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LOCAL BOUNDARY MORERA THEOREMS

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Morera Theorems

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1 Introduction and the main results

Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a bounded open set with boundary of class C^2 and let $\Gamma \subset bD$ be a relatively open set. Suppose that an affine complex subspace Λ of complex dimension p intersects Γ in a compact set and transversely. We say that $f \in C(\Gamma)$ has the Morera property with respect to Λ if the integral $\int_{\Lambda\cap\Gamma} f\alpha$ vanishes for each $(p,p-1)$-form α on $\mathbb{C}^N$ with constant coefficients [G-S].

Functions that typically satisfy the Morera conditions are the ones that belong to $A(D)$, that is, are continuous on $\overline{D}$ and holomorphic on D. If f is such a function, then $f|bD$ has the Morera property with respect to every affine linear subspace of complex dimension p, $1 \leq p \leq N-1$, which intersects bD transversely. We denote by $\widetilde{G}_{\mathbb{C}}(N,p)$ the space of all complex affine subspaces in $\mathbb{C}^N$ of complex dimension p.

A function $f \in C(\Gamma)$ is said to be a CR function on Γ if it satisfies the weak tangential Cauchy - Riemann equations on Γ, that is, if $\int_\Gamma f\bar{\partial}\alpha = 0$ for every smooth $(N, N-2)$-form α on $\mathbb{C}^N$ whose support meets Γ in a compact set. If f has a continuous extension to an open neighbourhood U of Γ in $\overline{D}$ which is holomorphic on $U \cap D$, then f is a CR function on Γ [H-C]. If bD is connected, then every continuous CR function on bD continues through D as a member of $A(D)$ [W].

Several global Morera theorems are known. These are theorems which specify various subsets $\mathcal{S}$ of $\widetilde{G}_{\mathbb{C}}(N,p)$ such that if $f \in C(bD)$ has the Morera property with respect to each $\lambda \in \mathcal{S}$ which intersects bD transversely, then f in a CR function on bD [G-S], [Go], [K-M].

Local Morera theorems are more difficult to prove. These are theorems where the subspaces along which one assumes the Morera property do not sweep the entire boundary bD. The first one was proved in [G-S]. The one that we take as our starting point here can be formulated as follows.

Theorem 1.0.1 [[Gl1], Theorem 1.1] *Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a bounded open set with boundary of class C^2. Let $\mathcal{H}$ be an open connected set of complex affine hyperplanes which contains a hyperplane that misses $\overline{D}$. Write $\Gamma = bD \cap (\cup_{H \in \mathcal{H}} H)$. Suppose that $f \in C(\Gamma)$ has the Morera property with respect to every $\Lambda \in \mathcal{H}$ that meets bD transversely. Then f is a CR function on Γ.*

The condition that $\mathcal{H}$ contains a hyperplane that misses $\overline{D}$ cannot be dropped in general. For instance, if B_2 is the unit ball of $\mathbb{C}^2$ and if f on $bB_2 \setminus \{z = 0\}$ is defined by $f(z, w) = 1/\bar{z}$, then f has the Morera property with respect to all complex lines close to the z-axis, yet there is no open subset of bB_2 where f is a CR function [Gl1].

Nevertheless, the following theorem, which is one of the main results of our paper, shows that there are quite general geometric situations in which the condition that $\mathcal{H}$ contains a hyperplane missing $\overline{D}$ is not necessary.

Theorem 1.0.2 *Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a bounded open set with boundary of class C^2. Let $\mathcal{H}$ be an open connected set of complex affine hyperplanes in $\mathbb{C}^N$. Write $E = \cup_{H \in \mathcal{H}} H$, $\Gamma = E \cap bD$. Assume that $f \in C(\Gamma)$ has the Morera property with respect to every $H \in \mathcal{H}$ that meets bD transversely. Assume that $\Gamma = K \cup Q$ where K and Q are disjoint open subsets of bD and assume that $\mathcal{H}$ contains a hyperplane that misses K. Then f is a CR function on K.*

It is an open question whether Theorem 1.0.1 and Theorem 1.0.2 hold in $\mathbb{C}^N$, $N \geq 3$, in the case when $\mathcal{H}$ consists of complex lines. In this direction we prove the following theorem which shows that there are geometric situations in which one can give an affirmative answer to our question.

Theorem 1.0.3 *Let D be a bounded open set in $\mathbb{C}^N$, $N \geq 2$, with boundary of class C^2. Let $\mathcal{L}$ be an open connected set of complex lines in $\mathbb{C}^N$. Write $E = \cup_{L \in \mathcal{L}} L$, $\Gamma = bD \cap E$. Assume that $f \in C(\Gamma)$ has the Morera property with respect to every $\Lambda \in \mathcal{L}$ that meets bD transversely.*

Assume that $\Gamma = K \cup Q$ where K and Q are disjoint open subsets of bD. Assume that $\mathcal{L}$ contains a line that misses K and assume that for every $L \in \mathcal{L}$ that meets bD transversely and meets K, the intersection $L \cap K$ bounds an open set in $L \cap D$ whose every component is simply connected.

Then f is a CR function on K. In fact, under the above assumption on the intersections $L \cap K$, there is an open neighbourhood $\mathcal{U}$ of K in $E \cap \overline{D}$ such that $f|K$ extends holomorphically into $\mathcal{U} \cap D$.

Remark 1.0.4 To prove that f is a CR function on K in the case $N = 2$, the assumptions on the intersections $L \cap K$, $L \in \mathcal{L}$ that meets bD transversely, are not necessary. This follows from Theorem 1.0.2.

The proof of Theorem 1.0.3 is given in Section 3.

2 Proof of Theorem 1.0.2

Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a bounded open set with boundary of class C^2. Let $\mathcal{H}$ be an open connected set of complex hyperplanes in $\mathbb{C}^N$. Write $E = \cup_{H \in \mathcal{H}} H$, $\Gamma = E \cap bD$. Assume that $f \in C(\Gamma)$ has the Morera property with respect to every $H \in \mathcal{H}$ that meets bD transversely. Assume that $\Gamma = K \cup Q$ where K and Q are disjoint open subsets of bD and assume that $\mathcal{H}$ contains a hyperplane H_0 that misses K. We have to prove that f a CR function on K, that is, $\int_K f\bar{\partial}\alpha = 0$ for every smooth $(N, N-2)$-form α on $\mathbb{C}^N$ whose support intersects K in a compact set. It is enough to consider the case where α is a smooth $(N, N-2)$-form with compact support contained in E. Given such a form α, let λ_α be a smooth function on $\mathbb{C}^N$ such that $\lambda_\alpha \equiv 1$ in a neighbourhood of $K \cap \operatorname{supp} \alpha$ and $\lambda_\alpha \equiv 0$ in a neighbourhood of $Q \cap \operatorname{supp} \alpha$. To prove that $\int_K f\bar{\partial}\alpha = 0$ is the same as to prove that $\int_\Gamma f\bar{\partial}(\lambda_\alpha \alpha) = 0$. To do this, we, as in [G-S], [Gl1], approximate the form $\lambda_\alpha \alpha$ by special simple forms γ for which the assumed Morera property of f implies that $\int_\Gamma f\bar{\partial}\gamma = 0$. The coefficients of these special forms depend, after a suitable coordinate change in $\mathbb{C}^N$, only on one variable, that is, they are complex plane waves. To prove that coefficients of form $\lambda_\alpha \alpha$ can be approximated on Γ by plane waves, we use, as in [G-S], [Gl1], duality. Since now we do not assume that $\mathcal{H}$ contains a hyperplane that misses bD, we need for this approximation the following new local support theorem for the complex Radon transform.

Theorem 2.0.5 *Let g be a continuous function on $\mathbb{C}^N$ with compact support and let $\mathcal{S}$ be an open connected set of complex affine hyperplanes in $\mathbb{C}^N$ such that*

$\int_H f dm_H = 0$ for all $H \in \mathcal{S}$ where we denote by m_H the natural $(2N-2)$-dimensional Lebesgue measure on H. Assume that $(\cup_{H\in\mathcal{S}} H) \cap \operatorname{supp} g = G_1 \cup G_2$ where G_1 and G_2 have disjoint closures in $\mathbb{C}^N$. If $\mathcal{S}$ contains a hyperplane that misses G_1, then every hyperplane in $\mathcal{S}$ misses G_1.

The global version ($G_2 = \emptyset$) of this theorem follows from Theorem 1.1 of [Q].

Given a normed vector space Y, let $Y^{'}$ be the space of bounded linear functionals on Y. For a subset M of Y and a subset N of $Y^{'}$ let $M^{\perp}$ and ${}^{\perp}N$ be their annihilators, that is, $M^{\perp} = \{u \in Y^{'};\ u(m) = 0 \text{ for each } m \in M\}$ and ${}^{\perp}N = \{y \in Y;\ u(y) = 0 \text{ for each } u \in N\}$. We use the notation $Y = Y_1 \oplus Y_2$ when Y_1, Y_2 are closed subspaces of Y such that $Y = Y_1 + Y_2$, $Y_1 \cap Y_2 = \{0\}$ and the projection P on Y with range Y_1 and null space Y_2 is continuous. If $Y = Y_1 \oplus Y_2$, then $Y^{'} = Y_1^{'} \oplus Y_2^{'}$. If $f \in Y$ and $u \in Y^{'}$, then $u(f) = u_1(f_1) + u_2(f_2)$ where $f = f_1 + f_2$, $u = u_1 + u_2$, $u_1 \in Y_1^{'}$, $u_2 \in Y_2^{'}$, $f_1 \in Y_1$, $f_2 \in Y_2$.

Proposition 2.0.6 *Let Y be a normed vector space and let $Y = Y_1 \oplus Y_2$. Let V be a subspace of Y. The following two conditions are equivalent:*

(a) Let $u \in Y^{'}$ and let $u = u_1 + u_2$ where $u_1 \in Y_1^{'}$ and $u_2 \in Y_2^{'}$.
If u is such that $u(f) = 0$ for every $f \in V$, then $u_1 = 0$.

(b) There exists a closed subspace Z_2 of Y_2 such that $\overline{V} = Y_1 \oplus Z_2$ where $\overline{V}$ is the closure of V in Y.

