TIGHT DISTANCE-REGULAR GRAPHS WITH SMALL DIAMETER

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Abstract

We prove the following bound for a k-regular graph on n vertices with nontrivial eigenvalues from the interval $[r, s]$

$$n(k + rs) \leq (k - r)(k - s).$$

Equality holds if and only if the graph is strongly regular with eigenvalues in $\{k, s, r\}$. Nonbipartite distance-regular graphs with diameter $d \geq 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$, whose local graphs satisfy the above bound with equality for $s = -1 - b_1/(\theta_1 + 1)$, and $r = -1 - b_1/(\theta_d + 1)$ are called **tight graphs** and are characterized in many ways in [10]. For example, a distance-regular graph is tight if and only if it is 1-homogeneous and $a_d = 0$.

We study tight graphs of small diameter. It turns out that in the case of diameter three these are precisely the Taylor graphs and in the case of antipodal diameter four these are precisely the graphs for which the Krein parameter q_{11}^4 vanishes. We derive nonexistence conditions, which rule out twenty otherwise feasible arrays of distance-regular graphs from the list in [3, pp. 421-425].

We prove that in an antipodal distance-regular graph Γ with diameter four and vanishing Krein parameters q_{11}^4 and q_{44}^4 every second subconstituent graph is again an antipodal distance-regular graph of diameter four. Finally, if Γ is also a double-cover, i.e., Q-polynomial, then it is 2-homogeneous.

CONTENTS

1   Introduction

We motivate the reader of this paper in two very different but equally important ways. The first one concerns vanishing of Krein parameters. When one considers a certain class $\mathcal{C}$ of distance-regular graphs, the Krein parameters of interest are the ones which are not zero for the whole class $\mathcal{C}$, but do vanish for certain members of $\mathcal{C}$. In the case of strongly regular graphs only q_{11}^1 and q_{22}^2 are such Krein parameters. Cameron, Goethals and Seidel [5] showed that vanishing of either of them implies that first and second subconstituent graphs are strongly regular. In the case of antipodal distance-regular graphs of diameter three the Krein parameter q_{33}^3 is not always zero but does vanish for certain graphs. Godsil [9] showed that its vanishing implies the first subconstituent of any vertex is strongly regular.

We study antipodal distance-regular graphs with diameter four. Let Γ be such a graph. The antipodal quotient of Γ is a strongly regular graph, and each first subconstituent graph of Γ is isomorphic to some first subconstituent graph of the quotient. The Krein parameter $q_{2i,2j}^{2h}$ of Γ is equal to the Krein parameter q_{ij}^h of the quotient. Let us assume additionally that Γ is not bipartite. Then the Krein parameters q_{11}^4 and q_{44}^4 of Γ are of particular interest. We show that vanishing of the Krein parameter q_{11}^4 of Γ is equivalent to 1-homogeneous property and $a_1 \neq 0$, which is further equivalent to a first subconstituent graph being strongly regular with certain eigenvalues.

We show that for a nonbipartite antipodal distance-regular graph with diameter four the condition $q_{11}^4 = 0$ is equivalent to the graph being **tight** as it was defined in [10], see Section 2. We also show that the tight graphs of diameter three are the Taylor graphs, which are further equivalent to "regular 2-graphs". The Taylor graphs, see Taylor [17], Seidel and Taylor [15], received considerable attention, as well as the other two known infinite families of tight graphs, namely the Johnson graphs $J(2d, d)$ and the halved cubes $\frac{1}{2}H(2d, 2)$. The only other known example of a tight graph, which we have not mentioned yet, is the Patterson graph. It is an example of a primitive distance-regular graph of diameter four. Therefore, the study of tight graphs of small diameter is the other motivation for our study.

Let Γ be a tight antipodal distance-regular graph of diameter four, and let r be the size of its antipodal classes. The nontrivial eigenvalues of a first subconstituent graph are denoted by p and $-q$. We call Γ a **tight graph with parameters** (p, q, r). All the intersection parameters and eigenvalues of Γ are expressed in terms of the parameters p, q and r. Furthermore, all the intersection parameters corresponding to 1-homogeneous partition that were calculated in [10] in terms of cosine sequences are also expressed in terms of parameters p, q and r. This results in the nonexistence conditions, which rule out twenty otherwise feasible arrays of antipodal distance-regular graphs from the list in [3, pp. 421-425]. Known examples of tight graphs with parameters (p, q, r) and most of the remaining feasible arrays of these graphs belong to one or two parameter infinite families, one of which corresponds to the restriction $p = q - 2$. It turns out the Krein parameter q_{44}^4 of Γ vanishes if and only if $p = q - 2$. We prove that $q_{44}^4 = 0$ implies that every second subconstituent graph is again an antipodal distance-

regular graph of diameter four. The antipodal distance-regular graph with valency 56 and diameter four, constructed by Soicher [16], cf. Brouwer [2], is the tight graph with parameters $(2, 4, 3)$ and has $q_{44}^4 = 0$, therefore it has the above property. Soicher has already found this property with the aid of computer. Finally, in the case of a tight graph with parameters $(p, q, 2)$, vanishing of the Krein parameter q_{44}^4 implies the 2-homogeneous property. Other remarkable infinite families of feasible parameters of tight graphs with parameters (p, q, r) will be studied in a separate paper.

In the remainder of this section we recall some basic definitions and notation. An **equitable partition** of a graph Γ is a partition of its vertices into cells $C_1, C_2, \ldots, C_s$ such that for all i and j the number c_{ij} of neighbours, which a vertex in C_i has in the cell C_j, is independent of the choice of the vertex in C_i. In other words each cell C_i induces a regular graph of valency c_{ii}, and between any two cells C_i and C_j there is a biregular graph, with vertices of the cells C_i and C_j having valencies c_{ij} and c_{ji} respectively.

A graph $\Gamma = (X, R)$ with diameter d is **distance-regular** when the distance partition corresponding to any vertex $x \in X$ is equitable and the parameters of the equitable partition do not depend on x. In a distance-regular graph for a pair of vertices (x, y) at distance h, the number p_{ij}^h of vertices at distance i from x and j from y depends only on integers i, j, h, and not on (x, y). We denote the intersection numbers p_{ii}^i, $p_{i,i+1}^i$, $p_{i,i-1}^i$ and p_{ii}^0 respectively by a_i, b_i, c_i and k_i, for $i = 0, 1, \ldots, d$, note $b_0 = a_i + b_i + c_i$ is the valency of the graph Γ and call $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ the intersection array of Γ. For a detailed treatment and all the terms which are not defined here see [1], [3] and [8]. A graph is i-**homogeneous** when the distance partition corresponding to any pair of vertices at distance i is equitable, see Nomura [14].

A graph Γ of diameter d is **antipodal** if the vertices at distance d from a given vertex are all at distance d from each other. Then 'being at distance d or zero' induces an equivalence relation on the vertices of Γ, and the equivalence classes are called **antipodal classes**. For an antipodal graph Γ we define the **antipodal quotient** of Γ, to be the graph with the antipodal classes as vertices, where two classes are adjacent if they contain adjacent vertices.

2 Tight graphs

We show that strongly regular graphs with diameter two are a special kind of extremal graphs. From this, one can quickly derive an inequality for distance-regular graphs, see (3). At the end of this section we prove that in the nonbipartite, diameter three case this inequality holds with equality precisely for Taylor graphs.

A graph Γ is called **strongly regular** with parameters (k, λ, μ) if and only if its adjacency matrix A satisfies the relations

$$A^2 = kI + \lambda A + \mu(J - I - A), \quad \text{and} \quad AJ = kJ, \quad \text{for some integers } k, \ \lambda \text{ and } \mu,$$

where I is the identity matrix and J is the all-ones matrix, i.e., when Γ is k-regular and has at most three eigenvalues.

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter two. Then it is a connected strongly regular graph with $\mu \neq 0$ and three distinct eigenvalues: $k > r > s$. Eigenvectors of r and s are chosen to be orthogonal to the all-ones vector corresponding to the eigenvalue k. Therefore, r and s are the roots of the quadratic equation $x^2 - (\lambda - \mu)x + (\mu - k) = 0$, and thus

$$\lambda - \mu = r + s, \qquad \mu - k = rs. \tag{1}$$

The above relations show that the parameters (k, λ, μ) of the strongly regular graph Γ can be expressed in terms of the eigenvalues (k, r, s) of Γ. By counting the edges between the neighbours and non-neighbours of a vertex of Γ we obtain: $\mu(|X| - 1 - k) = k(k - \lambda - 1)$, and, by (1) and $\mu \neq 0$,

$$|X| = \frac{(k - r)(k - s)}{k + rs}. \tag{2}$$

We show that the right side of the equality (2) is an upper bound on the number of vertices of a k-regular graph with the eigenvalues other then k from the interval $[s, r]$.

Theorem 2.1 *Let Γ denote a k-regular graph on n vertices, with eigenvalues $k = \eta_1, \ldots, \eta_n$ (not necessarily distinct). Let r and s be such numbers that $r \leq \eta_i \leq s$, for $i = 2, \ldots, n$. Then $n(k + rs) \leq (k - r)(k - s)$. Equality holds if and only if Γ is strongly regular with eigenvalues in $\{k, r, s\}$.*

Proof. The trace of the adjacency matrix A of Γ equals the sum of its eigenvalues and is zero. The trace of A^2 equals the sum of squares of eigenvalues and is nk, i.e., the number of walks of length two which start and end in the same vertex. Summing the inequalities $(\eta_i - r)(\eta_i - s) \leq 0$ for $i = 2, \ldots, n$, and using the above two facts we obtain the desired inequality, where equality holds if and only if $\eta_i \in \{r, s\}$ for $i = 2, \ldots, n$. It follows that in the case of equality the graph Γ has at most three eigenvalues, namely k, s and r, and is therefore strongly regular. ∎

We apply this result to distance-regular graphs. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter d, and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. For a vertex $x \in X$ let $\Gamma_i(x)$ denote the set of vertices at distance i from x, and for a vertex $y \in X$ let $D_j^i(x, y) := \Gamma_i(x) \cap \Gamma_j(y)$. The graph induced on the vertices $\Gamma_i(x)$ is called the i-**th subconstituent graph** of x. It is an a_i-regular graph on k_i vertices. The first subconstituent graph of x will be also called the **local graph** of x, and will be denoted by $\Delta = \Delta(x)$. Let $\partial(x, y)$ denote the distance between the vertices x and y. Then for $\partial(x, y) = 2$ the graph induced on $D_1^1(x, y)$ is called the $\mu(x, y)$-graph, or just the μ-graph.

