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DISCS IN STEIN MANIFOLDS

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1. The result

Let Δ be the open unit disc in \mathbb{C} . In the paper we prove the following

THEOREM *Let M be a Stein manifold, $\dim M \geq 2$. Given a point $p \in M$ and a vector X tangent to M at p there is a proper holomorphic map $f: \Delta \rightarrow M$ such that $f(0) = p$ and such that $f'(0) = \lambda X$ for some $\lambda > 0$.*

This has been known in the special case when M is a bounded domain in \mathbb{C}^N with boundary of class \mathcal{C}^2 . It was proved in [FG] by using the fact that for such domains there are bounded strictly plurisubharmonic exhaustion functions without critical points near the boundary.

2. Outline of the proof

We describe the idea of the proof in the special case when M is a pseudoconvex domain in \mathbb{C}^N .

Let $D \subset \mathbb{C}^N$ be a pseudoconvex domain and let ρ be a strictly plurisubharmonic exhaustion function for D . The idea is to start with an analytic disc $f: \bar{\Delta} \rightarrow D$ such that $f(0) = p$, $f'(0) = \lambda X$ for some $\lambda > 0$, and then to push $f(\zeta)$, $\zeta \in b\Delta$ in directions that are approximately tangent to the level sets of ρ while keeping $f(0)$ and $f'(0)$ fixed, keeping $f(\zeta)$ essentially unchanged for all ζ belonging to a fixed compact subset of Δ , and lowering $\rho(f(\zeta))$, $\zeta \in \Delta$, only a little.

We perform this inductively. If we want to get a proper map in the limit we must, at each step, increase $\rho(f(\zeta))$, $\zeta \in b\Delta$, for a certain amount and the sum of these amounts must equal $+\infty$. However, this amount depends on the lower bound of $|\text{grad}\rho|$. In particular, if $a < b$ and if ρ has no critical value on $[a, b]$ then there is a $\delta > 0$ such that whenever $a \leq \rho(f(\zeta)) \leq b$ ($\zeta \in b\Delta$), then, in the pushing process one can increase $\rho(f(\zeta))$ for at least δ for each $\zeta \in b\Delta$. Thus, if ρ has no critical values on (b, ∞) one can go on and use the pushing process inductively to obtain a proper holomorphic map $f: \Delta \rightarrow D$ such that $f(0) = p$. In this way one can prove the theorem in the special case when M is a bounded pseudoconvex domain in \mathbb{C}^N with boundary of class \mathcal{C}^2 [FG]. In the convex setting the idea of pushing boundaries of analytic discs in directions tangent to the level sets of ρ was used earlier in [G1,G2,G3,G4].

In the present paper we consider the general case. Now we must show how to push $f(\zeta)$, $\zeta \in b\Delta$, through the critical levels of ρ . We describe this when $N = 2$.

First, we observe that in the pushing process described above the change is holomorphic on Δ and for this one does not need to assume that f is holomorphic on Δ . Perform such a pushing on f to get g . The change $g - f$ is holomorphic on Δ so if f is not holomorphic on Δ but satisfies $|f(\zeta) - F(\zeta)| < \tau$ ($\zeta \in \bar{\Delta}$) where F is continuous on $\bar{\Delta}$ and holomorphic on Δ then g satisfies $|g(\zeta) - G(\zeta)| < \tau$ ($\zeta \in \bar{\Delta}$) where $G (= F + (g - f))$ is continuous on $\bar{\Delta}$ and holomorphic on Δ .

With no loss of generality assume that all critical points of ρ are nondegenerate. Then

the set of critical points of ρ is discrete and so is the set of critical values of ρ .

