weakly flag-transitive non-self-polar configurations. In the theorem below we give a complete classification of all those triples $(s;t,r)$ for which $\mathcal{C}(s;t,r)$ is a non-self polar weakly-flag-transitive configuration.

Theorem 3.4 *Let $t \geq 3$ be an integer, $r \geq 3$ be an odd integer and let $s \in \mathbb{Z}_r^*$ satisfy $s^t = \pm 1$ and $s^2 \neq \pm 1$. The graph $X(s;t,r)$ gives rise to a non-self polar weakly flag-transitive $(\frac{tr}{2})_4$-configuration $\mathcal{C}(s;t,r)$ if and only if one of the following conditions holds true.*

(i) $t = 4$ *and* $s^4 = 1$, $1 + s + s^2 + s^3 \neq 0$ *and* $1 - s + s^2 - s^3 \neq 0$; *or*

(ii) $t = 4$ *and* $s^4 = -1$; *or*

(iii) $t = 4k$, $k \geq 2$; *or*

(iv) $t = 2(2k+1)$, $k \geq 1$, *and* $s^t = -1$.

PROOF. The conditions (i)-(iv) are clearly necessary. First, t must be even in order for $X = X(s;t,r)$ to be bipartite. Next if $t = 4$ and $s^4 = 1$, then X has girth 4 if either $1 + s + s^2 + s^3 = 0$ or $1 - s + s^2 - s^3 = 0$, as $v_0^0 v_1^1 v_2^{1+s} v_3^{1+s+s^2} v_0^0$ is a 4-cycle in the first case and $v_0^0 v_1^1 v_2^{1-s} v_3^{1-s+s^2} v_0^0$ is a 4-cycle in the second case. On the other hand, if $t = 2(2k+1)$, $k \geq 1$, and $s^t = 1$, then σ^{2k+1} is an involution interchanging the two bipartition sets.

To prove sufficiency of conditions (i)-(iv), we proceed as follows. Letting $W_i = \{v_i^j : j \in \mathbb{Z}_r\}$ for each $i \in \mathbb{Z}_t$, the two parts of bipartition are $B_0 = W_0 \cup W_2 \cup \ldots \cup W_{t-2}$ and $B_1 = W_1 \cup W_3 \cup \ldots \cup W_{t-1}$. Since weak flag-transitivity $X = X(s;t,r)$ follows directly from Theorem 3.2, only two things need checking (under the assumption that one of conditions (i)-(iv) holds). First, we have to prove that X has no 4-cycles, and second, that X has no involutions intechanging B_0 and B_1. Since, by assumption, $s^2 \neq \pm 1$ we clearly have no 4-cycles in X for $t > 4$. For $t = 4$, the existence of a 4-cycle corresponds to a relation of the form $1 \pm s \pm s^2 \pm s^3 = 0$ in $\mathbb{Z}_r^*$. By computation (we omit the tedious details), we may see that this can only happen when $s^4 = 1$ and when, in addition, either $1 + s + s^2 + s^3 = 0$ or $1 - s + s^2 - s^3 = 0$. As for the nonexistence of involutions intechanging B_0 and B_1, note that $\mathrm{Aut}\, X = \langle \rho, \sigma, \tau \rangle$ by Theorem 3.2. Hence the automorphisms which interchange B_0 and B_1 are of the form $\sigma^i \rho^j$ or $\sigma^i \rho^j \tau$ for some $i = 2k+1 \in \mathbb{Z}_t$ and $j \in \mathbb{Z}_r$.

By computation, $\rho^\sigma = \sigma^{-1}\rho\sigma = \rho^s$, and $\rho^\tau = \rho^{-1}$. On the other hand σ and τ commute. Thus $\rho^j \sigma^i = \sigma^i \rho^{js^i}$ for all $i \in \mathbb{Z}_t$ and $j \in \mathbb{Z}_r$. Now $(\sigma^i \rho^j)^2 = \sigma^{2i} \rho^{j(1+s^i)}$ and for this to be identity we would have to have $\sigma^{2i} = 1 = \rho^{j(1+s^i)}$. But this is impossible as i is odd. A similar contradiction is obtained for automorphisms of the form $\sigma^i \rho^j \tau$, where $i = 2k+1 \in \mathbb{Z}_t$ and $j \in \mathbb{Z}_r$. This completes the proof of Theorem 3.4. ■

Let us mention that the smallest non-self polar weakly flag-transitive configuration obtained by the above theorem is $\mathcal{C}(2;4,17)$ with 34 points, whereas the smallest triangle-free non-self polar weakly flag-transitive configuration is $\mathcal{C}(3;8,17)$. In Figure 1 the configuration $\mathcal{C}(2;4,17)$ is shown.

Appendix

Blocks of the four smallest known configurations with prescribed properties are as follows

$\mathcal{C}(2;6,9)$:	(1 -1 2' -2')	(4' -4' 1" -1")	(4 -4 2" -2")		mod 9
$\mathcal{D}(3;6,14)$:	(0 1 1' 4')	(1' 6' 0" 1")	(0 5 0" 3")		mod 14
$\mathcal{C}(2;4,17)$:	(1 3 0' 4')	(4 5 0' 9')			mod 17
$\mathcal{C}(3;8,17)$:	(1 -1 3' -3')	(8' -8' 7" -7")	(4" -4" 5"' -5"')	(6 -6 2"' -2"')	mod 17

where for example, notation `-4''` mod 17 stands for $17 - 4 + 2 \times 17 = 47$ and `(1 -1 2' -2')` mod 9 stands for the nine blocks

```
(1 8 11 16) (2 0 12 17) (3 1 13 9) (4 2 14 10) (5 3 15 11)
(6 4 16 12) (7 5 17 13) (8 6 9 14) (0 7 10 15)
```

(compare Figure 1).