For $d \geq 2$ an easy eigenvalue interlacing argument guarantees $\theta_1 \geq 0$ and $\theta_d \leq -\sqrt{2}$, so we can define

$$b^- = -1 - \frac{b_1}{\theta_1 + 1} \quad \text{and} \quad b^+ = -1 - \frac{b_1}{\theta_d + 1}.$$

Suppose the graph Γ is nonbipartite with diameter $d \geq 3$, and let $a_1 = \eta_1 \geq \eta_2 \geq \ldots \geq \eta_k$ be the eigenvalues of the local graph $\Delta(x)$. Then, by Terwilliger's result [3, Thm. 4.4.3 and Thm. 4.4.4], $b^+ \geq \eta_i \geq b^-$, for $i = 2, \ldots, d$, and therefore, by Theorem 2.1, we have

$$k(a_1 + b^+ b^-) \leq (a_1 - b^+)(a_1 - b^-). \tag{3}$$

Equality holds in (3) if and only if $\eta_i \in \{b^+, b^-\}$ for $i = 2, \ldots, k$, i.e., the local graph Δ is strongly regular with eigenvalues a_1, b^- and b^+. The nonbipartite distance-regular graphs for which the equality holds are called **tight graphs** and are characterized in many ways in [10]. For example, a distance-regular graph with diameter d is tight if and only if it is 1-homogeneous, $a_1 \neq 0$ and $a_d = 0$. With the exception of the Patterson graph all known tight graphs are antipodal, see [10]. For diameter larger than four there are only two infinite families of examples known, the Johnson graph $J(2d, d)$ and the halved cube $\frac{1}{2}H(2d, 2)$, both having diameter d. In this paper we focus on tight graphs of small diameter.

Taylor graphs are the distance-regular graphs with intersection array of the form $\{k, c, 1; 1, c, k\}$. For numerous examples see [3, Sect. 7.6.C]. We characterize Taylor graphs by tight graphs with diameter three.

Theorem 2.2 *A nonbipartite distance-regular graph with diameter three is tight if and only if it is a Taylor graph.*

Proof. Let Γ be a tight graph with diameter three. We will first prove $b_2 = 1$. In a tight graph with diameter three $a_3 = 0$ by [10, Thm. 19.4], so we obtain, by $p_{23}^3 = k(b_2 - 1)/c_2$,

$$\frac{b_1 b_2}{c_2} = k_3 = p_{03}^3 + p_{13}^3 + p_{23}^3 + p_{33}^3 \geq 1 + \frac{k(b_2 - 1)}{c_2}, \tag{4}$$

i.e., $a_2 \geq a_1 b_2$, with equality if and only if $p_{33}^3 = 0$.

Let x, y be a pair of vertices of Γ at distance two. Let us count the edges between the sets $D_2^1(x, y)$ and $D_3^1(x, y)$. The sizes of these sets are respectively a_2 and b_2. Since $a_3 = 0$ the set $D_3^1(x, y)$ is independent and each vertex in it has a_1 neighbours in $D_2^1(x, y)$. The size of intersection $\Gamma(z) \cap D_3^1(x, y)$, $z \in D_2^1$, is independent of z by [10, Thm. 11.1, eq. (83)] (for, note that the above intersection is equal to $D_1^1(y, z) \cap \Gamma_3(x)$). Counting in two different ways gives $a_1 b_2 = a_2 |\Gamma(z) \cap D_3^1(x, y)|$. This number cannot be zero, since in the tight graph $a_1 \neq 0$ [10, Prop.6.5], therefore $a_1 b_2 \geq a_2$, with equality if and only if $|\Gamma(z) \cap D_3^1(x, y)| = 1$. But this means $a_2 = a_1 b_2$ by (4), and so $p_{33}^3 = 0$ and $|\Gamma(z) \cap D_3^1(x, y)| = 1$. Hence, two vertices in the independent set $D_3^1(x, y)$ would have distance at least three in the local graph $\Delta(y)$, which is a connected strongly regular graph by [10, Thm. 12.4]. It follows $b_2 = |D_3^1| = 1$ and then also $a_1 = a_2$. Hence Γ is a Taylor graph. The converse follows from straightforward calculation. ∎

3 Antipodal distance-regular graphs with diameter three and four and their krein bounds

We mentioned in the introduction that Godsil had shown in the case of antipodal distance-regular graphs with diameter three that for $r \neq 2$ the Krein condition $q_{33}^3 = 0$ implies the local graphs are strongly regular. It turns out that this is the only interesting Krein parameter for these graphs.

Lemma 3.1 *Let $\Gamma = (X, R)$ be a nonbipartite antipodal distance-regular graph with diameter three, r being the size of antipodal classes and eigenvalues $k = \theta_0 > \theta_1 > \theta_3 = -1 > \theta_4$. Then the eigenvalues θ_1 and θ_3 are the roots of $x^2 - (a_1 - c_2)x - k = 0$, where $a_1 = n - 2 - (r - 1)c_2$ and $k = n - 1$, and the following holds:*

(i) $q_{12}^2 = 0$, $q_{22}^3 = 0$, $q_{11}^2 > 0$, $q_{12}^3 > 0$, $q_{22}^2 > 0$, $q_{23}^3 > 0$.

(ii) $r \geq 2$, *with equality iff* $q_{11}^1 = 0$ *iff* $q_{11}^3 = 0$ *iff* $q_{13}^3 = 0$.

(iii)   *If* $r = 2$ *then* $q_{33}^3 = 0$. *If* $r > 2$ *then* $q_{33}^3 = 0$ *if and only if* $k = \theta_1^3$. ∎

Motivated by the above we study diameter four case. Let Γ be an antipodal distance-regular graph with diameter four, eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$ and let r be the size of its antipodal classes. Then its intersection array is, by [3, Prop. 4.2.2], determined by parameters (r, k, a_1, c_2), and has the following form

$$\{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\} = \{k, k - a_1 - 1, (r - 1)c_2, 1; 1, c_2, k - a_1 - 1, k\}, \tag{5}$$

We summarize below the basic relations concerning the parameters of Γ.

Lemma 3.2 *Let $\Gamma = (X, R)$ be an antipodal distance-regular graph with diameter four, and let r be the size of its antipodal classes. Let $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$ denote its eigenvalues and let m_i denote the multiplicity of θ_i. Then the following holds.*

(i)   *The antipodal quotient is a strongly regular graph with parameters (k, a_1, rc_2). Its eigenvalues are $\theta_0 = k$ and θ_2, θ_4, which are the roots of $x^2 - (a_1 - rc_2)x - (k - rc_2) = 0$. The remaining eigenvalues θ_1 and θ_3 of Γ are the roots of $x^2 - a_1 x - k = 0$.*

(ii)   *The following relations hold for the eigenvalues:*

$$\theta_0 = -\theta_1 \theta_3, \quad \text{and} \quad (\theta_2 + 1)(\theta_4 + 1) = (\theta_1 + 1)(\theta_3 + 1).$$

(iii)   *Parameters in terms of eigenvalues and r:*

$$k = \theta_0, \quad a_1 = \theta_1 + \theta_3, \quad b_1 = -(\theta_2 + 1)(\theta_4 + 1), \quad c_2 = \frac{\theta_0 + \theta_2 \theta_4}{r}.$$

(iv)   *The multiplicities are $m_0 = 1$,*

$$m_2 = \frac{(\theta_4 + 1)k(k - \theta_4)}{rc_2(\theta_4 - \theta_2)}, \quad m_4 = \frac{|X|}{r} - m_2 - 1, \quad \text{and} \quad m_i = \frac{(r - 1)|X|}{r(2 + a_1 \theta_i / k)} \quad \text{for} \quad i = 1, 3.$$

(v) *The eigenvalues θ_2, θ_4 are integral, $\theta_4 \leq -2$, $0 \leq \theta_2$, with $\theta_2 = 0$ if and only if Γ is bipartite, $\theta_3 < -1$, and the eigenvalues θ_1, θ_3 are integral when $a_1 \neq 0$.*

Proof. (i), (iv) and the last part of (v) follow from [3, Thm.1.3.1, Prop.4.2.3, 4.2.4, Cor.4.2.5]. (ii) and (iii) follow from (i).

(v): The claim that θ_2 and θ_4 are integral follows from two well known facts. The first one is that the conference graphs, i.e., the strongly regular graphs with parameters $(2c, c - 1, c)$, are the only strongly regular graphs which can have nonintegral eigenvalues [3, Thm. 1.3.1(ii)]. The second fact is that a conference graph cannot be an antipodal quotient of an antipodal distance-regular graph with diameter four, see [3, p.180]. The latter can be derived directly from the fact that $\lambda^2 + 4k$ is a square, see [3, Cor.4.2.5]. We have already mentioned that an interlacing argument implies the least eigenvalue is at most $-\sqrt{2}$, therefore $\theta_4 \leq -2$ by its integrality.