Proof: Let us prove first that (a) implies (b). Let $u \in Y^{'}$ be such that $u(f) = 0$ for every $f \in V$. Write $u = u_1 + u_2$ where $u_1 \in Y_1^{'}$ and $u_2 \in Y_2^{'}$. Then by (a), $u_1 = 0$. This implies that $V^{\perp} \subset \{0\} \oplus Y_2^{'}$. Since the annihilator $V^{\perp}$ is a closed subspace of $Y^{'}$, there exists a closed subspace N_2 of $Y_2^{'}$ such that $V^{\perp} = \{0\} \oplus N_2$. Since ${}^{\perp}(V^{\perp})$ equals to the closure $\overline{V}$ of V in Y, it follows that $\overline{V} = Y_1 \oplus^{\perp} N_2$. Let us set $Z_2 =^{\perp} N_2$. Thus, (b) follows from (a).

Conversely, suppose that (b) holds, so that $\overline{V} = Y_1 \oplus Z_2$ where Z_2 is a closed subspace of Y_2. Let $u \in Y^{'}$ be such that $u(f) = 0$ for every $f \in V$. By continuity $u(f) = 0$ for every $f \in \overline{V}$. If $u = u_1 + u_2$ where $u_1 \in Y_1^{'}$ and $u_2 \in Y_2^{'}$, then it follows that $u_1(f_1) = 0$ for every $f_1 \in Y_1$. This implies that $u_1 = 0$ and thus (a) holds. $\square$

As in [Gl1], we need the notion of good direction with respect to D to apply Fubini's theorem in our context. Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a bounded open set. For a complex line Λ through the origin, let $\Pi_\Lambda : \mathbb{C}^N \to \Lambda$ be the orthogonal projection. We call Λ a good direction with respect to D if the following holds: For almost all $z \in \Lambda$, either $\Pi_\Lambda^{-1}(z)$ meets bD transversely or $\Pi_\Lambda^{-1}(z)$ misses bD. If D has boundary of class C^2, then almost every complex line through the origin is a good direction with respect to D [[G-S], p. 579]. If we wish that every (rather than almost every) complex line Λ through the origin is a good direction with respect to D, it is necessary to require that bD is of class C^r, $r > 2N - 3$ [[G-S], p. 576].

Denote by $\langle | \rangle$ the Hermitian inner product on $\mathbb{C}^N$. A *complex plane wave* is a smooth function on $\mathbb{C}^N$ which, after a suitable linear change of coordinates, depends only on one variable. Thus, if $\Lambda \subset \mathbb{C}^N$ is a complex line through the origin, if $x \in \Lambda \setminus \{0\}$ and if $\psi : \mathbb{C} \to \mathbb{C}$ is a smooth function, we shall call the function $w(z) = \psi(\langle z \mid x \rangle)$ on $\mathbb{C}^N$ a complex plane wave in direction Λ. If $\mathcal{F}$ is a family of affine complex hyperplanes, then a complex plane wave $w(z) = \psi(\langle z \mid x \rangle)$ is said to be carried by $\mathcal{F}$ if the complex hyperplane $\langle z \mid x \rangle = c$ belongs to $\mathcal{F}$ whenever $c \in \operatorname{supp} \psi$.

Using the Fubini's theorem, it is easy to see that the assumed Morera property of f implies that $\int_\Gamma f \bar{\partial} \gamma = 0$ for a large family of complex plane waves:

Lemma 2.0.7 [[Gl1], Lemma 2.1] *Let D, $\mathcal{H}$, E and f be as above. Suppose that γ is a $(N, N-2)$-form on $\mathbb{C}^N$ whose coefficients are smooth complex plane waves that are carried by $\mathcal{H}$ and have good directions with respect to D. Then $\int_\Gamma f \bar{\partial} \gamma = 0$*

Choose an open neighbourhood $W_1 \subset E$ of K and an open neighbourhood $W_2 \subset E$ of Q such that W_1 and W_2 have disjoint closures in E. Since $\mathcal{H}$ contains a hyperplane H_0 that misses K, we can choose W_1 such that H_0 misses $\overline{W}_1 \cap E$. Given such neighbourhoods W_1 and W_2, we write $\Omega_1 = W_1 \cap D$ and $\Omega_2 = W_2 \cap D$. Let $\Omega = \Omega_1 \cup \Omega_2$.

Let $C^1(\overline{\Omega})$ be the space of functions on Ω which, together with their first order derivatives, extend continuously to the (compact) set $\overline{\Omega}$ equipped with the norm

$$\| \phi \| = \sup_{\Omega} | \phi | + \sum_{j=1}^{2N} \sup_{\Omega} | \partial\phi/\partial x_j | .$$

Denote by X the subspace of $C^1(\overline{\Omega})$ consisting of the restrictions to Ω of smooth functions on $\mathbb{C}^N$ with compact support contained in E. Note that if w is a complex plane wave on $\mathbb{C}^N$ carried by $\mathcal{H}$, then $w \in X$ (more precisely, $w|\Omega \in X$). Further, denote by X_i, $i = 1, 2$, the subspace of $C^1(\overline{\Omega})$ consisting of the restrictions to Ω of smooth functions on $\mathbb{C}^N$ with compact support contained in E which vanish identically on $\overline{\Omega}_j$, $j \neq i$. Since $\overline{\Omega}_1 \cap \overline{\Omega}_2 \cap E = \emptyset$, it follows that $X = X_1 \oplus X_2$.

Applying Theorem 2.0.5 and Proposition 2.0.6, we now prove the following key statement concerning approximation of functions by plane waves:

Lemma 2.0.8 *Let V be the subspace of X consisting of linear combinations of complex plane waves that are carried by $\mathcal{H}$ and have good directions with respect to D.*

Then the closure $\overline{V}$ of V in X contains the direct sum $X_1 \oplus \{0\}$, that is, the space $\overline{V}$ contains the restrictions to Ω of smooth functions on $\mathbb{C}^N$ with compact support contained in E which vanish identically on $\overline{\Omega}_2$.

Proof of Lemma 2.0.8 Denote by B_N the unit ball in $\mathbb{C}^N$ and by $\triangle$ the unit disc in $\mathbb{C}$. By Proposition 2.0.6 it suffices to show that if $u = u_1 + u_2 \in X_1' \oplus X_2' = X'$ annihilates all complex plane waves which are carried by $\mathcal{H}$ and have good directions with respect to D, then $u|X_1 = 0$, that is, $u_1 = 0$. So assume that $u(h) = 0$ for every $h \in V$. By the Hahn-Banach theorem u extends to an element $\tilde{u} \in (C^1(\overline{\Omega}))'$ which may be viewed as the distribution on $\mathbb{C}^N$ that takes ϕ to $\tilde{u}(\phi|\overline{\Omega})$. The distribution $\tilde{u}$ has compact support contained in $\overline{\Omega}$. We have to prove that $u_1 = 0$, that is, that $\tilde{u}(\phi) = u_1(\phi) = 0$ for every $\phi \in C^\infty(\mathbb{C}^N)$ with compact support contained in E which vanishes identically on $\overline{\Omega}_2$.

By the assumption, $\mathcal{H}$ contains a hyperplane H_0 that misses $\overline{\Omega}_1 \cap E$. Let $H_1 \in \mathcal{H}$ and let $l : [0, 1] \to \mathcal{H}$ be a path such that $l(0) = H_0$, $l(1) = H_1$. It is easy to see that there are connected neighbourhood $\mathcal{H}_1$ of $l([0, 1])$ in $\mathcal{H}$ and $\varepsilon > 0$ such that

$$E \cap (\overline{\Omega}_1 + 3\varepsilon B_N) \cap H_0 = \emptyset, \tag{1}$$

$$E \cap (\overline{\Omega}_1 + 3\varepsilon B_N) \cap (\overline{\Omega}_2 + 3\varepsilon B_N) = \emptyset \tag{2}$$

and

$$\text{if a hyperplane } H \text{ is parallel to } H^{'} \in \mathcal{H}_1 \text{ and if } \operatorname{dist}(H, H^{'}) < 3\varepsilon, \text{ then } H^{'} \in \mathcal{H}. \quad (3)$$

Let ψ_n be an approximate identity on $\mathbb{C}^N$ consisting of smooth even functions such that $\operatorname{supp} \psi_n \subset \varepsilon B_N$ for all n. Fix $n \in \mathbb{N}$ and $H \in \mathcal{H}_1$. Since the set of good directions with respect to D is dense in the set of all directions, it follows that arbitrary close to H there is a hyperplane $H^{'} \in \mathcal{H}_1$ such that complex direction orthogonal to $H^{'}$ is a good direction with respect to D. Let Σ be a complex plane wave carried by the family of hyperplanes parallel to $H^{'}$ and contained in $H^{'} + \varepsilon B_N$. Then $\Sigma * \psi_n$ is a complex plane wave carried by the family of hyperplanes parallel to $H^{'}$ and contained in $H^{'} + 2\varepsilon B_N$ and thus, by (3), $\Sigma * \psi_n$ is carried by $\mathcal{H}$ and has good direction with respect to D. By the assumption, $\widetilde{u}(\Sigma * \psi_n) = 0$ which, since ψ_n is even, implies that $(\widetilde{u} * \psi_n)(\Sigma) = 0$ [Hö]. Let $\Sigma_k(z) = \omega_k(\langle z \mid x \rangle - c)$ where $\langle z \mid x \rangle = c$ is the equation of $H^{'}$ and where ω_k is a smooth approximate identity on $\mathbb{C}$, $\operatorname{supp} \omega_k \subset \varepsilon \triangle$, for all k. As in the proof of Lemma 3.1.3 of [G-S], passing to the limit as $k \to \infty$, we get $\int_{H'} \widetilde{u} * \psi_n dm_{H'} = 0$. Since we can choose $H^{'}$ arbitrarily close to H, it follows by the continuity of $\widetilde{u} * \psi_n$ that $\int_H (\widetilde{u} * \psi_n) dm_H = 0$. Now, we want to apply Theorem 2.0.5. To do this, write $E_{\mathcal{H}_1} = \cup_{H \in \mathcal{H}_1} H$ and observe first that $\mathcal{H}_1$ is open and connected and

$$\int_H f dm_H = 0 \quad (H \in \mathcal{H}_1). \tag{4}$$

Further, $\operatorname{supp}(\widetilde{u} * \psi_n) \cap E_{H_1} \subset ((\overline{\Omega}_1 + \varepsilon B_N) \cup (\overline{\Omega}_2 + \varepsilon B_N)) \cap E_{\mathcal{H}_1}$ where by (2) and (3), $(\overline{\Omega}_1 + \varepsilon B_N) \cap E_{\mathcal{H}_1}$ and $(\overline{\Omega}_2 + \varepsilon B_N) \cap E_{\mathcal{H}_1}$ have disjoint closures in $\mathbb{C}^N$. Finally, by (1), the hyperplane $H_0 \in \mathcal{H}_1$ misses $(\overline{\Omega}_1 + \varepsilon B_N) \cap E_{\mathcal{H}_1}$. So Theorem 2.0.5 implies that $\widetilde{u} * \psi_n$ vanishes identically on $(\overline{\Omega}_1 + \varepsilon B_N) \cap E_{\mathcal{H}_1}$. Since $\widetilde{u} * \psi_n$ converges to $\widetilde{u}$, it follows that $\widetilde{u}(\phi) = 0$ for every $\phi \in C^\infty(\mathbb{C}^N)$ with compact support in $E_{\mathcal{H}_1}$ which vanishes identically on $\overline{\Omega}_2$. Since $H_1 \in \mathcal{H}$ was arbitrary, it follows that $\widetilde{u}(\phi) = 0$ for every $\phi \in C^\infty(\mathbb{C}^N)$ with compact support in E which vanishes identically on $\overline{\Omega}_2$. This completes the proof of Lemma 2.0.8. □

The proof of Theorem 2.0.5 is similar to the proof of Theorem 1.1 of [Q] and uses the following lemma.