Suppose that $a_1 < b_1 < c < a_2 < b_2$ and let c be the only critical value of ρ on $[a_1, b_2]$. Suppose that $f: \overline{\Delta} \rightarrow D$ is continuous and holomorphic on Δ and satisfies $a_1 < \rho(f(\zeta)) < b_1$ ($\zeta \in b\Delta$). Given $\lambda, b_1 < \lambda < c$, very close to c , one uses the pushing process above to get a continuous map $f: \overline{\Delta} \rightarrow D$, holomorphic on Δ and such that $\lambda < \rho(f_1(\zeta)) < c$ ($\zeta \in b\Delta$). Slightly changing f_1 we may assume that f_1 is smooth on $b\Delta$.

Let $a_2 < \alpha < \beta < b_2$. Let w_0 be a critical point of ρ . It is known that if $w - w_0 = (x_1 + iy_1, x_2 + iy_2)$ then, after a smooth change of coordinates near w_0 , the function ρ has the form $\rho(w_0) + x_1^2 + x_2^2 \pm y_1^2 \pm y_2^2$, that is, the index of w_0 does not exceed 2. This, together with the fact that f_1 is smooth on $b\Delta$, enables us to push those $f_1(\zeta), \zeta \in b\Delta$, that are near the critical points of ρ contained in $\{z \in D: \rho(z) = c\}$, through this level to get a continuous map $f_2: b\Delta \rightarrow D$ with the property that either $f_2(\zeta)$ is away from the critical points or $\rho(f_2(\zeta)) > c$. Away from the critical points we then push in the direction of $\text{grad}\rho$ to get a continuous map $f_3: b\Delta \rightarrow D$ such that $\rho(f_3(\zeta)) > c$ ($\zeta \in b\Delta$). In fact, we show that given $\delta > 0$ there is a $\lambda < c$ such that if $\lambda < \rho(f_1(\zeta)) < c$ ($\zeta \in b\Delta$) then there is a continuous map $f_3: \overline{\Delta} \rightarrow D$ such that $|f_3(\zeta) - f_1(\zeta)| < \delta$ ($\zeta \in \overline{\Delta}$) and $c < \rho(f_3(\zeta)) < a_2$ ($\zeta \in b\Delta$). For our purpose we choose $\delta > 0$ so small that if $z \in D$ then $\rho(z) \leq \beta$ and $|z - w| < \delta$ imply that $w \in D$ and $\rho(z) < b_2$ and $|\rho(z) - \rho(w)| < \alpha - a_2$.

The map f_3 is continuous on $\overline{\Delta}$ but is not holomorphic on Δ . However, there is a map $F(= f_1)$, continuous on $\overline{\Delta}$, holomorphic on Δ and such that $|f_3(\zeta) - F(\zeta)| < \delta$ ($\zeta \in \overline{\Delta}$). We now perform the pushing process to get a continuous map $f_4: \overline{\Delta} \rightarrow D$ such that $\rho(f_4(\zeta)) < \beta$ ($\zeta \in \overline{\Delta}$), $\alpha < \rho(f_4(\zeta)) < \beta$ ($\zeta \in b\Delta$), and such that there is a continuous map $g: \overline{\Delta} \rightarrow \mathbb{C}^2$, holomorphic on Δ and such that $|g(\zeta) - f_4(\zeta)| < \delta$ ($\zeta \in \overline{\Delta}$). Our choice of δ now implies that g maps $\overline{\Delta}$ into D and satisfies $a_2 < \rho(g(\zeta)) < b_2$ ($\zeta \in b\Delta$).

Thus, starting with a continuous map $f: \overline{\Delta} \rightarrow D$, holomorphic on Δ and satisfying $a_1 < \rho(f(\zeta)) < b_1$ ($\zeta \in b\Delta$), after a detour through nonholomorphic maps we end up with a continuous map $g: \overline{\Delta} \rightarrow D$, holomorphic on Δ and such that $a_2 < \rho(g(\zeta)) < b_2$ ($\zeta \in b\Delta$).

Now one builds this into an inductive process to obtain a proper holomorphic map $f: \Delta \rightarrow D$ such that $f(0) = p$.