Since $k = -\theta_1\theta_3$ and $k > \theta_1 > \theta_3$ we conclude $-1 > \theta_3$. On the other hand, $k - rc_2$ is equal to a_2 of the antipodal quotient and $-\theta_2\theta_4 = k - rc_2$ by (i). Therefore $\theta_2 \geq 0$, with equality if and only if $a_2 = 0$, which means that the antipodal quotient is a complete multipartite graph. But such a graph can be an antipodal quotient only when it is a complete bipartite graph, see [11, Thm. 2.7], in which case the original graph is bipartite. ∎

Let Γ be an antipodal distance-regular graph of diameter four. The above result shows that we can express its intersection parameters by the size r of its antipodal classes and any three of its eigenvalues.

The matrix $P(\Gamma)$ of eigenvalues of the distance matrices of Γ is defined by $(P)_{ij} = v_j(\theta_i)$, $i, j \in \{0, 1, 2, 3, 4\}$, where $v_i(x)$ is the polynomial for which the i-th distance matrix $A_i = v_i(A)$, and it has the following form:

$$P(\Gamma) = \begin{pmatrix} 1 & \theta_0 & \theta_0 b_1/c_2 & \theta_0(r-1) & r-1 \\ 1 & \theta_1 & 0 & -\theta_1 & -1 \\ 1 & \theta_2 & -r(\theta_2+1) & \theta_2(r-1) & r-1 \\ 1 & \theta_3 & 0 & -\theta_3 & -1 \\ 1 & \theta_4 & -r(\theta_4+1) & \theta_4(r-1) & r-1 \end{pmatrix}. \tag{6}$$

As mentioned in the introduction, vanishing of either of the Krein parameters q_{ii}^i, $i \in \{2, 4\}$ implies that the first subconstituent graph of each vertex x of Γ is strongly regular. This motivates us to examine other Krein bounds of Γ. Since the Krein parameter q_{ij}^h does not change sign when we permute superscripts and subscripts, see (7), we need to check, in the case of Γ, the twenty Krein bounds $q_{ij}^h \geq 0$ with $i, j, h \in \{1, 2, 3, 4\}$ and $i \leq j \leq h$. They correspond to the following triples ijh: 111, 112, 113, 114, 122, 123, 124, 133, 134, 144, 222, 223, 224, 233, 234, 244, 333, 334, 344, 444, all of which are described in the result below.

Theorem 3.3 *Let $\Gamma = (X, R)$ be a nonbipartite antipodal distance-regular graph with diameter four and eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$. Then the following holds:*

(i) $(\theta_2 + 1)^2(k^2 + \theta_4^3) \geq (\theta_4 + 1)(k + \theta_2\theta_4)^2$, with equality if and only if $q_{44}^4 = 0$.

(ii) $\theta_3^2 \geq -\theta_4$, with equality if and only if $q_{11}^4 = 0$.

(iii) $q_{11}^2 > 0$, $q_{12}^3 > 0$, $q_{13}^4 > 0$, $q_{22}^2 > 0$, $q_{22}^4 > 0$, $q_{23}^3 > 0$, $q_{24}^4 > 0$, $q_{33}^4 > 0$.

(iv) $r \geq 2$, with equality iff $q_{11}^1 = 0$ iff $q_{11}^3 = 0$ iff $q_{13}^3 = 0$ iff $q_{33}^3 = 0$.

(v) $q_{12}^2 = 0$, $q_{12}^4 = 0$, $q_{14}^4 = 0$, $q_{22}^3 = 0$, $q_{23}^4 = 0$, $q_{34}^4 = 0$.

Proof. We compute Krein parameters by using the entries of matrix P and the following identity

$$q_{ij}^h \;=\; \frac{m_i m_j}{|X|} \sum_{a=0}^{d} \frac{v_a(\theta_i) v_a(\theta_j) v_a(\theta_h)}{k_a^2}. \tag{7}$$

Then, by Lemma 3.2, we obtain (i), (ii), (iv) for $r = 2$, (v) and

$$(\theta_4 + 1)^2(k^2 + \theta_2^3) \geq (\theta_2 + 1)(k + \theta_2\theta_4)^2, \tag{8}$$

with equality if and only if $q_{22}^2 = 0$. We show that $q_{22}^2 \neq 0$. We use Lemma 3.2(i),(ii) to rewrite (8). In the case $a_1 = 0$ we do it in terms of k and θ_4 and transform (8) in the following inequality

$$\Big(k - \theta_4(\theta_4^2 + 3\theta_4 + 1)\Big)\Big(k(1 + 2/\theta_4) + 1\Big) \geq 0,$$

which follows from $\theta_4 \leq -2$. In the case $a_1 \neq 0$ we do it in terms of $p := \theta_2$, $q := -\theta_3$ and θ_4:

$$\Big(q(p+1)\theta_4 + q^2 + p(2q-1)\Big)\Big[q(p+1)\theta_4^3 + (q^2 + p(2q+1) + 2q)\theta_4^2 + (p^2(q-2) + (p+2q)q)\theta_4 - p(p+q^2)\Big] \;\geq\; 0. \tag{9}$$

Note $q \geq 2$ and $\theta_4 \leq \theta_3 - 1 = -q - 1 \leq -3$ by Lemma 3.2. For $q = 2$ we have by (ii) $\theta_4 = -3$ or $\theta_4 = -4$, in which case it is easy to verify $q_{22}^2 \neq 0$ directly. Hence we assume $q > 2$. The sum of the first two terms of the second factor on the left side of (9) is nonpositive when

$$q(p + 1)\theta_4 + (q^2 + p(2q + 1) + 2q) \;\leq\; 0. \tag{10}$$

To verify this inequality substitute θ_4 with its upper bound $-q - 1$ and get $pq(1 - q) + p + q \leq 0$, which is obviously true for $q > 2$. Therefore, the second factor of on the left side of (9) is always negative. The first factor on the left side of (9) is smaller than the left side of (10), thus negative as well and $q_{22}^2 \neq 0$.

We are going to prove $\theta_1^2 > -\theta_4$. By Lemma 3.2, $a_1 = \theta_1 + \theta_3$ and $\theta_1 > 0 > \theta_3$, so we conclude $\theta_1^2 \geq \theta_3^2$ with equality if and only if $a_1 = 0$. On the other hand we have, by (ii), $\theta_3^2 \geq -\theta_4$. Suppose both inequalities hold with equality, then $\theta_1 = -\theta_3$ and thus also $k = \theta_3^2 = -\theta_4$. But this contradicts the assumption that Γ is nonbipartite. Hence

$$\theta_1^2 > -\theta_4 \quad \text{and then obviously also} \quad \theta_1^2 > -\theta_3. \tag{11}$$

The cases q_{33}^3 in (iv) for $r \neq 2$ and q_{33}^4 in (iii) follow from (11). All other cases of (iii) and (iv) follow directly from Lemma 3.2. ∎

We conclude this section with a corollary which is a special case of the result in [19, Thm. 3].

Corollary 3.4 *An antipodal distance-regular graph with diameter four is Q-polynomial if and only if $r = 2$ and $q_{11}^4 = 0$. If $r = 2$ and $q_{11}^4 = 0$, then θ_0, θ_1, θ_2, θ_3, θ_4 is a unique Q-polynomial ordering, and the Q-polynomial structure is dual bipartite, i.e., $q_{ij}^h = 0$ when $i + j + h$ is odd.*

Proof. Based on the information from Theorem 3.3 it is not diffucult to draw the representation diagrams $\Delta(E)$ defined in [20], cf. [3, Thm. 2.11.6], corresponding to the primitive idempotent $E = E_1, E_2, E_3, E_4$. See Figure 3.1 for the representation diagram $\Delta(E_1)$. In the representation diagram $\Delta(E_2)$ the edge 13 is a connected component, the representation diagram $\Delta(E_4)$ has a cycle on vertices 1,2,3, and the representation diagram $\Delta(E_3)$ has a cycle on vertices 1,2,3,4. ∎
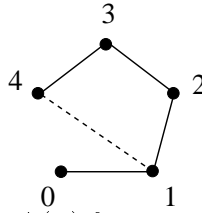

Figure 3.1: The representation diagram $\Delta(E)$ for $E = E_h$ is the undirected graph with vertices $0, 1, 2, 3, 4$, where two are joined whenever $i \neq j$ and $q_{ij}^h = q_{ji}^h \neq 0$. Only the representation diagram $\Delta(E_1)$ can be a path, and this is the case if and only if $r = 2$ and $q_{11}^4 = 0$.

4 Krein parameter q_{11}^4, 1-homogeneous property and the first sub-constituent

We characterize the vanishing of the Krein parameter q_{11}^4 for antipodal distance-reugular graphs of diameter four by the 1-homogoeneous property.

Lemma 4.1 *Let Γ be an antipodal distance-regular graph with diameter four and eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$. Then $b^+ = \theta_2$ and $b^- = \theta_3$.*

Proof. By Lemma 3.2(iii), we have $b_1 = (\theta_2 + 1)(-\theta_4 - 1)$, and one immediately calculates b^+ and by Lemma 3.2(ii) also b^-. ∎

Theorem 4.2 *Let Γ be an antipodal distance-regular graph with diameter four. Then the following are equivalent: (i) $q_{11}^4 = 0$. (ii) Equality holds in the inequality (3).*

Proof. Immediate consequence of Lemma 3.2 and Lemma 4.1. ∎

Suppose that for a nonbipartite antipodal distance-regular graph Γ with diameter four the conditions (i) and (ii) of Theorem 4.2 hold. Then we say Γ is a **tight** graph with parameters (p, q, r), where r is the size of its antipodal classes and $p := \theta_2$, $q := -\theta_3$.