Lemma 2.0.9 *Let $H \in \mathbb{C}^N$ be an affine complex hyperplane. Let $\mathcal{S}$ be an open set of complex hyperplanes in $\mathbb{C}^N$ containing H and let $r > 0$ be such that $H' \in \mathcal{S}$ whenever H' is an affine complex hyperplane parallel to H, $\text{dist}(H', H) < r$. Write $P(H, r) = \{z \in \mathbb{C}^N; \text{dist}(H, r) < r\}$. Let g be a continuous function with compact support on $\mathbb{C}^N$. Assume that $\int_{H'} g dm_{H'} = 0$ for each $H' \in \mathcal{S}$. The following is impossible:*

There are ρ, $0 < \rho < r$, a point $x \in bP(H, \rho) \cap \text{supp}\, g$ and $\varepsilon > 0$ such that $P(H, \rho) \cap [x + \varepsilon B_N] \cap \text{supp}\, g = \emptyset$.

A global version of Lemma 2.0.9, where the condition $P(H, \rho) \cap [x + \varepsilon B_N] \cap \text{supp}\, g = \emptyset$ is replaced by the global condition $P(H, \rho) \cap \text{supp}\, g = \emptyset$, was, as the key statement, applied in the proof of Theorem 1.1 of [Q].

Proof of Lemma 2.0.9 The proof is essentially contained in [Q] and uses microlocal arguments from [Q] ([[Q], Proposition 2.2] and [[Hö], Theorem 8.5.6]) and the fact that $WF_A(g) \cap [\{x\} \times \mathbb{C}^N]$ depends only on the behavior of g in a neighbourhood of x where we denote by $WF_A(g)$ the analytic wave front set of a function g [Hö]. Denote by $H^\perp$ the subspace (passing through the origin) of $\mathbb{C}^N$ that is orthogonal to H. Assume that the lemma is false, that is, there are ρ, x and ε with the above properties. By the assumption, $\mathcal{S}$ is open and contains the hyperplane H' through x that is parallel to H. Since $\int_{H'} g dm_{H'} = 0$ for each $H' \in \mathcal{S}$, [[Q], Proposition 2.2] implies that $(x, \omega) \notin WF_A(f)$ for every $\omega \in H^\perp$. Let ω be a normal vector to $bP(H, \rho)$. By the construction of $P(H, \rho)$, $\omega \in H^\perp$. So, [[Hö], Theorem 8.5.6] and the fact that $WF_A(f) \cap [x \times \mathbb{C}^N]$ depends only on the behavior of f in a neighbourhood of x imply that $(x, \omega) \in WF_A(f)$, which is a contradiction. This completes the proof. □

To complete the proof of Theorem 1.0.2, we have to show that $\int_K f \bar{\partial} \alpha = 0$ for every smooth $(N, N-2)$-form α on $\mathbb{C}^N$ whose support intersects K in a compact set. As usual, write $\omega(z) = dz_1 \wedge \cdots \wedge dz_N$, $\omega_{[j,k]}(z) = dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_N$. It is enough to consider the case when $\alpha = \phi(z)\omega(z) \wedge \omega_{[j,k]}(\bar{z})$ where ϕ is a smooth function on $\mathbb{C}^N$ with compact support contained in E. Let λ_ϕ be a smooth function with compact support on $\mathbb{C}^N$ such that $\lambda_\phi \equiv 1$ in a neighbourhood of $\text{supp}\, \phi \cap \overline{\Omega}_1$ and $\lambda_\phi \equiv 0$ in a neighbourhood of $\text{supp}\, \phi \cap \overline{\Omega}_2$. Lemma 2.0.8 applies to show that on $\overline{\Omega}$ the function $\lambda_\phi \phi$ can be, together with its first derivatives, uniformly approximated by linear combinations Φ of complex plane waves which are carried by $\mathcal{H}$ and which have good directions with respect to D. By Lemma 2.0.7, $\int_\Gamma f\bar{\partial}[\Phi(z)\omega(z)\wedge\omega_{[j,k]}(\bar{z})]=0$, so it follows that $\int_K f\bar{\partial}\alpha=\int_\Gamma f\bar{\partial}[\lambda_\phi(z)\phi(z)\omega(z)\wedge\omega_{[j,k]}(\bar{z})]=0$. This completes the proof of Theorem 1.0.2.

Remark 2.0.10 By using the same method of proving, provided that Theorem 2.0.5 is replaced by Theorem 1.1 of [Gl2], one can easily show that Theorem 1.0.2 also holds if $\mathcal{H}$ consists of affine real hyperplanes instead of affine complex hyperplanes and with the Morera property defined in a natural way, as in [Go]: If an affine real hyperplane $H\in\mathcal{H}$ intersects bD transversely, then $f\in C(\Gamma)$ is said to have the Morera property with respect to H if $\int_{H\cap bD} f\alpha=0$ for every $(N,N-2)$-form α on $\mathbb{C}^N$ with constant coefficients.

3 Proof of Theorem 1.0.3

3.1 The idea of the proof

Let D be a bounded open set in $\mathbb{C}^N$, $N\geq 2$, with boundary bD of class C^2. Let $\mathcal{L}$ be an open connected set of complex lines in $\mathbb{C}^N$. Write $E=\cup_{L\in\mathcal{L}}L$, $\Gamma=bD\cap E$.

We describe the idea of the proof in the case when $Q=\emptyset$. In this case, we assume that $\mathcal{L}$ contains a line that misses $\overline{D}$ and that for every $L\in\mathcal{L}$ that meets bD transversely, the intersection $L\cap bD$ bounds an open set in $L\cap D$ whose every component is simply connected. Assume that $f\in C(\Gamma)$ has the Morera property with respect to every $\Lambda\in\mathcal{L}$ that meets bD transversely. We have to prove that f is a CR function on $bD\cap E$. We know that the method applied in the proof of Theorem 1.0.1 works only in the case $N=2$. When $N>2$, the proof works only locally in two dimensional slices. That is, given an affine two dimensional complex subspace Π that intersects bD transversely, Theorem 1.0.1 can be applied to each component of the set $\mathcal{L}_\Pi=\{\Lambda\in\mathcal{L};\Lambda\subset\Pi\}$. Suppose $\mathcal{W}$ is such a component. In general, $\mathcal{W}$ need not contain a line Λ which misses $\overline{D}$, so Theorem 1.0.1 cannot be applied. However, suppose that $\mathcal{W}$ contains a line Λ_0 that meets bD transversely such that there is a neighbourhood $\mathcal{P}_{\Lambda_0}$ with the property that f has a continuous extension F to $\mathcal{P}_{\Lambda_0}\cap\overline{D}$ which is holomorphic on $\mathcal{P}_{\Lambda_0}\cap D$. In this case, one finds another bounded open set Ω with boundary of class C^2 which outside a slight neighbourhood of $\Lambda_0 \cap D$ coincides with D and which misses Λ_0. Then one can modify f in $\mathcal{P}_{\Lambda_0} \cap b\Omega$ to equal F. The modified f still has the Morera property along each line $\Lambda \in \mathcal{W}$, but now there is a line $\Lambda_0 \in \mathcal{W}$ missing $\overline{\Omega}$. So, by Theorem 1.0.1 one can conclude that $f|(bD \cap (\cup_{\Lambda \in \mathcal{W}} \Lambda))$ is a CR function.

We want to use such a method step by step. To do this, we need to propagate analytic extendibility and here we need the assumption that for every $\Lambda \in \mathcal{L}$ that meets bD transversely, $\Lambda \setminus \overline{D}$ is connected.

3.2 Holomorphic extension by the Cauchy integral

Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a bounded open set with boundary of class C^2.

In this section, we list some known facts about holomorphic extensions of CR functions from a part of bD by Cauchy integrals.

Since we work with a domain with boundary of class C^2 rather than of class C^∞, we give a remark on transverse intersections. Let L be a complex line in $\mathbb{C}^N$ and let $\mathcal{P}$ be the set of complex lines parallel to L. By Sard's Theorem [A-R], for almost all $\Lambda \in \mathcal{P}$, either Λ meets bD transversely or $\Lambda \cap bD = \emptyset$. The same is true if $\mathcal{P}$ is the set of all complex lines passing through a point.

Given $L \in \widetilde{G}_{\mathbb{C}}(N,1)$ and $r > 0$, we denote by $\mathcal{K}(L,r)$ the set of all $\Lambda \in \widetilde{G}_{\mathbb{C}}(N,1)$ parallel to L and at distance less than r from L and by $\mathcal{K}_1(L,r)$ the subset of these $\Lambda \in \mathcal{K}(L,r)$ that either miss bD or meet bD transversely. We write $P(L,r) = \{z \in \mathbb{C}^N;\ \mathrm{dist}(z,L) < r\} = \cup_{\Lambda \in \mathcal{K}(L,r)} \Lambda$, $P_1(L,r) = \cup_{\Lambda \in \mathcal{K}_1(L,r)} \Lambda$. Note that $P_1(L,r)$ is an open subset of $P(L,r)$. If $z \in \mathbb{C}^N$, then we denote by $L(z)$ the complex line through z which is parallel to L.