The proof for general Stein manifolds is technically more complicated but uses the same idea.

3. Preliminaries

By the embedding theorem for Stein manifolds [Ho] we may assume that M is a closed submanifold of \mathbb{C}^N for some $N \in \mathbb{N}$.

We first show that it is enough to consider the case $\dim M = 2$. Let $p \in M$ and let $X \in \mathbb{C}^N$ be tangent to M at p . If $X = 0$ then let V be a complex line through p . If $X \neq 0$ then let V be the complex line passing through p and $p + X$. One may assume that V is not contained in M . If it is, then, with no loss of generality, replace M with $\Phi(M)$, where $\Phi: \mathbb{C}^N \rightarrow \mathbb{C}^N$ is a biholomorphic map such that $\Phi(p) = p$, $(D\Phi)(p)(X) = X$ and such that $\Phi(M)$ does not contain V . Let $\mathcal{L}(V)$ be the family of all affine complex subspaces of \mathbb{C}^N of dimension $N - 1$ which contain V . By the transversality theorem, almost every $\Sigma \in \mathcal{L}(V)$ meets $M \setminus V$ transversely. Further, since M does not contain V , $M \cap V$ is at most countable and so almost every $\Sigma \in \mathcal{L}(V)$ is transverse to each of the tangent spaces

$T_p(M)$, $p \in M \cap V$. Thus, there is a $\Sigma \in \mathcal{L}(V)$ that intersects M transversely. Obviously $p \in \Sigma \cap M$ and X is tangent to $\Sigma \cap M$ at p . Thus, we may replace M by $M \cap \Sigma$, a closed submanifold of Σ of dimension $\dim M - 1$. Repeating the process, we may, with no loss of generality, assume that $\dim M = 2$. With no loss of generality assume that $0 \notin M$.

The transversality theorem implies that after rotating M around p if necessary we may assume that the sphere $\{z \in \mathbb{C}^N : |z| = |p|\}$ intersects M transversely and so do all the spheres $\{z \in \mathbb{C}^N : |z - a| = |p - a|\}$ for sufficiently small $a \in \mathbb{C}^N$. Sard's implies that for almost every $a \in \mathbb{C}^N$ the function $z \mapsto |z - a|^2$ is a Morse function on M . Thus, after translating M for a suitable small a we may assume that the function $\rho: M \rightarrow \mathbb{R}$ defined by $\rho(z) = |z|^2$ ($z \in M$) is a Morse function and that $\rho(p)$ is not a critical value of ρ . Since ρ is a Morse function all its critical points are nondegenerate. In particular, the set of critical points of ρ is discrete and so is the set of critical values of ρ .

By the theorem of Docquier and Grauert [GR, p. 257] there are an open neighbourhood Ω of M in \mathbb{C}^N and a holomorphic map $\pi: \Omega \rightarrow M$ such that $\pi(z) = z$ ($z \in M$).

4. Small discs tangent to the level sets of ρ

Denote by \mathbb{B} the open unit ball in \mathbb{C}^N . For each $q \in \mathbb{C}^N \setminus \{0\}$ let $E(q) = \{z \in \mathbb{C}^N : \langle z - q, q \rangle = 0\}$ be the affine complex hyperplane passing through q and tangent to the sphere $b|q|\mathbb{B}$, and for each $q \in M$ let $T(q)$ be the affine complex subspace of dimension 2 passing through q and tangent to M at q .