Proposition 4.3 *Let Γ be a tight graph with parameters (p, q, r), and eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$. Then $\theta_0 = q(pq + p + q)$, $\theta_1 = pq + p + q$, $\theta_2 = p$, $\theta_3 = -q$, $\theta_4 = -q^2$, the parameters p and q*

are integral, $p \geq 1$, $q \geq 2$, and the eigenvalue multiplicities are $m_0 = 1$,

$$m_1 = (r-1)\frac{q(pq^2 + q^2 + pq - p)}{p+q}, \qquad m_2 = \frac{q(pq + p + q)(q^2 - 1)(2\,q + pq + p)}{(p+q)(q^2+p)},$$

$$m_3 = (r-1)\frac{(pq^2 + q^2 + pq - p)(pq + p + q)}{p+q}, \qquad m_4 = \frac{(p+1)(pq + p + q)(pq^2 + q^2 + pq - p)}{(p+q)(q^2+p)}.$$

Proof. In a tight graph $a_1 \neq 0$, see [10, Cor. 6.3], so all the eigenvalues of Γ are integral by Lemma 3.2. By Lemma 3.2, $-q = \theta_3 < -1$, i.e., $q \geq 2$, and since Γ is not bipartite also $p = \theta_2 \geq 1$. All the above formulas are obtained directly from Lemma 3.2. ∎

By (5) and Lemma 3.2 we derive an alternative characterization of tight graphs with parameters (p, q, r).

Theorem 4.4 *Let $\Gamma = (X, R)$ be an antipodal distance-regular graph with diameter four, and let p, q and r be some real numbers. Then the following are equivalent.*

(i)   *Γ is a tight graph with parameters (p, q, r).*

(ii)   *The intersection array of Γ equals*

$$\{q(pq + p + q), (q^2 - 1)(p+1), \frac{(r-1)q(p+q)}{r}, 1; 1, \frac{q(p+q)}{r}, (q^2 - 1)(p+1), q(pq + p + q)\}.$$

(iii)   *The antipodal quotient of Γ has the following parameters*

$$(k, \lambda, \mu) = \Big(q(pq + p + q), p(q+1), q(p+q)\Big). \quad \blacksquare$$

Remark 4.5 *For convenience we list here the remaining standard parameters of a tight graph with parameters (p, q, r): $a_1 = a_3 = p(q+1)$, $a_2 = pq^2$, $a_4 = 0$, the valencies of the distance graphs are*

$$k_1 = q(pq + p + q), \qquad k_2 = \frac{(pq + p + q)(q^2 - 1)(p+1)r}{p+q}, \qquad k_3 = (r-1)q(pq + p + q), \qquad k_4 = r - 1,$$

and they sum to $v = |X| = r(2\,q + pq + p)(pq^2 + q^2 + pq - p)/(p+q)$.

The above two results imply yet another characterization of tight graphs with parameters (p, q, r).

Corollary 4.6 *Let $\Gamma = (X, R)$ be an antipodal distance-regular graph with diameter four, and let p, q and r be some real numbers. Then the following are equivalent.*

(i)   *Γ is a tight graph with parameters (p, q, r).*

(ii)   *The local graph of some vertex $x \in X$ is strongly regular with parameters*
    $(k', \lambda', \mu') = \Big(p(q+1), 2p - q, p\Big).$

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Suppose (i)-(ii) hold, then p, q and r are integral, and the nontrivial eigenvalues of local graphs in Γ are $(\theta'_1, \theta'_2) = (p, -q)$, with their multiplicities

$$m'_1 \;=\; \frac{(q^2-1)(pq+p+q)}{p+q}, \quad \text{and} \quad m'_2 \;=\; k-1-m'_1 \;=\; \frac{pq(q+1)(p+1)}{p+q}. \quad \blacksquare$$

We are now able to interpret, by Theorem 4.2 and [10, Thm. 11.7], the vanishing of the Krein parameter q^4_{11}.

Corollary 4.7 *An antipodal distance-regular graph with diameter four is 1-homogeneous with $a_1 \neq 0$ if and only if $q^4_{11} = 0$.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\blacksquare$

The **cosine sequence** $\{\psi_j\}$ corresponding to θ_i is defined by $\psi_j = v_j(\theta_i)/k_j$, $i,j \in \{0,1,2,3,4\}$, see [3, p.128 and p.142]. The cosine sequences $\{\sigma_i\}$ and $\{\rho_i\}$ of a tight graph with parameters (p,q,r) corresponding to eigenvalues θ_1 and θ_4 respectively are, by (6),

$$\left(1, \frac{1}{q}, 0, \frac{-1}{(r-1)q}, \frac{-1}{r-1}\right) \qquad \left(1, \frac{-q}{(pq+p+q)}, \frac{p+q}{(p+1)(pq+p+q)}, \frac{-q}{(pq+p+q)}, 1\right). \qquad (12)$$

Let Γ be a tight graph with parameters (p,q,r). Bellow we calculate parameters corresponding to 1-homogeneous partition of Γ in terms of the parameters p, q and r, using formulas in [10, Thm. 11.2] and notation described in Figure 4.1, cf. [10, Lemma 2.14]:

$$\beta_1 = \gamma_3 = q(p+1)(q-1), \quad \beta_2/(r-1) = \gamma_2 = (p+q)/r, \quad \delta_1 = \mu' = p, \quad \delta_2 = pq(q-1), \quad \delta_3 = a_1 = p(q+1).$$

The number beside edges connecting cells D^j_i, see Figure 4.1, indicates how many neighbours a vertex from the closer cell has in the other cell.
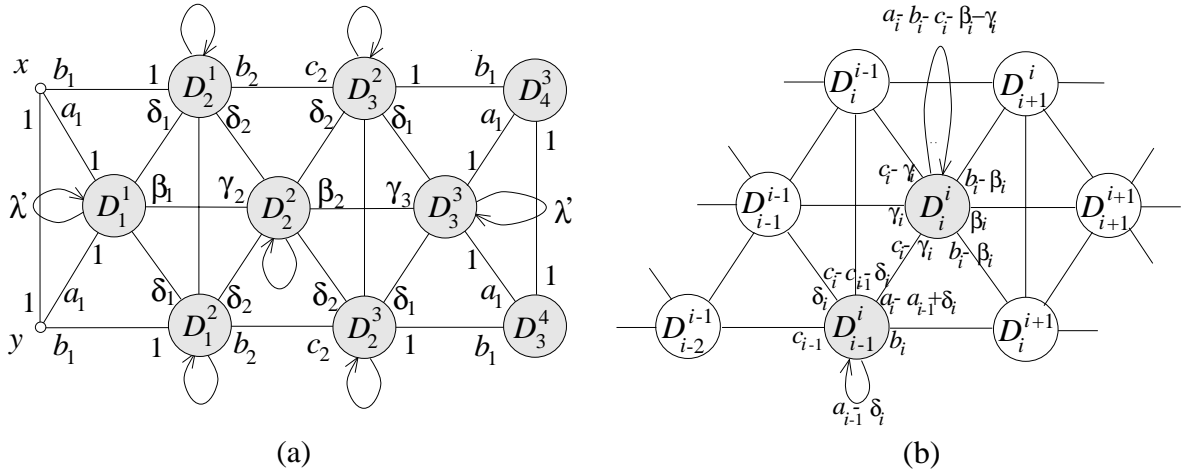

(a)   (b)

Figure 4.1: 1-homogeneous distance distribution diagram, corresponding to an edge $xy \in R$. $D^i_j = D^i_j(x,y) = \Gamma_i(x) \cap \Gamma_j(y)$. For convenience we mention here the intersection numbers needed for a 1-homogeneous partition: $p^1_{11} = a_1 = p(q+1)$, $p^1_{12} = b_1 = (q^2-1)(p+1)$, $p^1_{23} = (r-1)b_1 = (r-1)(q^2-1)(p+1)$, $p^1_{34} = r-1$, $p^1_{22} = rpq(q^2-1)(p+1)/(p+q)$.

Gardiner [7] proved that in an antipodal distance-regular graph of diameter d a vertex x, which is at distance $i \leq \lfloor d/2 \rfloor$ from one vertex in an antipodal class, is at distance $d-i$ from all other vertices in

this antipodal class. This implies the following identity:

$$\Gamma_{d-i}(x) = \cup\{\Gamma_d(y) \,|\, y \in \Gamma_i(x)\} \quad \text{for } i = 0, 1, \ldots, \lfloor d/2 \rfloor \,. \tag{13}$$

Let us fold the Figure 4.1(a), i.e., take the antipodal quotient. Then D_3^4 folds to $D_2^0 = \{x\}$, D_3^2 folds to $D_0^2 = \{y\}$, D_3^3 folds to D_1^1, D_2^3 folds to D_2^1, D_3^2 folds to D_1^2, by (13), and we obtain the following result.

Corollary 4.8 *The antipodal quotient of a tight graph with parameters (p, q, r) is 1-homogeneous, with parameters as in Figure 4.2.* ∎
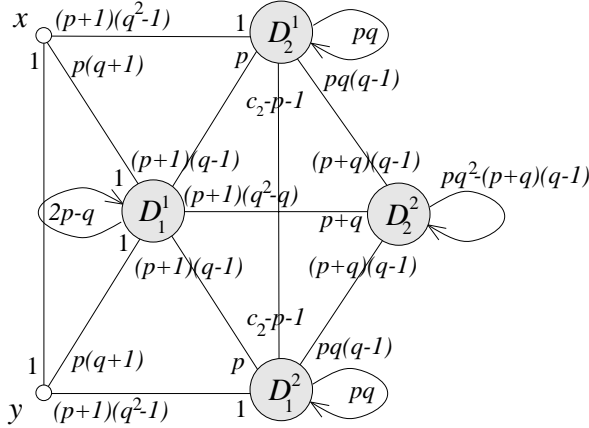

Figure 4.2: The distance distribution corresponding to a pair of adjacent vertices of the antipodal quotient of Γ.