Denote by B_{N-1} the unit ball in $\mathbb{C}^{N-1}$ centered at the origin. Let $L \in \widetilde{G}_{\mathbb{C}}(N,1)$, $r > 0$ and choose the coordinate system in $\mathbb{C}^N$ so that L coincides with the z_1 axis. In this case, $P(L,r) = \{(\zeta_1, \zeta_2) \in \mathbb{C}^N;\ \zeta_1 \in \mathbb{C},\ \zeta_2 \in rB_{N-1}\}$. For each $w \in rB_{N-1}$, write $L_w = L((0,w))$.

Let $S \subset bD$ be a relatively open subset such that every $\Lambda \in \mathcal{K}_1(L,r)$ meets S in a compact set which can be empty. If $\Lambda \in \mathcal{K}_1(L,r)$ meets S, then $\Lambda \cap S$ is a finite union of simple closed curves of class C^2. If f is a continuous function on $S \cap P(L,r)$, we define the function $\widehat{f}_{L,r}$ on $P_1(L,r) \setminus S$ by the Cauchy integral as follows

$$\widehat{f}_{L,r}(z,w) = \begin{cases} \frac{1}{2\pi i}\int_{L_w\cap S}\frac{f(\zeta,w)d\zeta}{\zeta-w} & ((z,w)\in P_1(L,r)\setminus S;\ L_w \text{ meets } S) \\ 0 & ((z,w)\in P_1(L,r);\ L_w \text{ misses } S). \end{cases}$$

By Sard's theorem, the set $P(L,r)\setminus P_1(L,r)$ is of measure zero. Thus, $\widehat{f}_{L,r}$ is in fact defined almost everywhere on $P(L,r)\setminus S$. By transversality, $\Lambda\cap S$ depends smoothly on $\Lambda\in\mathcal{K}_1(L,r)$. Since f is a continuous function on S, it follows that $\widehat{f}_{L,r}$ is a continuous function on $P_1(L,r)\setminus S$.

If f is a CR function, then the function $\widehat{f}_{L,r}$ is holomorphic. This was proved by H. Rossi [Ro](See also [H-C]):

Lemma 3.2.1 *If f is a continuous CR function on $S\cap P(L,r)$, then there is a holomorphic function $F_{L,r}$ on $P(L,r)\setminus S$ such that $F_{L,r}=\widehat{f}_{L,r}$ almost everywhere on $P(L,r)\setminus S$.*

Under the additional assumption on holomopphic extendibility along complex lines, we get holomorphic extendibility of f:

Assume that there is a bounded open set G in $\mathbb{C}^N$ such that $P(L,r)\cap bG = S\cap P(L,r)$.

Lemma 3.2.2 *Let f be a continuous CR function on $bG\cap P(L,r)$. Assume that $f|(\Lambda\cap bG)$ has a continuous extension to $\Lambda\cap\overline{G}$ which is holomorphic on $\Lambda\cap G$ for each $\Lambda\in\mathcal{K}_1(L,r)$. Then the holomorphic function $F_{L,r}$ extends continuously from $G\cap P_1(L,r)$ to $\overline{G}\cap P_1(L,r)$ and the extension coincides with f on $bG\cap P_1(L,r)$.*

Proof: By Lemma 3.2.1, $F_{L,r}$ is holomorphic on $P(L,r)\setminus bG$ and $F_{L,r}=\widehat{f}_{L,r}$ almost everywhere on $P(L,r)\setminus bG$. Since $\widehat{f}_{L,r}$ is a continuous function on $P_1(L,r)\setminus bG$, $F_{L,r}\equiv\widehat{f}_{L,r}$ on $P_1(L,r)\setminus bG$. The assumption implies that $\widehat{f}_{L,r}|(\Lambda\setminus\overline{G})\equiv 0$ for each $\Lambda\in\mathcal{K}_1(L,r)$. By the jump formulas for the Cauchy integral it follows that $\widehat{f}_{L,r}|(\Lambda\cap G)$ extends continuously to $\Lambda\cap\overline{G}$ and the extension coincides with f on $\Lambda\cap bG$ for each $\Lambda\in\mathcal{K}_1(L,r)$. Thus, everywhere on $bG\cap P_1(L,r)$ the nontangential limits of $\widehat{f}_{L,r}$, a bounded holomorphic function on $P_1(L,r)\cap G$, coincide with f. Since f is continuous, this completes the proof. □

To show that f extends holomorphically into $P(L,r)\cap G$, we have to look at families of parallel lines in slightly different directions:

Lemma 3.2.3 *Let f be a continuous CR function on $P(L,r)\cap bG$. Let $\mathcal{U}\subset \widetilde{G}_{\mathbb{C}}(N,1)$ be an open neighbourhood of L such that $\mathcal{K}(L,r)\subset\mathcal{U}$ and such that every $\Lambda\in\mathcal{U}$ which meets bD transversely meets $P(L,r)\cap bG$ in a compact set. Assume that for every $\Lambda\in\mathcal{U}$ that meets bD transversely, $f|(\Lambda\cap bG)$ has a continuous extension to $\Lambda\cap\overline{G}$ which is holomorphic on $\Lambda\cap G$. Then f has a continuous extension to $\overline{G}\cap P(L,r)$ which is holomorphic on $G\cap P(L,r)$.*

Proof: By Lemma 3.2.2 there is nothing to prove if every line $\Lambda\in\mathcal{K}(L,r)$ meets bD transversely. Let $z\in P(L,r)\cap bG$. We will show that there is a neighbourhood W of z in $P(L,r)\cap bG$ such that $F_{L,r}$ extends continuously from G to $G\cup W$ and the extension coincides with f on W. By Lemma 3.2.2 our assumptions imply that this is the case if $z\in P_1(L,r)\cap bG$. If $z\notin P_1(L,r)\cap bG$, then by Sard's theorem there are a line $L^{'}$ close to L and an $r^{'}<r$ such that $\mathcal{K}(L^{'},r^{'})\subset\mathcal{U}$ and such that a neighbourhood $W\subset bD$ of z is contained in $P_1(L^{'},r^{'})\cap bG$. Our assumptions and Lemma 3.2.2 imply that $F_{L^{'},r^{'}}$ extends continuously to W and the extension coincides with f on W. Since $F_{L,r}=F_{L^{'},r^{'}}$ near z, the same will hold for $F_{L,r}$. This completes the proof. $\square$

Given a complex line L and $r>0$, we have already defined $P(L,r)$ as the union of all complex lines parallel to L and at a distance less than r from L. Let $p\in L$ and $R>0$. We denote by $P(L,p,R,r)$ the set of all points $z\in P(L,r)$ such that the orthogonal projection of z to L lies in the open disc of radius R centered at p. Thus, if L is the z_1 axis and $p=0$, then $P(L,p,R,r)=\{(z_1,\dots,z_N);|z_1|<R,\ |z_2|^2+\dots+|z_N|^2<r^2\}$. Further, we define the open neighbourhood $\mathcal{K}(L,p,R,r)$ of L in $\widetilde{G}_{\mathbb{C}}(N,1)$ as the set of all complex lines Λ in $\mathbb{C}^N$ such that the orthogonal projection of $\Lambda\cap P(L,p,R,r)$ to L is the open disc of radius R centered at p. Thus, if L is the z_1 axis and $p=0$, then $\Lambda\in\mathcal{K}(L,p,R,r)$ if and only if Λ intersects $P(L,p,R,r)$ and if $\zeta\in\Lambda\cap bP(L,p,R,r)$, then $|\zeta_1|=R$ and $|\zeta_1|^2+\dots+|\zeta_N|^2<r^2$.

To prove Theorem 1.0.3, we shall show that under the Morera conditions, the CR property and holomorphic extendibility propagate along a path of complex lines.

We will do this in small steps.

3.3 Local propagation of holomorphic extendibility under CR property

The following fact is obvious:

Proposition 3.3.1 *Let Λ be a complex line and let $\rho > 0$. Assume that $\Lambda \setminus \overline{G}$ is connected for every $\Lambda' \in \mathcal{K}_1(\Lambda, \rho)$. Then $P(\Lambda, \rho) \setminus \overline{G}$ is connected.*

Locally, under the assumption that f is a CR function, holomorphic extendibility propagates:

Lemma 3.3.2 *Let $R > 0$ and $p \in L$. Assume that $\Lambda \cap \overline{G} \subset P(L, r)$ if $\Lambda \in \mathcal{K}(L, p, R, r)$. Assume that $\Lambda \setminus \overline{G}$ is connected whenever $\Lambda \in \mathcal{K}(L, p, R, r)$ meets bD transversely. Let f be a continuous CR function on $P(L, r) \cap bG$. Suppose there is an $r_0 \in (0, r)$ such that*

f has a continuous extension to $P(L, r_0) \cap \overline{G}$ which is holomorphic on $P(L, r_0) \cap G$. *(a)*

Then f has a continuous extension to $\overline{G} \cap P(L, r)$ which is holomorphic on $G \cap P(L, r)$.

We understand that (a) is also satisfied in the case when $P(L, r_0)$ misses bG.

Proof: There are a neighbourhood $\mathcal{U} \in \mathcal{K}(L, p, R, r)$ of L and a $\rho > 0$ such that if $\Lambda \in \mathcal{U}$ then $\mathcal{K}(\Lambda, \rho) \subset \mathcal{K}(L, p, R, r)$ and $P(\Lambda, \rho) \cap \overline{G} \subset P(L, r_0) \cap \overline{G}$. In particular, f has a continuous extension to $P(\Lambda, \rho) \cap \overline{G}$ which is holomorphic on $P(\Lambda, \rho) \cap G$ for each $\Lambda \in \mathcal{U}$. If $\Lambda \in \mathcal{U}$, then $F_{\Lambda,\rho} \equiv 0$ on $P(\Lambda, \rho) \setminus \overline{G}$. Note that if $\Lambda \in \mathcal{U}$ and if $\rho_1 > \rho$ is such that $\mathcal{K}(\Lambda, \rho_1) \subset \mathcal{K}(L, p, R, r)$, then F_{Λ,ρ_1} vanishes identically on $P(\Lambda, \rho_1) \setminus \overline{G}$. This is so as both $F_{\Lambda,\rho}$ and F_{Λ,ρ_1} are holomorphic, $F_{\Lambda,\rho} \equiv F_{\Lambda,\rho_1}$ on $P(\Lambda, \rho) \setminus \overline{G}$ and since by Proposition 3.3.1, $P(\Lambda, \rho_1) \setminus \overline{G}$ is connected. Now Lemma 3.2.3 completes the proof. $\square$