Assume that $Q \subset M$ is a compact set consisting of regular points of ρ . For each $q \in Q$, $T(q)$ intersects $E(q)$ transversely so $E(q) \cap T(q) = L(q)$ is a complex line and there is a $\delta > 0$ such that $(q + \delta\mathbb{B}) \cap E(q) \cap M = \Lambda(q)$ is a closed, one dimensional complex submanifold of $q + \delta\mathbb{B}$, which is tangent to $L(q)$ at q . Since Q is compact, a $\delta > 0$ can be chosen that works for all $q \in Q$. For each $q \in Q$ there is a ν , $0 < \nu < \delta$, such that for each r , $0 < r \leq \nu$, $\Lambda(q)$ intersects $b(q + r\mathbb{B})$ transversely, and the intersection $(q + r\mathbb{B}) \cap \Lambda(q)$ is biholomorphically equivalent to a disc. Again, by the compactness of Q , a $\nu > 0$ can be chosen that works for all $q \in Q$. Since $E(q)$ is orthogonal to q it follows that the spheres in $E(q)$ centered at q are the level sets of the function $z \mapsto |z|^2$ restricted to $E(q)$. In particular,

$$\rho(z) = |q|^2 + r^2 = \rho(q) + r^2 \quad (z \in \Lambda(q) \cap b(q + r\mathbb{B})). \quad (4.1)$$

By transversality everything varies smoothly if we vary $q \in M$ and r , $0 < r < \nu$, smoothly.

Lemma 4.1 *Given a compact set $Q \subset M$ of regular points of ρ there is a $\mu_0 > 0$ such that for every positive continuous function μ on $b\Delta$ that satisfies $\mu(\zeta) < \mu_0$ ($\zeta \in b\Delta$) there is a continuous map $F: b\Delta \times \overline{\Delta} \rightarrow M$ such that*

- (i) *for each $\zeta \in b\Delta$ the function $\eta \mapsto F(\zeta, \eta)$ is holomorphic on Δ*
- (ii) *$F(\zeta, 0) = f(\zeta)$ ($\zeta \in b\Delta$)*
- (iii) *$\rho(F(\zeta, \eta)) > \rho(f(\zeta))$ ($\zeta \in b\Delta, \eta \in \overline{\Delta} \setminus \{0\}$)*
- (iv) *$\rho(F(\zeta, \eta)) = \rho(f(\zeta)) + \mu(\zeta)$ ($\zeta \in b\Delta, \eta \in b\Delta$).*

Proof. Let ν be as in the preceding discussion and put $\mu_0 = \nu^2$. Let μ be a positive continuous function on $b\Delta$ such that $\mu(\zeta) < \mu_0$ ($\zeta \in b\Delta$) and let $f: b\Delta \rightarrow Q$ be a continuous map. The preceding discussion shows that for each $\zeta \in b\Delta$, $b(f(\zeta) + \mu(\zeta)^{1/2}\mathbb{B})$ intersects $\Lambda(f(\zeta))$

transversely and $D(\zeta) = \Lambda(f(\zeta)) \cap (f(\zeta) + \mu(\zeta)^{1/2}\mathbb{B})$ is biholomorphically equivalent to a disc. If z belongs to the boundary of this disc, that is, if $z \in \Lambda(f(\zeta)) \cap b(f(\zeta) + \mu(\zeta)^{1/2}\mathbb{B})$ then $\rho(z) = \rho(f(\zeta)) + \mu(\zeta)$. By the transversality and by the continuity of f and μ the discs $D(\zeta)$ change continuously with ζ .

Let $a: b\Delta \rightarrow b\mathbb{B}$ be a continuous map such that for each $\zeta \in b\Delta$, $a(\zeta) \in T_{f(\zeta)}^{\mathbb{C}}\Lambda(f(\zeta)) = -f(\zeta) + L(f(\zeta))$. For each $\zeta \in b\Delta$, let Ψ_ζ be the biholomorphic map from Δ to $D(\zeta)$ such that $\Psi_\zeta(0) = f(\zeta)$ and that $\Psi'_\zeta(0)$ is a positive multiple of $a(\zeta)$. The map Ψ_ζ is unique and by the well known properties of conformal maps [Po] extends continuously to $\overline{\Delta}$. Obviously $\rho(\Psi_\zeta(\eta)) = \rho(f(\zeta)) + \mu(\zeta)$ ($\eta \in b\Delta$). Define $F(\zeta, \eta) = \Psi_\zeta(\eta)$ ($\zeta \in b\Delta$, $\eta \in \overline{\Delta}$). By construction, F satisfies (i), (ii) and (iv) and (4.1) implies (iii).