Let xy be an edge of a tight graph with parameters (p, q, r), $D_j^i = D_j^i(x, y)$, $z \in D_{i-1}^i$ and $w \in D_i^i$. We express the number of vertices in D_1^1 at distance $i - 1$ (respectively i) from z, for $i = 1, 2, 3, 4$ and the number of vertices in D_1^1 at distance $i - 1$ (respectively $i + 1$) from w, for $i = 1, 2, 3$, which were calculated in [10, Thm. 4.1 and Thm. 11.1], in terms of the parameters (p, q, r) and list them in Table 4.1.

i	$|\Gamma_{i-1}(z) \cap D_1^1|$	$|\Gamma_i(z) \cap D_1^1|$	$|\Gamma_{i-1}(w) \cap D_1^1|$	$|\Gamma_{i+1}(w) \cap D_1^1|$
1	0	$p(q+1)$	1	$(p+1)(q-1)$
2	p	pq	$(p+q)/r$	$(r-1)(p+q)/r$
3	pq	p	$(p+1)(q-1)$	1
4	$p(q+1)$	0	n.a.	n.a.

Table 4.1

5   Feasibility Conditions

Lemma 5.1 *Let Γ be a distance-regular graph, whose local graphs are strongly regular with parameters (k', μ', λ'). Then the μ-graphs of Γ are μ'-regular.*

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Proof. Let u and v be vertices of Γ at distance two and let w be any of their common neighbours. Then the number of neighbours of w in the $\mu(u, v)$-graph is equal to the number of the common neighbours of u and v in the local graph of w. ∎

A graph with diameter at least two is called **Terwilliger graph** when every μ-graph has the same number of vertices and is complete. We now give new feasibility conditions for the parameters of tight graphs with parameters (p, q, r) and group them with all previously known conditions in the following result.

Theorem 5.2 *Let $\Gamma = (X, R)$ be a tight graph with parameters (p, q, r). Then p, q, r are integers, such that $p \geq 1$, $q \geq 2$, $r \geq 2$ and*

> *(i)* $pq(p + q)/r$ *is even,*
> *(ii)* $r(p + 1) \leq q(p + q)$, *with equality if and only if Γ is a Terwilliger graph,*
> *(iii)* $r \mid p + q$,
> *(iv)* $p \geq q - 2$, *with equality if and only if $q_{44}^4 = 0$,*
> *(v)* $p + q \mid q^2(q^2 - 1)$,
> *(vi)* $p + q^2 \mid q^2(q^2 - 1)(q^2 + q - 1)(q + 2)$.

Proof. (i): The μ-graphs in Γ have valency μ' by Lemma 5.1, therefore $\mu\mu' = pq(p + q)/r$ must be even (since it is twice the number of edges in a μ-graph).

(ii): $p = \mu' < \mu = q(p + q)/r$, as the valency μ' must be smaller than the number of vertices μ in the μ-graph. In the case $\mu = \mu' + 1$ the graph Γ is a Terwilliger graph by definition.

(iii): Since Γ is a tight graph, it is 1-homogeneous. Consider the distance partition corresponding to an edge $xy \in R$ and let $z \in D_2^2$. Then the number of neighbours of z in $D_1^1(x, y)$ equals $\gamma_2 = (p + q)/r$ and must therefore be an integer.

(iv): We express all the parameters in Theorem 3.3(i) in terms of p and q and obtain the desired inequality.

(v) and (vi) follow from the integrality of the multiplicities of a local graph and the antipodal quotient. The integrality of the nontrivial eigenvalue multiplicities of the local graph implies $p + q \mid q^2(q^2 - 1)$. We can express m_2 in the following way:

$$m_2 = -q(q^2 - 1)(q + 1)^2 + \frac{(q^2 - 1)q^3}{p + q} - \frac{q^2(q^2 - 1)(q^2 + q - 1)(q + 2)}{p + q^2}$$

Therefore $p + q^2 \mid q^2(q^2 - 1)(q^2 + q - 1)(q + 2)$. ∎

Remark 5.3 *The remaining eigenvalue multiplicities of Γ give no new divisibility conditions.*

Many infinite families of feasible intersection arrays are ruled out. In particular the following parameters from the tables in Brouwer et al. [3, pp. 421-425] are ruled out, see Table 5.1. Dickie and

Terwilliger [6] studied P and Q polynomial antipodal graphs. They showed that all such graphs with diameter at least five are already known. In the diameter four case the condition (i) of the above result rules out about one quarter of the feasible parameters of such graphs.

#	k	p	q	r	(i)	(ii)	(iii)
1	45	3	3	2	*		
2	45	3	3	6	*	*	
3	56	2	4	4			*
4	56	2	4	8			*
5	81	6	3	9		*	
6	96	4	4	8		*	
7	115	3	5	5			*
8	115	3	5	8	*		
9	115	3	5	10			*
10	117	9	3	4	*	*	
11	117	9	3	6		*	
12	117	9	3	9		*	*
13	175	5	5	2	*		
14	176	8	4	6		*	
15	189	15	3	6	*	*	
16	189	15	3	2	*		
17	204	4	6	3			*
18	204	4	6	4			*
19	261	21	3	4		*	
20	414	9	6	2	*		*

(a)

#	k	p	q	r
1	81	6	3	3
2	96	4	4	4
3	96	4	4	2
4	115	3	5	2
5	115	3	5	4
6	117	9	3	2
7	175	5	5	5
8	176	8	4	3
9	189	15	3	3
10	204	4	6	2
11	204	4	6	5
12	261	21	3	2
13	288	6	6	2
14	288	6	6	3
15	329	5	7	2
16	336	16	4	2
17	416	20	4	2

(b)

#	graph	k	p	q	r	μ
1	! 3.Sym(7).2	10	1	2	3	2
2	! $J(8,4)$	16	2	2	2	4
3	! halved 8-cube	28	4	2	2	6
5	$3.O_6^-(3)$	45	3	3	3	6
4	! Soicher1 [16]	56	2	4	3	8
6	$3.O_7(3)$	117	9	3	3	12
7	Meixner1 [13]	176	8	4	2	24
8	Meixner2 [13]	176	8	4	4	12
9	Soicher2 [16]	416	20	4	3	32
10	$3.Fi_{24}^-$	31671	351	9	3	1080

(c)

Tables 5.1: (a) Ruled out cases (the cases (3) and (4) have already been excluded by Brouwer [2], [4, 11.4I]). (b) Feasible parameters. (c) Known examples, where "!" indicates the uniqueness of the corresponding graph.

Theorem 5.4 *Let Γ be a tight graph with diameter d and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. If the second largest eigenvalue $\theta_1 = b_1 - 1$, i.e., $b^- = -2$, then Γ is either the Johnson graph, the Halved cube, the Gosset graph or the Conway-Smith graph. If $b^+ = 1$ then Γ is the Conway-Smith graph.*

Proof. Each of the local graphs is connected, so we have $c_2 \geq 2$ and $a_1 \geq 2$. Now for the b^+ part see Koolen [12] and for the b^- part see Terwilliger [18], cf. [3, Theorem 4.4.11.]. ∎

The Conway-Smith graph, the Johnson graph and the Halved cube are the first three examples in the Table 5.1(c). The tight graphs with parameters $(p, 2, r)$ or $(1, q, r)$ are therefore known by the above result, and we can assume $p \geq 2$ and $q \geq 3$ when looking for new examples. Let us now consider the case of equality in Theorem 5.2(ii).

Proposition 5.5 *If a tight graph with parameters (p,q,r) is a Terwilliger graph, then $(p, q, r) = (1, 2, 3)$.*

Proof. Let Γ be a tight graph with parameters (p, q, r). Then the μ-graphs have valency p by Lemma 5.1 and $c_2 = q(p + q)/r$, where $(p + q)/r = \gamma_2$, is an integer. Let Γ also be a Terwilliger graph, i.e.,

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$c_2 = p + 1$, i.e., $p + 1 = q(p + q)/r$, i.e., $r = q(p + q)/(p + 1)$. Hence q divides $p + 1$ and $p + 1$ divides $q(q - 1)$.

The local graphs of Γ are strongly regular graphs with parameters $(k', \lambda', \mu') = (p(q+1), 2p - q, p)$ and are also Terwilliger graphs. Therefore, by [3, Thm. 1.16.3],

$$\lambda'(\lambda' - 1) \geq (\mu' - 1)(k' - \lambda' - 1), \quad \text{i.e.,} \quad p^2(5 - q) - 2p(2q + 1) + q^2 + 2q - 1 \geq 0.$$

Since $\lambda' \geq 0$ implies $2p \geq q$, it follows $5 - q > 0$. By Theorem 4.3, $q \geq 2$, so $q \in \{2, 3, 4\}$. The conditions $q | p + 1$, $p + 1 | q(q - 1)$ and, by Theorem 5.2(v)(vi), the conditions $p + q | q^2(q^2 - 1)$ and $p + q^2 | q^2(q^2 - 1)(q^2 + q - 1)(q + 2)$ are satisfied only by $(p, q, r) = (1, 2, 3)$. ∎

Theorem 5.2(iii) implies $r \leq p + q$. Let us consider the case of equality.

Proposition 5.6 *If $p + q = r$ for a tight graph with parameters (p, q, r), then $(p, q, r) = (1, 2, 3)$.*

Proof. Let $\Gamma = (X, R)$ be a tight graph with parameters (p, q, r) such that $r = p + q$. Then $c_2 = q$ and since the μ-graphs have valency p by Lemma 5.1, we conclude $q \geq p + 1$. On the other hand, by Theorem 5.2(iv), $q \leq p + 2$. Thus $q \in \{p + 1, p + 2\}$. If $q = p + 1$ then Γ is a Terwilliger graph and we get, by Proposition 5.5, $(p, q, r) = (1, 2, 3)$.