3.4 CR functions and Morera property along complex lines in $\mathbb{C}^N$

Let $\mathcal{L}$ be an open connected family of complex lines in $\mathbb{C}^N$ and let $f \in C(\Gamma)$ have the Morera property with respect to every complex line $L \in \mathcal{L}$ that meets bD transversely. Suppose that $(\cup_{\Lambda\in\mathcal{L}}\Lambda) \cap bD = K \cup Q$ where K and Q are disjoint open subsets of bD and suppose that a line $L_0 \in \mathcal{L}$ misses K. If $N = 2$, Theorem 1.0.2 implies that f is a CR function on K. The method applied in the proof of Theorem 1.0.2 works only in the case $N = 2$. Nevertheless, in the case $N > 2$, one can prove a similar fact for families $\mathcal{L}$ of the form $\mathcal{K}(L,p,R,r)$. In this case, one can use Theorem 1.0.2 in dimenison two in sufficiently many two dimensional affine complex planes:

Lemma 3.4.1 *Let D be a bounded open set in $\mathbb{C}^N$, $N \geq 2$, with boundary of class C^2. Let L be a complex line in $\mathbb{C}^N$, $p \in L$ and let $r > 0$, $R > 0$. Assume that $\Lambda \cap \overline{D} \subset P(L,p,R,r)$ if $\Lambda \in \mathcal{K}(L,p,R,r)$. Suppose that $f \in C(P(L,r) \cap bD)$ has the Morera property with respect to every line $\Lambda \in \mathcal{K}(L,p,R,r)$ that meets bD transversely. Assume that $P(L,r) \cap bD = K^{'} \cup Q^{'}$ where $K^{'}$ and $Q^{'}$ are disjoint open subsets of bD. Assume that there is a complex line $L_0 \in \mathcal{K}(L,p,R,r)$ parallel to L which misses $K^{'}$. Then f is a CR function on $K^{'}$.*

Lemma 3.4.1 will be used locally in the proof of Theorem 1.0.3.

Proof: By Theorem 1.0.2 there is nothing to prove if $N = 2$. Thus, let $N > 2$. We will show by applying Theorem 1.0.2 locally in two dimensional slices that for each $z_0 \in K^{'}$, there is neighbourhood $\Omega^{'}(z_0)$ of z_0 in $P(L,p,R,r)$ such that f is a CR function on $K^{'} \cap \Omega^{'}(z_0)$. This will imply that f is a CR function on $K^{'}$.

Since $L_0 \cap \overline{K^{'}} = \emptyset$, there is a neighbourhood $\mathcal{U} \subset \mathcal{K}(L,p,R,r)$ of L_0 such that every line $\Lambda \in \mathcal{U}$ misses $K^{'}$. Denote by $\mathcal{E}$ the set of all two dimensional affine complex planes Π such that $\Lambda \subset \Pi$ for some $\Lambda \in \mathcal{U}$. Since $\mathcal{U}$ is open, $\mathcal{E}$ is an open subset of $\widetilde{G}_{\mathbb{C}}(N,2)$. Let $\Pi \in \mathcal{E}$ and let $\Lambda_0 \subset \Pi$ where $\Lambda_0 \in \mathcal{U}$. There is an open connected subset $\mathcal{W}_{\Lambda_0}(\Pi)$ of the set of all complex lines $\Lambda \in \mathcal{K}(L,p,R,r)$ contained in Π such that $\mathcal{W}_{\Lambda_0}(\Pi)$ contains all complex lines $\Lambda \in \mathcal{K}(L,p,R,r)$ contained in Π and parallel to Λ_0. Since every line from the set $\mathcal{U}$ misses $K^{'}$ and since f has the Morera property with respect to every line $\Phi \in \mathcal{W}_{\Lambda_0}(\Pi)$ that meets bD transversely, it follows by Theorem 1.0.2 that f is a CR function on $K^{'} \cap (\cup_{\Phi\in\mathcal{W}_{\Lambda_0}(\Pi)}\Phi)$ whenever

Π meets bD transversely.

Let $z_0 \in K'$. Denote by Π_0 the affine complex two dimensional plane that passes through z_0 and L_0. There are a neighbourhood $\mathcal{P}(\Pi_0)$ of Π_0 in $\mathcal{E}$ and a neighbourhood $\Omega(z_0)$ of z_0 in $P(L,p,R,r)$ such that for every $\Pi \in \mathcal{P}(\Pi_0)$ that passes through z_0, the set $\mathcal{P}(\Pi_0)$ contains all planes $z+\Pi$, $z \in \Omega(z_0)$. For each $\Pi \in \mathcal{P}(\Pi_0)$, there is $\Lambda_0 \in \mathcal{U}$, $\Lambda_0 \subset \Pi$, such that $\Pi \cap \Omega(z_0) \subset \cup_{\Phi \in \mathcal{W}_{\Lambda_0}(\Pi)} \Phi$. This implies that f is a CR function on $\Pi \cap \Omega(z_0) \cap K'$ for each $\Pi \in \mathcal{P}(\Pi_0)$ that meets bD transversely. We will now show by using the Fubini's theorem that there is a neighbourhood $\Omega'(z_0) \subset \Omega(z_0)$ such that f is a CR function on $K' \cap \Omega'(z_0)$.

The set of all planes $\Pi \in \mathcal{P}(\Pi_0)$ that pass through z_0 is open. Since the boundary bD is of class C^2, for almost every $\Pi \in \mathcal{P}(\Pi_0)$ that passes through z_0 the following holds: almost every affine complex 2-plane Π' parallel to Π either meets bD transversely or misses bD [[G-S], p. 579]. Thus, by translation and a linear change of coordinates in $\mathbb{C}^N$, we can assume that in new coordinates z_j, $1 \le j \le N$, in $\mathbb{C}^N$ the following holds: $z_0 = 0$ and the linear span $\Pi_{i,j}$ of any pair of coordinate axes z_i, z_j belongs to $\mathcal{P}(\Pi_0)$ and each plane $\Pi_{i,j}$ has the following property: almost every affine complex 2-plane $\Pi'_{i,j}$ parallel to $\Pi_{i,j}$ either meets bD transversely or misses bD. Let $\varepsilon > 0$ be such that $z + \Pi_{i,j} \in \mathcal{P}(\Pi_0)$ for each $z \in \varepsilon B_N$ and such that $\varepsilon B_N \cap bD = \varepsilon B_N \cap K'$. We have to show that $\int_K f\bar\partial\alpha = 0$ for every smooth $(N, N-2)$-form α on $\mathbb{C}^N$ with compact support contained in εB_N. As usual, write $\omega(z) = dz_1 \wedge \cdots \wedge dz_N$, $\omega_{[j,k]}(z) = dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_N$. It is enough to consider the case when $\alpha = \phi(z)\omega(z) \wedge \omega_{[j,k]}(\bar z)$, $1 \le j,k \le N$, where ϕ is a smooth function on $\mathbb{C}^N$ with compact support contained in εB_N. Let $\pi(z) = (z_1, \cdots, \widehat{z_j}, \cdots, \widehat{z_k}, \cdots, z_N) = z'$. Applying Fubini's theorem yields

$$\int_{K' \cap \varepsilon B_N} f\bar\partial(\phi(z)\omega(z)\omega_{[j,k]}(\overline{z})) = \tag{5}$$
$$= \pm \int_{\pi(\mathrm{supp}\,\phi)} \left[\int_{\pi^{-1}(z') \cap K' \cap \varepsilon B_N} f\overline{\partial}(\phi(z)dz_j \wedge dz_k)\right] \omega_{[j,k]}(z) \wedge \omega_{[j,k]}(\bar z).$$

Since $\pi^{-1}(z')$ is of the form $z' + \Pi_{j,k}$ and since f is a CR function on $(z' + \Pi_{j,k}) \cap \varepsilon B_N \cap K'$ whenever the plane $z' + \Pi_{j,k}$, meets bD transversely, it follows that the inner integral vanishes for almost every $z' \in \mathrm{supp}\,\phi$. This implies that $\int_K f\bar\partial\alpha = 0$ for every smooth $(N, N-2)$-form α on $\mathbb{C}^N$ with compact support contained in εB_N. This completes the proof. $\square$

3.5 Local propagation of CR property under Morera conditions

Locally, under the Morera property along lines and under the assumption that f extends into a small neighbourhood of a complex line, CR property propagates:

Lemma 3.5.1 *Let L be a complex line, $p \in L$ and $r > 0$, $R > 0$. Assume that $\Lambda \cap \overline{D} \subset P(L,p,R,r)$ if $\Lambda \in \mathcal{K}(L,p,R,r)$. Let $f \in C(P(L,r) \cap bD)$ have the Morera property with respect to every line $\Lambda \in \mathcal{K}(L,p,R,r)$ that meets bD transversely. Let G be either a union of components of $P(L,r) \cap D$ or a union of bounded components of $P(L,r) \cap (\mathbb{C}^N \setminus \overline{D})$. Assume that there are a complex line L_0 parallel to L and an $r_0 > 0$ such that $P(L_0, r_0) \subset P(L,r)$ and such that*

f has a continuous extension F to $P(L_0, r_0) \cap \overline{G}$ which is holomorphic on $P(L_0, r_0) \cap G$. *(b)*

Then f is a CR function on $bG \cap P(L,r)$.

We understand that (b) is also satisfied in the case when $P(L_0, r_0)$ misses $\overline{G}$.

Proof: The boundary $P(L,r) \cap bD$ consists of two parts $\Sigma_1 = bG \cap P(L,r)$ and $\Sigma_2 = [P(L,r) \setminus \overline{G}] \cap bD$. By Lemma 3.4.1 there is nothing to prove if there is a line $\Lambda \in \mathcal{K}(L,p,R,r)$ parallel to L which misses Σ_1. Thus, suppose that $\mathcal{K}(L,p,R,r)$ does not contain such a line.