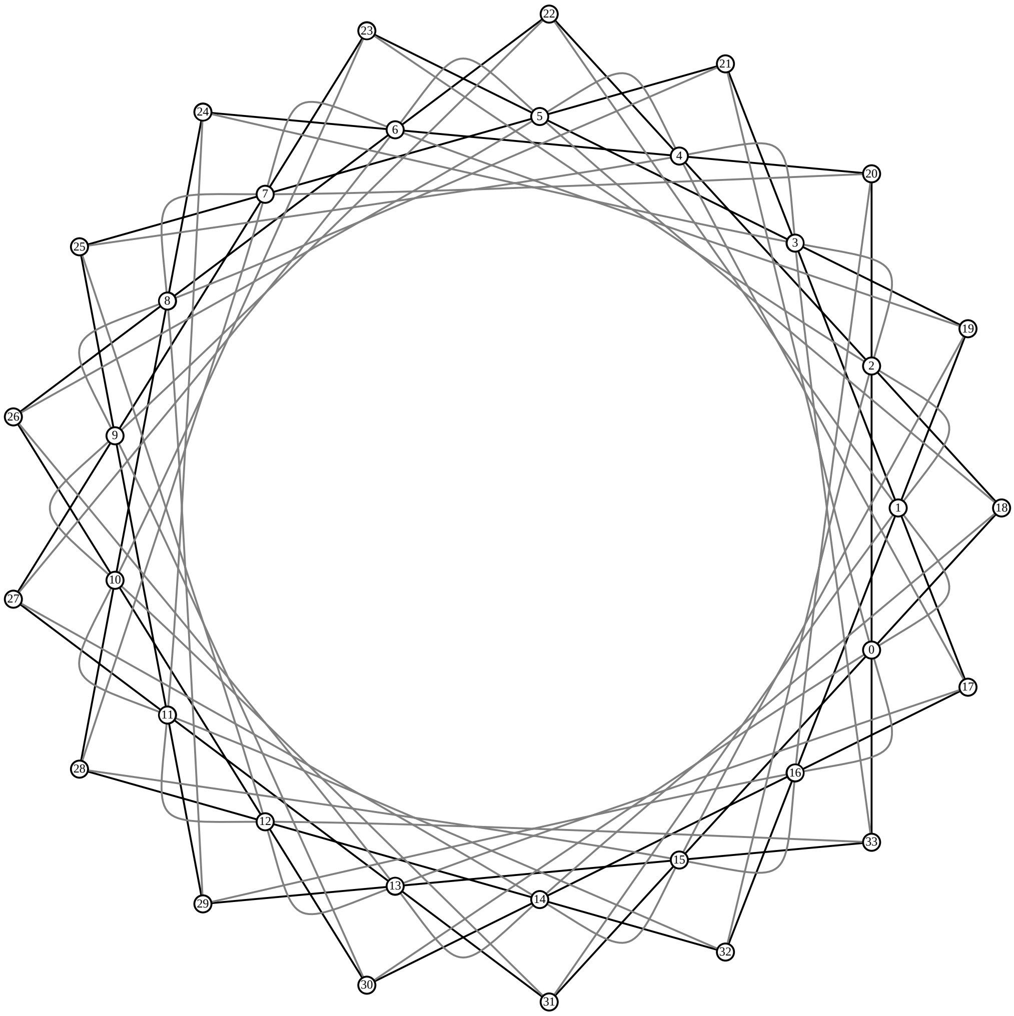

Figure 1: The smallest known weakly flag-transitive non-self polar configuration $\mathcal{C}(2;4,17)$

References

[1] N. Biggs, *Algebraic Graph Theory*, Second Edition, Cambridge Univ. Press, Cambridge, 1993.

[2] N. Biggs and A. T. White, *Permutation groups and combinatorial structures*, Cambridge University Press, 1979.

[3] A. Betten, G. Brinkmann, and T. Pisanski, Counting symmetric configurations v_3, submitted.

[4] J. Bokowski, B. Sturmfels, *Computational Synthetic Geometry*, Springer-Verlag, New York, LNM 1355, 1989.

[5] A. Bondy, U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Co., New York, 1976.

[6] I. Z. Bouwer, Vertex and edge-transitive but not 1-transitive graphs, *Canad. Math. Bull.*, **13** (1970), 231–237.

[7] H.S.M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* **56**, (1950) 413–455.

[8] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relators for Discrete Groups*, Fourth Edition, Springer Verlag, 1980.

[9] J. Folkman, Regular line-symmetric graphs, *J. Combin. Theory*, **3** (1967), 215–232.

[10] H. Gropp, *Configurations*, in The CRC Handbook of Combinatorial Designs, C.J. Colburn and J.H. Dinitz, eds., 253–255.

[11] D. Marušič, Recent developments in half-transitive graphs, *Discrete Math.* **182** (1998) 219–231.

[12] D. Marušič, Half-transitive group actions on finite graphs of valency 4, *J. Combin. Theory, B* **73** (1998) 41–76.

[13] D. Marušič and C. E. Praeger, Tetravalent graphs admitting half-transitive group actions; alternating cycles, submitted

[14] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, Berlin, 1982.

[15] W. T. Tutte, *Connectivity in graphs*, University of Toronto Press, Toronto, 1966.

[16] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.