Suppose now $q = p + 2$. Then the complement of a μ-graph consists of $(p + 2)/2$ copies of K_2, i.e., the μ-graph is $K_{(p+2)/2 \times 2}$ and λ' (λ of the local graph) is $2p - q = p - 2$, therefore $p \geq 2$. Brouwer [2] proved that $(p, q) = (2, 4)$ implies $r = 3$, thus $p \geq 3$ and $r \geq 8$.

Let x and y be vertices of Γ at distance two. Let u and v be nonadjacent vertices in a local graph Δ of y. The μ-graph corresponding to u and v contains y, so it has only y and its antipodal vertex outside Δ. Hence, the set $D_1^1(u, v) \cap \Gamma(y)$ induces $K_{p/2 \times 2}$ and the set $\{u, v\} \cup (D_1^1(u, v) \cap \Gamma(y))$ induces $K_{(p+2)/2 \times 2}$ in Δ. The parameters λ and μ of the graph $K_{(p+2)/2 \times 2}$ are the same as the parameters of the local graph Δ and $p \geq 3$, so each edge of Δ lies in at most one $K_{(p+2)/2 \times 2}$ in Δ.

Let z be a vertex of $D_1^3(x, y)$. For each vertex $t \in D_1^3(x, y)$, the set $\{z, t\} \cup (D_1^1(z, t) \cap \Gamma(y))$ induces $K_{(p+2)/2 \times 2}$ inside $\Gamma(y)$. Since the graphs $K_{(p+2)/2 \times 2}$ inside $\Gamma(y)$ have no common edges and since z has $a_1 - \mu' = p(p + 2)$ neighbours in $D_1^2(x, y)$, there are at most $p + 2$ copies of $K_{(p+2)/2 \times 2}$ containing z. But we have $|D_1^3(x, y)| = p_{13}^2 = b_2 = (r - 1)(p + 2)$ choices for t, so $r = 2$. Contradiction! ∎

Remark 5.7 *This result rules out an infinite family of feasible parameters of tight graphs: $(p, q, r) = (2s - 2, 2s, 4s - 2)$, $s = 2, 3, 4, \ldots$*

There are some very interesting families of parameters (p, q, r) in the tables (b) and (c), for example $(q - 2, q, q - 1)$, $(q - 2, q, 2)$, (qs, q, q), $(p, p, 2)$. We study the first two families in the following section.

6 Krein parameter q_{44}^4, 2-homogeneous property and the second subconstituent

Let Γ be a tight graph with parameters (p, q, r). We first determine how close is Γ to having 2-homogeneous property. Then we find that the vanishing of the Krein parameter q_{44}^4 implies the second subconstituents of Γ are antipodal distance-reugular graphs of diameter four, and that the vanishing of the Krein parameter q_{44}^4 and $r = 2$ imply the 2-homogoeneous property.

Proposition 6.1 *Let $\Gamma = (X, R)$ be a tight graph with parameters (p, q, r) and let $x, y \in X$ be vertices at distance two. The subgraph induced by the vertices which are not at distance two from both x and y has an equitable partition $\pi = \{D_j^i(x, y) \mid i = 0, 1, 2; \ j = 0, 1, 2; \ (i, j) \neq (2, 2)\}$, with parameters as in Figure 6.1.*
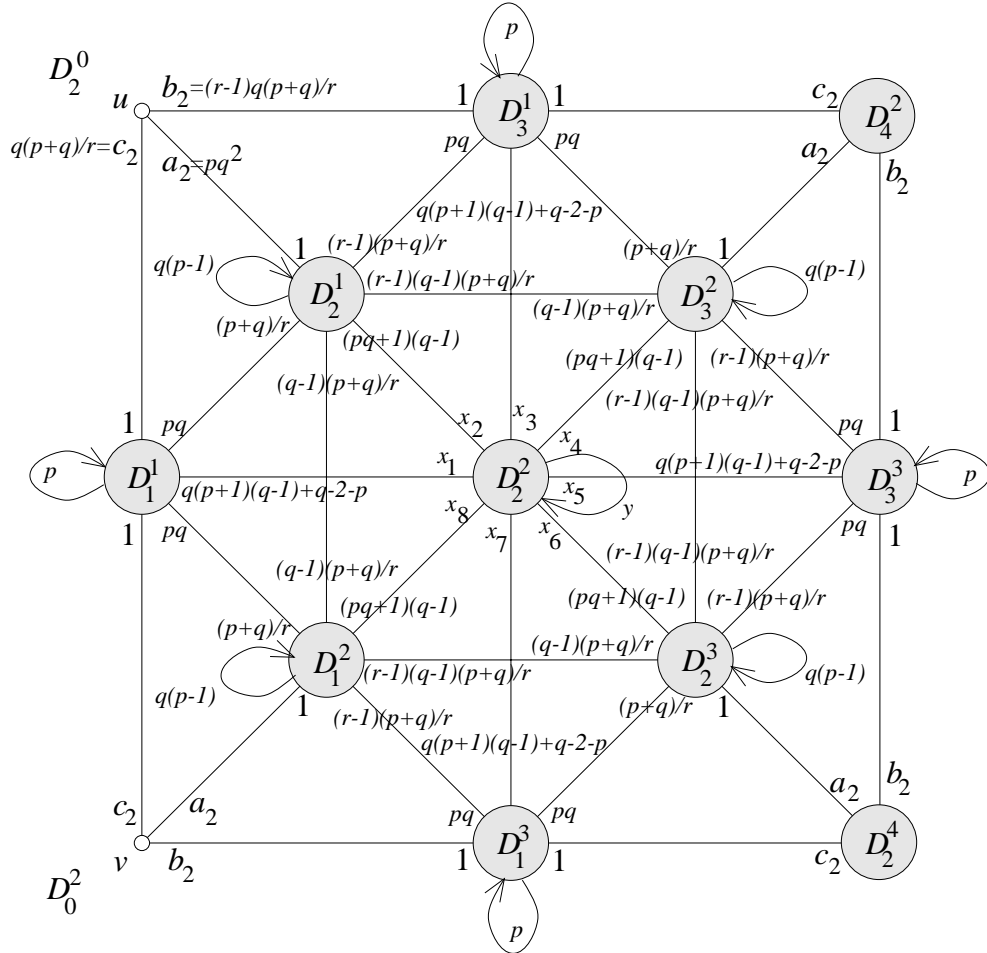

Figure 6.1: The distance distribution corresponding to a pair of vertices at distance two. For convenience we list here the intersection numbers needed for a 2-homogeneous partition: $p_{11}^2 = c_2 = q(p + q)/r$, $p_{12}^2 = a_2 = pq^2$, $p_{13}^2 = b_2 = (r - 1)q(p + q)/r$, $p_{22}^2 = k_2 - r(a_1 + 1)$, $p_{23}^2 = (r - 1)(k - rc_2) = pq^2(r - 1)$, $p_{24}^2 = r - 1$, $p_{33}^2 = b_2(r - 1)$.

Proof. Consider the distance partition of Γ corresponding to a pair (u, v) of vertices at distance two, and let $D_j^i := D_j^i(u, v) = \Gamma_i(u) \cap \Gamma_j(v)$, see Figure 6.1. Then the distance-regularity and antipodality of

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Γ imply that the parameters of the vertices in the sets D_2^0, D_0^2, D_2^4, D_4^2, corresponding to the partition π are as presented in Figure 6.1. Furthermore, each vertex in D_1^1 has, by Lemma 5.1, $\mu' = p$ neighbours in D_1^1, $a_1 - \mu' = pq$ neighbours in D_1^2 (as well as in D_2^1) and $k - \mu' - 2(a_1 - \mu') - 2 = q(p+1)(q-1) + q - 2 - p$ neighbours in D_2^2.

Take a vertex $w \in D_2^1$, and consider the neighbours of w in D_1^1. Then these are exactly the common neighbours of u and w which are at distance $\partial(w, v) - 1 = \partial(u, v) - 1 = 1$ from v. By [10, Thm. 11.1], there is exactly $\gamma_2 = (p+q)/r$ of them. Then the number of neighbours of w in D_1^2 is $c_2 - (p+q)/r = (q-1)(p+q)/r$.

Based on (13), we draw Figure 6.2 and realize that $D_2^1(u, v) = D_2^1(u, t)$ for each $t \in D_4^2(u, v)$. Furthermore, we express the sets $D_3^2(u, v)$ and $D_3^1(u, v)$ as unions of disjoint sets for which we already know the cardinality of the intersection with $\Gamma(w)$:

$$D_3^2(u, v) = \cup\{D_2^1(t, u) \,|\, t \in D_4^2(u, v)\} \quad \text{and} \quad D_3^1(u, v) = \cup\{D_1^1(t, u) \,|\, t \in D_4^2(u, v)\}. \tag{14}$$
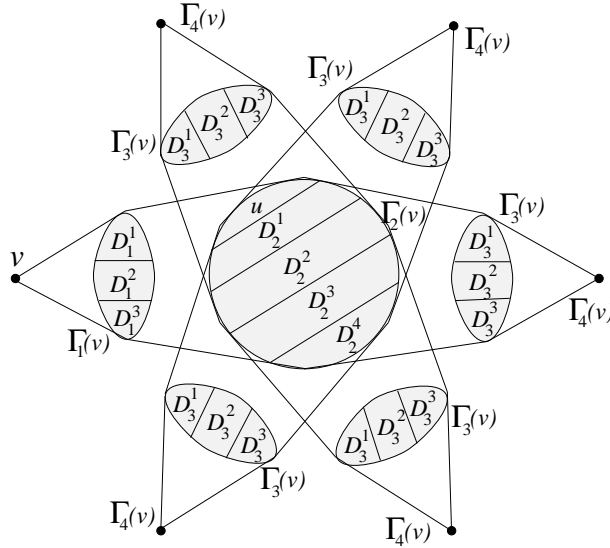

Figure 6.2: Three partitions of an antipodal distance-regular graph with diameter $d = 4$. First there is the distance partition corresponding to a vertex v, with its cells *denoted/labeled* by $\{v\} = \Gamma_0(v), \ldots, \Gamma_4(v)$; the second partiton corresponds to the distance distribution of the antipodal class containing v, i.e., $\{v\} \cup \Gamma_4(v)$, with its cells being actual bubbles; the third partition is the distance partition corresponding to v and $u \in \Gamma_2(v)$ from Figure 6.1, its cells are *denoted/labeled* by D_i^j, and we have $\Gamma_4(v) = D_4^2$ and $\{u\} = D_2^0$. (We have chosen the covering index r to be six, as this is the least common multiple of covering indices of known examples.)