Since boundary values of holomorphic functions always satisfy the Morera conditions, our assumptions imply that the functions $f|\Sigma_1$ and $f|\Sigma_2$ satisfy the Morera conditions with respect to all complex lines Λ in a neighbourhood of L_0 that meet bD transversely. Choose a complex line Λ in this neighbourhood which is parallel to L and meets bD transversely. Now, we want to apply Lemma 3.4.1. To do this, we replace D by another bounded open set Ω in $\mathbb{C}^N$ with boundary of class C^2. Consider first the case when G is a union of components of $P(L,r) \cap D$. In this case, we cut out $\Lambda \cap \overline{G}$ from D: Let $\rho > 0$ be so small that $P(\Lambda, \rho) \subset P(L_0, r_0)$ and such that every line in $\mathcal{K}(\Lambda, \rho)$ meets bD transversely. We now modify G inside $P(\Lambda, \rho)$, that is, we replace G by an open subset G' of G such that $G' \setminus P(\Lambda, \rho) = G \setminus P(\Lambda, \rho)$, such that $bG' \cap P(L,r)$ is of class C^2 and such that $\overline{G'}$ misses Λ. We denote by Ω the union $(D \setminus G) \cup G'$. In the case when G is a union of bounded components of $P(L,r) \cap (\mathbb{C}^N \setminus \overline{D})$, we add $\Lambda \cap \overline{G}$ to D: we use G' as above and we denote by Ω

the union $D \cup (\overline{G} \setminus \overline{G'})$. In both cases, the open set Ω outside of $P(\Lambda, \rho)$ coincides with D, $b\Omega$ is of class C^2 and $b\Omega \cap P(L, r)$ consists of two parts $bG' \cap P(L, r)$ and Σ_2. Now, a line Λ misses $bG' \cap P(L, r)$. We now work with Ω in place of D.

Denote by f the function on $bG' \cap P(L, r)$ which coincides with old f on $P(L, r) \setminus P(\Lambda, \rho)$ and which coincides with F on $bG' \cap P(L_0, r_0)$.

Since Λ misses $\overline{G'}$, there is a neighbourhood $\mathcal{U} \subset \mathcal{K}(L, p, R, r)$ such that every line $\Phi \in \mathcal{U}$ misses $\overline{G'}$. Let $\mathcal{U}$ be so small that for every $\lambda \in \mathcal{K}(L, p, R, r)$ parallel to some line from the set $\mathcal{U}$, the following holds: either $\lambda \cap \overline{\Omega} \subset P(L_0, r_0)$ or $\lambda \cap \overline{\Omega} \subset P(L, r) \setminus P(\Lambda, \rho)$. Since F is holomorphic on $P(L_0, r_0) \cap G$ and since Ω coincides with D on $P(L, r) \setminus P(\Lambda, \rho)$, it follows that $f|b\Omega$ has the Morera property with respect to all lines $\lambda \in \mathcal{K}(L, p, R, r)$ parallel to some line from the set $\mathcal{U}$ that meet bD and $b\Omega$ transversely. Now, as in the proof of Lemma 3.4.1, we see that f is a CR function on $bG' \cap P(L, r)$. This implies that f is a CR function on $bG \cap P(L, r)$ since by our assumptions f is a CR function on $bG \cap P(L_0, r_0)$ and since $bG' = bG$ in $P(L, r) \setminus P(\Lambda, \rho)$.

To construct G' above, we choose a very small ρ', $0 < \rho' < \rho$ and let $G" = G \setminus \overline{P(\Lambda, \rho')}$. The boundary $bP(\Lambda, \rho')$ intersects $bG \cap P(L, r)$ transversely. Now, we cut of the corners of $G"$ in smooth way by the standard smoothing procedure. So, there is an open set $G' \subset G"$ with boundary $bG' \cap P(L, r)$ of class C^2 which outside a small neighbourhood $U \subset P(\Lambda, \rho)$ of the intersection $bP(\Lambda, \rho') \cap [bG \cap P(L, r)]$ coincides with the boundary $bG" \cap P(L, r)$. The open set G' has the desired properties. This completes the proof. $\square$

3.6 Some technical lemmas

In this section, we list a few simple facts that we will use in the step by step application of Lemma 3.3.2, Lemma 3.4.1 and Lemma 3.5.1 above to prove Theorem 1.0.3.

The space $\widetilde{G}_{\mathbb{C}}(N, 1)$ of all complex lines in $\mathbb{C}^N$ becomes a manifold in a natural way. Denote by $\mathcal{G}_i \subset \widetilde{G}_{\mathbb{C}}(N, 1)$, $1 \leq i \leq N$, the set of those lines that are not perpendicular to z_i axis. Each $\Lambda \in \mathcal{G}_i$ can be written uniquely in the form $\Lambda = \{\zeta(\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_N) + (\beta_1, \ldots, \beta_{i-1}, 0, \beta_{i+1}, \ldots, \beta_N) : \zeta \in \mathbb{C}\}$ with $(\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_N) \in \mathbb{C}^N$ and

$(\beta_1,\ldots,\beta_{i-1},0,\beta_{i+1},\ldots,\beta_N)\in\mathbb{C}^N$. Define $\Phi_i:\mathcal{G}_i\to\mathbb{C}^{2(N-1)}$, $i=1,\ldots,N$, by $\Phi_1(\{\zeta(\alpha_1,\ldots,\alpha_{i-1},1,\alpha_{i+1},\ldots,\alpha_N)+(\beta_1,\ldots,\beta_{i-1},0,\beta_{i+1},\ldots,\beta_N):\zeta\in\mathbb{C}\})=(\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_N,\beta_1,\ldots,\beta_{i-1},\beta_{i+1},\ldots,\beta_N)$. Then $(\mathcal{G}_i,\Phi_i)$, $i=1,\ldots,N$, are the coordinate neighbourhoods on $\widetilde{G}_{\mathbb{C}}(N,1)$. In particular, every open connected set in $\widetilde{G}_{\mathbb{C}}(N,1)$ is path-connected. Denote by I the closed interval $[0,1]$.

The following fact is obvious:

Proposition 3.6.1 *Given a path $\Lambda:I\to\widetilde{G}_{\mathbb{C}}(N,1)$, there are paths $a:I\to\mathbb{C}^N$, $b:I\to\mathbb{C}^N\setminus\{0\}$ such that $\Lambda(t)=\{a(t)+\zeta b(t),\zeta\in\mathbb{C}\}$, $t\in I$.*

Proposition 3.6.2 *Let $R>0$, $r>0$. Let $0<\varepsilon<\min\{1,r(r+\sqrt{N-1}(1+R))^{-1}\}$, $p_1,p_2,u_1,u_2\in\mathbb{C}^N$, $|u_1|=|u_2|=1$, $|p_1-p_2|<\varepsilon$, $|u_1-u_2|<\varepsilon$, $\Lambda_1=\{p_1+\zeta u_1;\zeta\in\mathbb{C}\}$ and $\Lambda_2=\{p_2+\zeta u_2;\zeta\in\mathbb{C}\}$. Then $\Lambda_2\in\mathcal{K}(\Lambda_1,z_1,R,r)$.*

Proof: Choose a coordinate system in $\mathbb{C}^N$ so that Λ_1 coincides with z_1 axis. Let $\Lambda_2=\{\zeta(\alpha_1,\ldots,\alpha_N)+(\beta_1,\ldots,\beta_N);\zeta\in\mathbb{C}\}$ where $|\alpha_1-1|^2+|\alpha_2|^2+\cdots+|\alpha_N|^2<\varepsilon^2$, $|\beta_1|^2+\cdots+|\beta_N|^2<\varepsilon^2$. Since $\alpha_1\neq 0$, Λ_2 intersects $\{z\in\mathbb{C}^N;|z_1|\leq R\}$ in a compact set. Suppose that $|\zeta\alpha_1+\beta_1|=R$. Then $|\zeta|<(R+|\beta_1|)/|\alpha_1|<(R+\varepsilon)/(1-\varepsilon)$ and it follows that for each i, $2\leq i\leq N$, $|\zeta\alpha_i+\beta_i|<(R+\varepsilon)\varepsilon/(1-\varepsilon)+\varepsilon=(1+R)\varepsilon/(1-\varepsilon)$. Since $\varepsilon<\min\{1,r/(r+\sqrt{N-1}(1+R))\}$, it follows that $\sqrt{N-1}(1+R)\varepsilon/(1-\varepsilon)<r$, which completes the proof. □

Proposition 3.6.3 *Let $a:I\to\mathbb{C}^N$, $b:I\to\mathbb{C}^N\setminus\{0\}$ be paths and let $\Lambda(t)=\{a(t)+\zeta b(t);\zeta\in\mathbb{C}\}$, $t\in I$. Let $\mathcal{L}\subset\widetilde{G}_{\mathbb{C}}(N,1)$ be an open set containing $\Lambda(I)$. Given $R>0$, there are $r>0$ and $\delta>0$ such that*

(i) $\mathcal{K}(\Lambda(t),a(t),R,r)\subset\subset\mathcal{L}$ $(t\in I)$,

(ii) $\Lambda(t')\in\mathcal{K}(\Lambda(t),a(t),R,r)$ $(|t-t'|<\delta,\ t,t'\in I)$.

In particular, there are t_i, $0=t_1<t_2<\cdots<t_s=1$, such that

$$\Lambda(I)\subset\cup_{i=1}^{s}\mathcal{K}(\Lambda(t_i),a(t_i),R,r)\subset\mathcal{L}$$

and that

$$\Lambda(t)\in\mathcal{K}(\Lambda(t_i),a(t_i),R,r)\qquad (t\in[t_1,t_2],\ i=1;\ t\in[t_{i-1},t_{i+1}],\ 2\leq i\leq(s-1);\ t\in[t_{s-1},t_s],\ i=s).$$

Proof: By compactness there is an $r > 0$ such that $\mathcal{K}(\Lambda(t), a(t), R, r) \subset \mathcal{L}$ $(t \in I)$. Let $\varepsilon > 0$ be the one given by Proposition 3.6.2. The uniform continuity of a, b yields a $\delta > 0$ such that $|a(t) - a(t^{'})| < \varepsilon$, $|b(t) - b(t^{'})| < \varepsilon$ whenever $|t - t^{'}| < \delta$ so (ii) follows by Proposition 3.6.2. We complete the proof by choosing t_i so that $|t_{i+1} - t_i| < \delta$ for all i, $1 \leq i \leq (s-1)$. □