Hence, the number of neighbours of w in the set D_3^2 is $(r-1)(q-1)(p+q)/r$, and in D_3^1 is $(r-1)(p+q)/r$. Finally, the number of neighbours of w in D_2^1 is $a_1 - (p+q) = q(p-1)$, and in D_2^2 is $k - a_1 - (q-1)(p+q) - 1 = (pq+1)(q-1)$.

Let z be a vertex in D_3^1. By antipodality, there are no edges between the sets in the second union of (14). Therefore, z has p neighbours in D_3^1, $a_1 - p = pq$ neighbours in D_2^1 and, by $a_1 = a_3$, also $a_3 - p = pq$ neighbours in D_3^2. Finally, z has $q(p+1)(q-1) + q - 2 - p$ neighbours in D_2^2 and $b_3 = 1$

neighbours in D_4^2. By the symmetry between u and v, the vertices in D_1^2 and D_1^3 could be handled similarly as the vertices in D_2^1 and D_3^1.

Since Γ is antipodal with diameter four, the vertices in the sets D_4^2 and D_2^4 fold respectively to v and u, and therefore $D_3^3 \cup D_1^3$ folds onto $D_1^1 \cup D_3^1$, and D_3^2, D_2^3 fold respectively onto D_1^2 and D_2^1. Furtermore, the neighbourhoods of antipodal vertices are disjoint, in particular the neighbourhoods of vertices in D_2^4 or neighbourhoods of vertices in D_4^2, and

$$D_3^3(u,v) \text{ is a disjoint union of } (r-1)^2 \text{ sets } D_1^1(x,y), \ x \in D_4^2, \ y \in D_2^4. \tag{15}$$

Hence, each vertex in the set D_3^3 has one neighbour in D_2^4 (as well as in D_4^2), p neighbours in D_3^3, pq neighbours in D_3^2 (as well as in D_2^3) and $q(p+1)(q-1) + q - 2 - p$ neighbours in D_2^2. Finally, a vertex in D_3^2 has one neighbour in D_4^2, $q(p-1)$ neighbours in D_3^2, $\gamma_2 = (p+q)/r$ neighbours in D_3^1, $(r-1)\gamma_2 = (r-1)(p+q)/r$ neighbours in D_3^3, $c_2 - (p+q)/r = (q-1)(p+q)/r$ neighbours in D_2^1, $b_2 - (r-1)(p+q)/r = (r-1)(q-1)(p+q)/r$ neighbours in D_2^3 and $(pq+1)(q-1)$ neighbours in D_2^2. By the symmetry between u and v, the vertices in D_2^3 could be handled similarly as the vertices in D_3^2. ∎

It remains to consider vertices in the set D_2^2. We will need the following result, cf. Terwilliger [19, Equation (12)].

Theorem 6.2 *Let Γ be a distance-regular graph with the cosine sequence $\{\nu_i\}$ corresponding to a primitive idempotent E and let μ' be the average valency of the μ-graph corresponding to a pair of vertices (u,v) with $\partial(u,v) = 2$, i.e., the graph induced by $D_1^1(u,v)$. Then*

$$1 - \nu_2^2 + c_2(\nu_2 + \nu_2^2 - 2\nu_1^2) + \mu'(\nu_1 - \nu_2)(1 + \nu_2) \geq 0, \tag{16}$$

with equality if and only if

$$(\nu_2 + 1) \sum_{w \in D_1^1(u,v)} E\hat{w} = c_2 \nu_1 (E\hat{u} + E\hat{v}). \tag{17}$$

Proof. For a pair of vertices u and v at distance two the following Cauchy-Schwartz inequality

$$||w_{11}||^2 ||E\hat{u} + E\hat{v}||^2 - w_{11}(E\hat{u} + E\hat{v}) \geq 0, \quad \text{where} \quad w_{11}(u,v) = \sum_{w \in D_1^1(u,v)} E\hat{w},$$

simplifies, by [10, Lemma 2.], after division by $2c_2$ to (16). The inequality (16) holds with equality, when the vectors $w_{11}(u,v)$ and $E\hat{u} + E\hat{v}$ are collinear: $w_{11}(u,v) = c(E\hat{u} + E\hat{v})$. By taking the inner product of the left and the right side of the last equation with $E\hat{u}$, we obtain $c_2\nu_1 = (\nu_2 + 1)c$, as desired. ∎

Remark 6.3 *If we consider a pair (u, v) of adjacent vertices (instead of a pair of vertices at distance two), the above Cauchy-Schwartz inequality can be expressed as*

$$a_1(1 - \nu_1)(1 + \nu_1 + a_1\nu_1) - \mu'(\nu_1 - \nu_2)(1 + \nu_1)b_1 \geq 0, \tag{18}$$

with equality if and only if the vectors $w_{11}(u, v)$ and $E\hat{u} + E\hat{v}$ are collinear, cf. Terwilliger [19, Equation (13)]. The above inequality gives us the least upper bound (respectively the greatest lower bound) on μ' when E corresponds to θ_1 (respectively θ_d). The inequality, which consists of this upper bound being bigger than or equal to this lower bound, can be easily transformed to the inequality (3) in Section 2, and was called in [10] the Fundamental Bound. In case of equality we get two linear dependencies, which have already been used in [10].

Remark 6.4 *In that case when E corresponds to θ_1 or θ_3 of an antipodal distance-regular graph with diameter four, we have $\nu_2 = 0$, and the inequality (16) simplifies considerably:*

$$1 - 2c_2\nu_1^2 + \mu'\nu_1 \geq 0.$$

Corollary 6.5 *Let Γ be a tight graph with parameters (p, q, r). Then the inequality (16) holds with equality*

(i)   for the cosine sequence $\{\nu_i\}$ corresponding to θ_1 if and only if $r = 2$,

(ii) for the cosine sequence $\{\nu_i\}$ corresponding to θ_4 if and only if $q_{44}^4 = 0$. ∎

Theorem 6.6 *Let $\Gamma = (X, R)$ be a tight graph with parameters (p, q, r). Then the following holds.*

(i) $q_{44}^4 = 0$ implies that second subconstituent graphs, i.e., the graphs induced by $\Gamma_2(x)$, $x \in X$, are antipodal distance-regular graphs with diameter four, intersection array

$$\left\{ (q-2)q^2, (q-1)^3, 2(r-1)(q-1)(q-2)/r, 1; 1, 2(q-1)(q-2)/r, (q-1)^3, (q-2)q^2 \right\},$$

eigenvalues $(q-2)q^2$, $(q-2)q$, $q-2$, $-q$, $-q(q-2)-2$, and their respective multiplicities 1, $(r-1)(q^2-2)(q+1)/2$, $(q-2)q(q+1)/2$, $(r-1)(q^2-2)(q-2)(q+1)/2$, $(q-1)q(q^2-2)/2$.

(ii) $q_{44}^4 = 0$ and $r = 2$ imply Γ is 2-homogeneous. with parameters as in Figure 6.1 and Figure 6.3(b)

Proof.   By Proposition 6.1, we need to consider only the vertices in the set D_2^2 of a distance partition corresponding to a pair of vertices (u, v) in Γ at distance two, see Figure 6.1 and Figure 6.3. Let $w \in D_2^2(u, v)$, and let x_i be the number of vertices in $D_1^1(u, w)$ which are at distance i from v. Then x_i is the number of neighbours of w in $D_i^1(u, v)$, see Figure 6.1, and

$$x_1 + x_2 + x_3 = c_2. \tag{19}$$

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We derive a relation on x_i's by taking the scalar product of $E\hat{w}$ with the left and the right side of the equation (17):

$$(\nu_2 + 1)(\nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3) = 2c_2\nu_1\nu_2 \tag{20}$$

(i): Assume $q_{44}^4 = 0$, i.e., $p = q-2$, and let E correspond to θ_4. Then $c_2\nu_1/(\nu_2+1) = -2(q-1)/r$ and as $\nu_1 = \nu_3$ we solve the system (19), (20) for $x_1 + x_3$ and x_2 using (12): $x_1 + x_3 = 4(q-1)/r$ and $x_2 = 2(q-1)(q-2)/r$.

Now we repeat these calculations for the triple (w, v, t), where $t \in D_2^4$, instead of the triple (w, u, v), and obtain by antipodality of Γ that the number of neighbours of w in D_2^3 is $x_6 = |D_2^4|x_2 = 2(r-1)(q-1)(q-2)/r$. The number of neighbours of w in D_2^2 is $y = a_2 - x_2 - x_6 = (q-2)(q^2 - 2q + 2)$.