3.7 Completion of the proof of Theorem 1.0.3

Proof of Theorem 1.0.3 Let D, E, $\mathcal{L}$, L, f K, Q, be as in Theorem 1.0.3. By our assumption there is a line Λ_0 in $\mathcal{L}$ that misses K. Given a line $L \in \mathcal{L}$ and an $r > 0$ such that $\mathcal{K}(L, r) \subset \mathcal{L}$, denote by $G_{L,r}$ the union of components of the set $P(L,r) \cap D$ that abut K. Our assumptions on K imply that $P(L,r) \cap bG_{L,r} = K \cap P(L,r)$. We will show that for each $\Lambda_1 \in \mathcal{L}$, there are an $r > 0$ such that $\mathcal{K}(\Lambda_1, r) \subset \mathcal{L}$ and a continuous function on $P(\Lambda_1, r) \cap \overline{G}_{\Lambda_1,r}$, holomorphic on $G_{\Lambda_1,r}$ which agrees with f on $P(\Lambda_1, r) \cap bG_{\Lambda_1,r}$. Let $\Lambda_1 \in \mathcal{L}$ and let $\Lambda : I \to \mathcal{L}$ be a path from Λ_0 to Λ_1. By Proposition 3.6.3, there are paths a, $b : I \to \mathbb{C}^2$ such that $\Lambda(t) = \{a(t) + \zeta b(t);\ \zeta \in \mathbb{C}\}$ $(t \in I)$. Since D is bounded, there is an $R > 0$ such that if $r > 0$ and if $\mathcal{K}(\Lambda(t), a(t), R, r)$, then $\Lambda \cap \overline{D} \in P(\Lambda(t), a(t), R, r)$ $(t \in I)$. By compactness, there is an $r > 0$ such that $\mathcal{K}(\Lambda(t), a(t), R, r) \subset \mathcal{L}$ $(t \in I)$. By Proposition 3.6.3, there are t_j, $0 = t_1 < \cdots < t_s = 1$, such that $\Lambda(I) \subset \cup_{j=1}^{s} \mathcal{K}(\Lambda(t_j), a(t_j), R, r)$ and such that

$$\Lambda(t_{j+1}) \in \mathcal{K}(\Lambda(t_j), a(t_j), R, r),\ \ 1 \leq j \leq s-1. \tag{6}$$

Our assumptions on the intersections $L \cap K$, $L \in \mathcal{L}$, imply that $L \setminus \overline{G}_{\Lambda(t_j),r}$ is connected whenever $L \in \mathcal{K}(\Lambda(t_j), a(t_j), R, r)$ meets bD transversely for each j, $1 \leq j \leq s$. Then our assumptions, Lemma 3.3.2, Lemma 3.4.1 and Lemma 3.5.1 used inductively imply that for each j, $1 \leq j \leq s$, f has a continuous extension to $P(\Lambda(t_j), r) \cap \overline{G}_{\Lambda(t_j),r}$ which is holomorphic on $G_{\Lambda(t_j),r}$. This shows that for each $\Lambda_1 \in \mathcal{L}$, there is an $r > 0$ such that $\mathcal{K}(\Lambda_1, r) \subset \mathcal{L}$ and such that f has a continuous extension to $P(\Lambda_1, r) \cap \overline{G}_{\Lambda_1,r}$ which is holomorphic on $G_{\Lambda_1,r}$. Thus, given $p \in K$, there is a neighbourhood U_p of p in E such that f extends holomorphically into $U_p \cap D$. This gives an open neighbourhood $\mathcal{U}$ of K in $E \cap \overline{D}$ such that $f|K$ continues holomorphically into $\mathcal{U} \cap \mathcal{D}$ and consequently, f is a CR function on K. This completes the proof. $\square$

Remark 3.7.1 The above method of the proof of Theorem 1.0.3 also works if the assumption that there is a complex line Λ_0 in $\mathcal{L}$ that misses K is replaced by the assumption that for some $\Lambda_0 \in \mathcal{L}$, there is a neighbourhood $\mathcal{P}_{\Lambda_0}$ of Λ_0 in $\mathbb{C}^N$ such that f extends holomorphically into the union of components of the set $\mathcal{P}_{\Lambda_0} \cap D$ that abut K.

Remark 3.7.2 We proved Lemma 3.4.1 and Lemma 3.5.1 in greater generality than needed in the proof of Theorem 1.0.3. We did this since essentially the same proof then shows that the assumption in Theorem 1.0.3 that for every $L \in \mathcal{L}$ that meets bD transversely and meets K, the intersection $L \cap K$ bounds an open set in $L \cap D$ with every component simply connected can be replaced by the assumption that for every $L \in \mathcal{L}$ that meets bD transversely and meets K, the intersection $L \cap K$ bounds a bounded open set in $L \cap (\mathbb{C}^N \setminus \overline{D})$ with every component simply connected. In this case, under our assumptions on the intersections $L \cap K$, there is an open neighbourhood $\mathcal{U}$ of K in $E \cap (\mathbb{C}^N \setminus D)$ such that $f|K$ extends holomorphically into $\mathcal{U} \cap (\mathbb{C}^N \setminus \overline{D})$.

To see this, we show similarly as above that for every $L \in \mathcal{L}$ that meets K, there is a neighbourhood $\mathcal{P}_L$ of L such that f extends holomorphically into the union of bounded components of $\mathcal{P}_L \cap (\mathbb{C}^N \setminus \overline{D})$ that abut K.

4 Two examples

We describe the use of Theorem 1.0.2 and Theorem 1.0.3 on two simple examples.

Let D be a bounded open set in $\mathbb{C}^N$, $N \geq 2$, with boundary of class C^2 and let $\mathcal{H}$ be an open connected set of complex affine hyperplanes. Write $E = \cup_{H \in \mathcal{H}} H$. Let f be a continuous function on $\Gamma = bD \cap E$.

Example 1. We describe a simple geometric situation in which one does not need to assume that $\mathcal{H}$ contains a hyperplane missing $\overline{D}$ and still conclude that if f has the Morera property with respect to the hyperplanes in $\mathcal{H}$ that meet bD transversely, then f is a CR function on Γ. Let D be the union of two open balls Ω_1, Ω_2 in $\mathbb{C}^N$ with disjoint closures in $\mathbb{C}^N$ such that for each $j = 1, 2$ there is a hyperplane $H_j \in \mathcal{H}$ which misses $\overline{\Omega}_j$. If f has the Morera property with respect to all complex hyperplanes in $\mathcal{H}$ which meet bD transversely, then by Theorem 1.0.2 f is a CR function on Γ. Further, if $\mathcal{H}$ consists of complex lines rather than of complex hyperplanes and if f has the Morera property with respect to each complex line in $\mathcal{H}$ which meets bD transversely, then by Theorem 1.0.2 f is a CR function on Γ and in fact, there is a neighbourhood $\mathcal{U}$ of Γ in $\overline{D} \cap E$ such that f extends holomorphically into $\mathcal{U} \cap D$.

Example 2. To give the second example, suppose that $D = B_N \setminus r\overline{B}_N$ with $0 < r < 1$ and that the set $\mathcal{H}$ contains a hyperplane that misses $r\overline{B}_N$. Suppose that f has the Morera property with respect to all hyperplanes in $\mathcal{H}$ which meet bD transversely. By Theorem 1.0.2 f is a CR function on $E \cap b(rB_N)$. Note that in this case we assume that f satisfies the Morera conditions along hyperplanes in $\mathcal{H}$ on all Γ (that is, on $b(B_N \setminus r\overline{B}_N)$) and we did not assume that $\mathcal{H}$ contains a hyperplane missing $\overline{D}$. However, if $\mathcal{H}$ does contain such a hyperplane, then by Theorem 1.0.2 the Morera property along hyperplanes in $\mathcal{H}$ implies that f is a CR function on all Γ. Assume now that the set $\mathcal{H}$ consists of complex lines and that it contains a complex line which misses $r\overline{B}_N$. If $f \in C(\Gamma)$ has the Morera property with respect to all complex lines in $\mathcal{H}$ which meet bD transversely, then by Theorem 1.0.3 and by Remark 3.7.2, f is a CR function on $E \cap b(rB_N)$ and there are a neighbourhood $\mathcal{U}$ of $E \cap rB_N$ in E and a continuous function on $\mathcal{U} \cap r\overline{B}_N$, holomorphic on $\mathcal{U} \cap rB_N$ which agrees with f on $E \cap rB_N$. This implies that both functions $f|(b(rB_N) \cap E)$ and $f|(bB_N \cap E)$ have the Morera property with respect to all complex lines in $\mathcal{H}$ that meet bD transversely. By Theorem 1.0.3, the Morera property of $f|(bB_N \cap E)$ along complex lines in $\mathcal{H}$ is sufficient that f is a CR function $bB_N \cap E$ and that f extends holomorphically into a neighbourhood of $bB_N \cap E$ in $\overline{B}_N \cap E$ provided that $\mathcal{L}$ contains a complex line that misses $\overline{B}_N$. Thus, if $\mathcal{H}$ contains a line missing $\overline{B}_N$, the Morera property of f along all lines in $\mathcal{H}$ that meet bD transversely implies that f is a CR function on the whole Γ.

5 Concluding Remarks

In this section, we give a remark to Theorem 1.0.3.

Let D be a bounded open set in $\mathbb{C}^N$, $N \geq 2$ with boundary of class C^2. Let $\mathcal{L}$

be an open connected set of complex lines in $\mathbb{C}^N$. Write $E = \cup_{L\in\mathcal{L}} L$, $\Gamma = bD \cap E$. Assume that $f \in C(\Gamma)$ has the Morera property with respect to every $\Lambda \in \mathcal{L}$ that meets bD transversely. The proof of Theorem 1.0.3 (applied to the case when $K = bD \cap E$) shows that under the assumption that $\Lambda \setminus \overline{D}$ is connected whenever $\Lambda \in \mathcal{L}$ meets bD transversely, the holomorphic extendibility propagate under Morera conditions, that is, if for some $L \in \mathcal{L}$,

there are an open neighbourhood $\mathcal{P}_L$ of L and a continuous function on $\mathcal{P}_L \cap \overline{D}$, holomorphic on $\mathcal{P}_L \cap D$ which agrees with f on $\mathcal{P}_L \cap bD$, (c)

then (c) holds for every $L \in \mathcal{L}$. In particular, there are an open neighbourhood $\mathcal{U}$ of Γ in E and a continuous function on $\mathcal{U} \cap \overline{D}$, holomorphic on $\mathcal{U} \cap D$ which agrees with f on Γ and consequently, f is a CR function on Γ.

We understand that (c) is also satisfied in the case when $\mathcal{P}_L \cap \overline{D} = \emptyset$.