Finally, the distance partition corresponding to u of the second subconstituent graph of the vertex v is equitable with the desired parameters, which do not depend on u. It follows that every second subconstituent graph of Γ is an antipodal distance-regular graph with diameter four and the desired intersection array. Hence, $x_8 = x_2$ and $x_4 = x_6$.
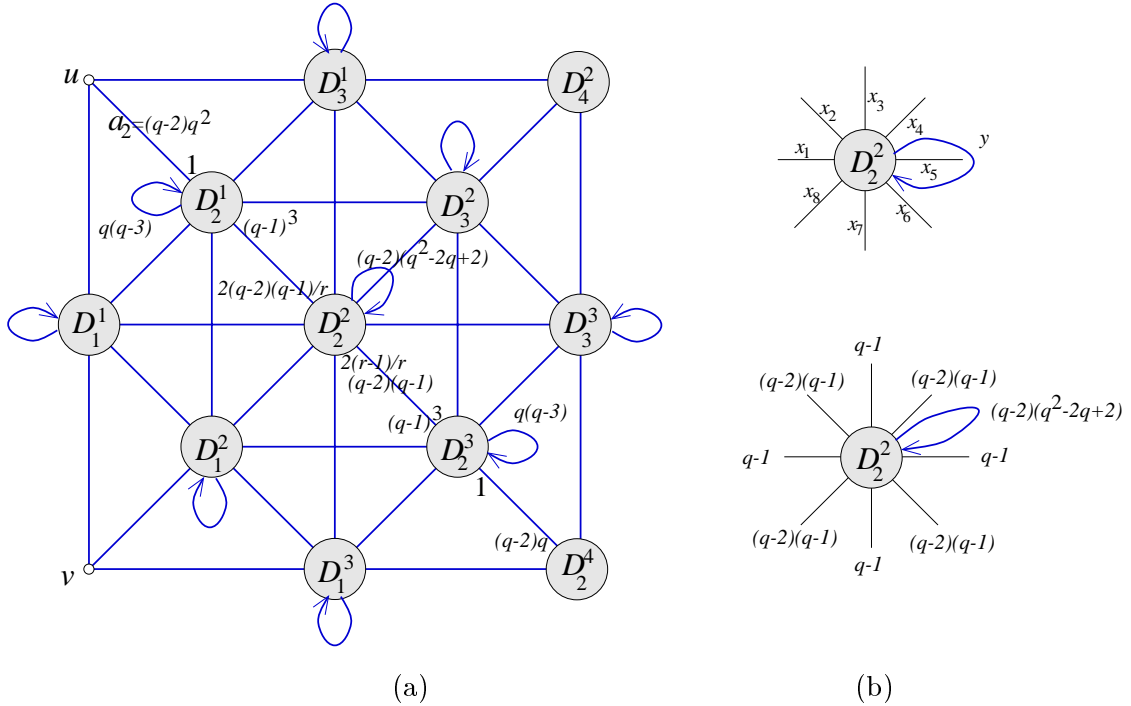

(a) (b)

Figure 6.3: (a) The distance distribution of a second subconstituent in the case when $p = q - 2$. (b) The distance distribution corresponding to a pair of vertices at distance two is equitable for $p = q - 2$ and $r = 2$.

(ii): Now we assume $r = 2$ (which implies $|D_2^4| = |D_4^2| = 1$) and let E correspond to θ_1. Then $c_2\nu_1/(\nu_2 + 1) = (p + q)/2$, $\nu_1 = -\nu_3$ and $\nu_2 = 0$ by (12), therefore, by (20), $x_1 = x_3$. Finally, we assume also $q_{44}^4 = 0$, i.e., $p = q - 2$. Then $x_1 = x_3 = q - 1$, $x_2 = x_6 = (q - 2)(q - 1)$ and $y = (q - 2)(q^2 - 2q + 2)$ by (i), $x_7 = x_3$, $x_8 = x_2$ and $x_4 = x_6$ by symmetry between u and v, and $x_5 = b_2 - x_3 - x_4 = q - 1$. Hence, the distance partition corresponding to u and v, $\partial(u, v) = 2$ is equitable and the graph Γ is 2-homogeneous with the desired parameters. ∎

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Remark 6.7 *For all the feasible intersection arrays of tight graphs with parameters $(q-2, q, r)$ from the tables in [3] the second subconstituent graph has also feasible intersection array. There are five such feasible intersection arrays which are the candidates for a second subconstituent graph, namely for $(3, 5, 2)$, $(3, 5, 4)$, $(4, 6, 2)$, $(4, 6, 5)$ and $(5, 7, 2)$.*

Remark 6.8 *Soicher1 graph satisfies Theorem 6.6(i) (with $q = 4$ and $r = 3$). Soicher has already noticed that in the case of his graph the second subconstituent graph is an antipodal distance-regular graph with diameter four and intersection array $\{32, 27, 8, 1; 1, 4, 27, 32\}$. He verified its distance-regularity with the aid of computer, see [16]. The antipodal quotient of this graph is the strongly regular graph which is the second subconstituent graph of the second subconstituent graph of the McLaughlin graph, see [4, Section 11.4.I]. All local graphs are the incidence graphs of the affine plane $AG(2, 4)$ with a parallel class deleted (their intersection arrays are $\{4, 3, 3, 1; 1, 1, 3, 4\}$ and they are the antipodal 4-covers of $K_{4,4}$).*

Theorem 6.9 *Let $\Gamma = (X, R)$ be a tight graph with parameters (p, q, r). If $q_{44}^4 = 0$ then the antipodal quotient of Γ is 2-homogeneous, with parameters as in Figure 6.4(b).*
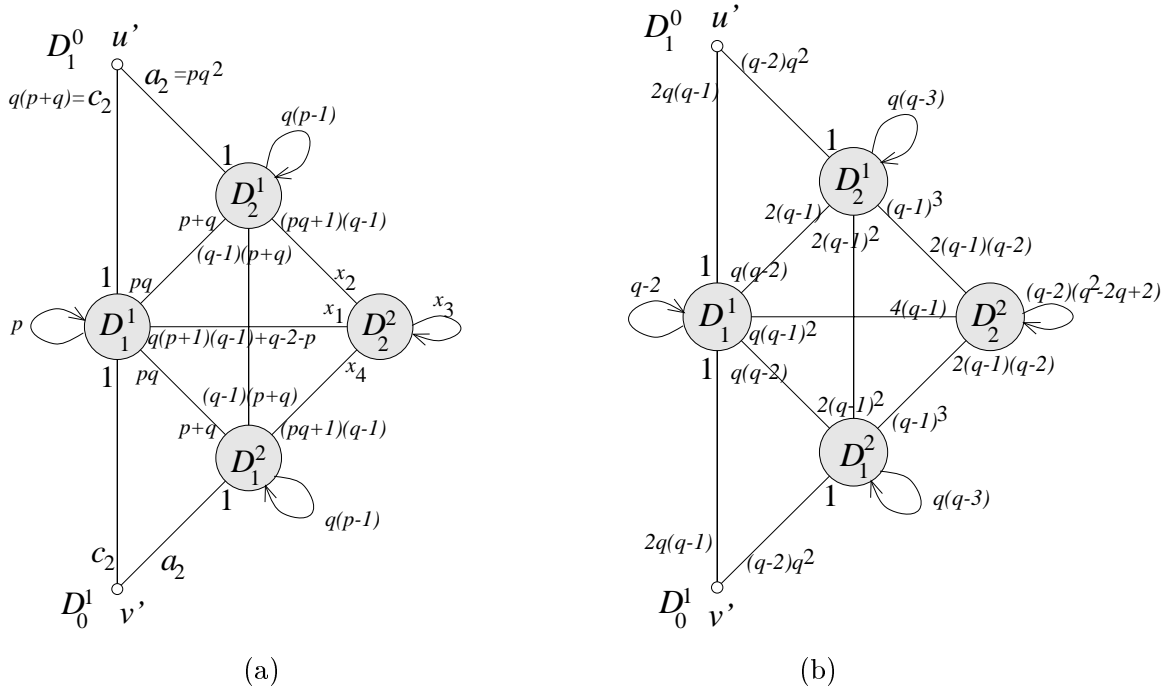

(a)　　　　　　　　　　　(b)

Figure 6.4: (a) The distance distribution corresponding to a pair (u', v') of vertices at distance two corresponding to the antipodal quotient of a tight graph with parameters (p, q, r).
(b) 2-homogeneous partition for $p = q - 2$. For convenience we list here the intersection numbers needed for a 2-homogeneous partition: $p_{11}^2 = c_2 = q(p+q)$, $p_{12}^2 = a_2 = pq^2$ and $p_{22}^2 = k_2 - r(a_1 + 1)$.

Proof. Let us fold the Figure 6.1, i.e., take the antipodal quotient of Γ. Then D_2^4 and $\{u\}$ go to $D_2^0(u', v')$, D_4^2 and $\{v\}$ go to $D_0^2(u', v')$, D_2^1 and D_2^3 go to $D_2^1(u', v')$, D_1^2 and D_3^2 go to $D_1^2(u', v')$, $D_1^1 \cup D_1^3$, D_3^1 and D_3^3 go to $D_1^1(u', v')$, by (13), and we obtain Figure 6.4(a). Then $x_1 + x_2 = c_2 = x_1 + x_4$,

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hence $x_2 = x_4$. If we add the condition $q_{44}^4 = 0$, then, by the result of Cameron, Goethals and Seidel [5], the second subcostituent is also strongly regular. Therefore, we get $x_2 = |D_2^1|(pq+1)(q-1)/|D_2^2| = 2(q-1)(q-2)$ and $x_1 = c_2 - x_2 = 4(q-1)$. We can make the same conclusion using Theorem 6.6(i) instead of [5]. Finally, $x_3 = k - x_1 - x_2 - x_4 = (q-2)(q^2 - 2q + 2)$, see Figure 6.4(b). ∎

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