In general, in the case $N \geq 3$, we do not get holomorphic extendibility of f into the whole set $E \cap D$ since in this case the holomorphic extensions of f along lines in $\mathcal{L}$ that pass through a fixed point $z \in D$ do not necessarily coincide at z if z lies outside a neighbourhood $\mathcal{U}$ of Γ as shown by the following example:

Suppose that $D = B_3$. Let L_1, L_2 be the z_1, z_2 axes. Translate L_1 in the direction of $(0,0,1)$ to get L_1' that misses $\overline{B}_3$. Translate L_2 in the direction of $(0,0,i)$ to get L_2' that misses $\overline{B}_3$ and then join L_1' with L_2' by a path of lines that misses $\overline{B}_3$. Thus we get a path that joins L_1 with L_2. Denote by Π_1 the real subspace spanned by $(1,0,0)$, $(0,i,0)$, $(0,0,1)$ and by Π_2 the real subspace spanned by $(0,1,0)$, $(0,i,0)$, $(0,0,i)$. The subspaces Π_1 and Π_2 meet only at the origin. A small open neighbourhood $\mathcal{L}$ of Λ sweeps bB_3 in an open set consisting of Σ_1, an open neighbourhood of $\Pi_1 \cap bB_3$, and Σ_2, an open neighbourhood of $\Pi_2 \cap bB_3$, such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Put $f|\Sigma_1 = 1$, $f|\Sigma_2 = 2$. Then (c) holds for every $L \in \mathcal{L}$ and consequently there there are an open neighbourhood $\mathcal{U}$ of Γ in E and a continuous function on $\mathcal{U} \cap \overline{B}_N$, holomorphic on $\mathcal{U} \cap B_N$ which agrees with f on Γ. Since the holomorphic extensions along lines L_1, L_2 do not coincide at the origin, f does not continue holomorphically into the whole set $B_N \cap E$.

However, the following proposition shows that in the case $N = 2$, we do get holomorphic extendibility of f into the whole set $D \cap E$.

Proposition 5.0.3 *Let D be a bounded open set in $\mathbb{C}^2$ with boundary of class C^2. Let $\mathcal{L}$ be an open connected set of complex lines in $\mathbb{C}^2$. Write $E = \cup_{\Lambda\in\mathcal{L}}\Lambda$ and $\Gamma = bD \cap E$. Let $f \in C(\Gamma)$ be a CR function. Assume that for every $\Lambda \in \mathcal{L}$ that meets bD transversely, $f|(\Lambda \cap bD)$ has a continuous extension to $\Lambda \cap \bar{D}$ which is holomorphic on $\Lambda \cap D$. Then f has a continuous extension to $\overline{D} \cap E$ which is holomorphic on $D \cap E$.*

Proof: By hypothesis, for every $\lambda \in \mathcal{L}$ that meets bD transversely, there is a function f_λ continuous on $\lambda\cap\overline{D}$ which is holomorphic on $D\cap\lambda$ and such that f_λ and f agree on $\lambda\cap bD$. For each $z \in D\cap E$, there is a line $\lambda \in \mathcal{L}$ that meets bD transversely such that $z \in \lambda$. We will define $\widetilde{F}(z) = f_\lambda(z)$. We will show that $\widetilde{F}(z)$ does not depend on λ, that is holomorphic on $D \cap E$ and if we set $\widetilde{F} = f$ on Γ, then $\widetilde{F}$ is continuous on $\overline{D} \cap E$ and holomorphic on $D \cap E$.

We first prove that $\widetilde{F}$ does not depend on λ. Let $z \in D$ and let L_1, $L_2 \in \mathcal{L}$ be such that $L_1 \cap L_2 = \{z\}$. Fix a complex line Λ_0 in $\mathcal{L}$ and let $\lambda_i : I \to \mathcal{L}$ be paths such that $\lambda_i(0) = \Lambda_0$, $\lambda_i(1) = L_i$, $i = 1, 2$. Define $T = \{t \in I;\ \lambda_1(t) \cap \lambda_2(t)$ is a point $\}$ and define $p : T \to \mathbb{C}^2$ by $\lambda_1(t) \cap \lambda_2(t) = \{p(t)\}$ $(t \in T)$. Then T is a subset of I on which p is continuous. Clearly $0 \notin T$, $1 \in T$. Let $(t^{'}, 1]$ be the component of T containing 1. Since $t^{'} \notin T$, we have either $\lambda_1(t^{'}) \cap \lambda_2(t^{'}) = \emptyset$ or $\lambda_1(t^{'}) = \lambda_2(t^{'})$. In the first case, $\lambda_1(t^{'})$ and $\lambda_2(t^{'})$ are parallel which implies that if $t > t^{'}$ is sufficiently close to $t^{'}$, then $|p(t)|$ is arbitrarily large. Thus, since D is bounded, $p(t^{'}, 1]$ is a path starting at z when $t = 1$ and leaving D when $t \searrow t^{'}$. Thus, there is a $t^{"} \in I$ such that either

(a) $p : [t^{"}, 1] \to \mathbb{C}^2$ is a path such that $p(t^{"}) \in \Gamma$, $p(t) \in D \cap E$ $(t^{"} < t \leq 1)$, $p(1) = z$

or

(b) $p : (t^{"}, 1] \to \mathbb{C}^2$ is a path such that $p(t) \in D \cap E$ $(t^{"} < t \leq 1)$, $p(1) = z$, $\lambda_1(t^{"}) = \lambda_2(t^{"})$ and given $\delta > 0$, $p(t) \in P(\lambda_1(t^{"}), \delta) \cap D$ if $t > t^{"}$ is sufficiently close to $t^{"}$.

By Proposition 3.6.1, there are paths a_i, $b_i : I \to \mathbb{C}^2$ such that $\lambda_i(t) = \{a_i(t) + \zeta b_i(t);\ \zeta \in \mathbb{C}\}$ $(t \in I,\ i = 1, 2)$. Since D is bounded, there is an $R < \infty$ such that for every $r > 0$ and for every $\Lambda \in \mathcal{K}(\lambda_i(t), a_i(t), R, r)$ we have $\Lambda \cap \bar{D} \subset$

$P(\lambda_i(t), a_i(t), R, r)$, $i = 1,\ 2$, $t \in I$. By compactness, there is an $r > 0$ such that $\mathcal{K}(\lambda_i(t), a_i(t), R, r) \subset \mathcal{L}$ $(t \in I,\ i = 1,\ 2)$. By Proposition 3.6.3, there are t_i, $0 = t_1 < \cdots < t_k = t" < t_{k+1} < \cdots t_s = 1$, such that $\lambda_i(I) \subset \cup_{j=1}^{s} \mathcal{K}(\lambda_i(t_j), a_i(t_j), R, r)$, such that

$$\lambda_i(t_{j+1}) \in \mathcal{K}(\lambda_i(t_j), a(t_j), R, r) \ \ (k \le j \le (s-1),\ \ i = 1,\ 2) \tag{7}$$

and such that

$$\begin{aligned} p(t_k, t_{k+1}] &\subset P(\lambda_i(t_k), r) \ \ (i = 1,\ 2), \\ p[t_j, t_{j+1}] &\subset P(\lambda_i(t_j), r) \ \ (k+1 \le j \le s-1,\ i = 1,\ 2). \end{aligned} \tag{8}$$

Our assumptions and Lemma 3.2.3 imply that for each i, $i = 1$, 2, each j, $k \le j \le s$, f has a continuous extension $F_{\lambda_i(t_j),r}$ to $\overline{D} \cap P(\lambda_i(t_j), r)$ which is holomorphic on $D \cap P(\lambda_i(t_j), r)$. By (7) it follows that

$$F_{\lambda_i(t_{j+1}),r} \equiv F_{\lambda_i(t_j),r} \text{ in a neighbourhood of } p(t_{j+1}),\ \ k+1 \le j \le s-1,\ \ i = 1,\ 2. \tag{9}$$

In the case (a), $p(t_k) \in bD \cap P(\lambda_1(t_k), r) \cap P(\lambda_2(t_k), r)$ and in the case (b), $\lambda_1(t_k) = \lambda_2(t_k)$. It follows by (8) that, in both cases (a) and (b), $F_{\lambda_1(t_k),r} \equiv F_{\lambda_2(t_k),r}$ in a neighbourhood of $p(t_k, t_{k+1}]$. This implies by (9) that $F_{\lambda_1(t_{k+1}),r} \equiv F_{\lambda_2(t_{k+1}),r}$ in a neighbourhood of $p(t_{k+1})$. By (8) it follows that $F_{\lambda_1(t_{k+1}),r} \equiv F_{\lambda_2(t_{k+1}),r}$ in a neighbourhood of $p([t_{k+1}, t_{k+2}])$. Repeating this process, we get in a finite number of steps that $F_{\lambda_1(t_s),r} \equiv F_{\lambda_2(t_s),r}$ in a neighbourhood of z. Since $\lambda_1(1) = L_1$ and $\lambda_2(1) = L_2$, this implies that $f_{L_1}(z) = f_{L_2}(z)$.

It remains to show that $\widetilde{F}$ is continuous on $\overline{D} \cap E$ and holomorphic on $D \cap E$. Given a line $L \in \mathcal{L}$ and $r > 0$ such that $\mathcal{K}(L, r) \subset \mathcal{L}$, it follows by Lemma 3.2.3 that f has a continuous extension $F_{L,r}$ to $P(L, r) \cap \overline{D}$ which is holomorphic on $P(L, r) \cap D$. Since $F_{L,r}|(D \cap P_1(L, r) = \widetilde{F}|(D \cap P_1(L, r)$, it follows that $\widetilde{F}$ is continuous on $\overline{D} \cap E$ and holomorphic on $D \cap E$. This completes the proof. □

6 Open questions

The methods used in the proof of Theorem 1.0.2 does not apply if one works with complex lines in $\mathbb{C}^N$ with $N \ge 3$. The proof of Theorem 1.0.2 given above breaks down if we replace an open connected set of complex affine hyperplanes by the lower dimensional complex affine subspaces [G-S], more precisely, Lemma 2.0.7 does not hold if we replace complex plane waves by the functions which, in some coordinate system, depend on k variables where $k \geq 2$. Thus, we pose the following two questions:

Let $N \geq 3$ and let $D \subset \mathbb{C}^N$ be a bounded set with boundary of class C^2. Suppose that $\mathcal{L}$ is an open connected set of complex lines in $\mathbb{C}^N$. Write $\Gamma = (\cup_{\Lambda \in \mathcal{L}} \Lambda) \cap bD$. Suppose that $f \in C(\Gamma)$ has the Morera property with respect to every line $\Lambda \in \mathcal{L}$ that meets bD transversely.

Question 1: Must f be a CR function on Γ?

Question 2: Suppose that for some $L \in \mathcal{L}$, there is a neighbourhood $\mathcal{P}_L$ of L such that f is a CR function on $\mathcal{P}_L \cap bD$. Does CR property propagate in the presence of Morera property of lines?

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