It remains to prove that F is continuous. To do this it is enough to prove that if $\zeta \in b\Delta$ then Ψ_ξ is arbitrarily close to Ψ_ζ , uniformly on $\overline{\Delta}$, provided that ξ is sufficiently close to ζ . Passing to a smaller μ_0 at the beginning if necessary we may assume that there is an $\omega > 0$ such that for each $\xi \in b\Delta$, $|\xi - \zeta| < \omega$, the orthogonal projection P_ζ onto $L(f(\zeta))$ is one to one and regular on $D(\xi)$. Since $D(\xi)$ vary continuously with ξ it is enough to prove that the conformal map that maps $P_\zeta(D(\zeta))$ onto $P_\zeta(D(\xi))$ that takes $P_\zeta(f(\zeta))$ to $P_\zeta(f(\xi))$, and whose derivative takes $P_\zeta(a(\zeta))$ to a positive multiple of $P_\zeta(a(\xi))$, is arbitrarily close to the identity, uniformly on Δ , provided that ξ is sufficiently close to ζ . This follows from [Po, p. 26]. The proof is complete.

5. Pushing the boundaries of discs in tangent directions

The following fact is essentially contained in [FG].

Lemma 5.1 *Let $G: b\Delta \times \overline{\Delta} \rightarrow \mathbb{C}^N$ be a continuous map such that for each $\zeta \in b\Delta$, the map $\eta \mapsto G(\zeta, \eta)$ is holomorphic on Δ and satisfies $G(\zeta, 0) = 0$. Given $\gamma > 0$ and r , $0 < r < 1$, there is a polynomial $P: \mathbb{C} \rightarrow \mathbb{C}^N$ such that*

- (i) $P(\zeta) \in G(\zeta, b\Delta) + \gamma\mathbb{B}$ ($\zeta \in b\Delta$)
- (ii) $P(t\zeta) \in G(\zeta, \overline{\Delta}) + \gamma\mathbb{B}$ ($\zeta \in b\Delta$, $0 \leq t \leq 1$)
- (iii) $|P(\zeta)| < \gamma$ ($|\zeta| \leq r$)
- (iv) $P(0) = 0$
- (v) $P'(0) = 0$.

Proof. It suffices to show that G can be approximated uniformly on $b\Delta \times \overline{\Delta}$ by maps of the form

$$\tilde{G}(\zeta, \eta) = \sum_{j=1}^n A_j(\zeta) \zeta^{-m} \eta^j$$

where $A_j: \mathbb{C} \rightarrow \mathbb{C}^N$ are polynomials and m, n are positive integers. The polynomial

$$P(\zeta) = \tilde{G}(\zeta, \zeta^k) = \sum_{j=1}^n A_j(\zeta) \zeta^{kj-m}$$

then satisfies (i)-(iv) provided that the approximation of G by \tilde{G} is sufficiently good and the integer $k > m$ is chosen sufficiently large.

Approximate first $G(\zeta, \eta)$ on $b\Delta \times \overline{\Delta}$ uniformly by $G(\zeta, r\eta)$ with $r < 1$ sufficiently close to 1. We have

$$\begin{aligned} G(\zeta, \eta) &= \frac{1}{2\pi i} \int_{b\Delta} \frac{G(\zeta, \nu) d\nu}{\nu - \eta} \\ &= \frac{1}{2\pi i} \int_{b\Delta} \left[\frac{\eta}{\nu^2} + \cdots + \frac{\eta^n}{\nu^{n+1}} \right] G(\zeta, \nu) d\nu + \eta^{n+1} \frac{1}{2\pi i} \int_{b\Delta} \frac{G(\zeta, \nu) d\nu}{\nu^{n+1}(\nu - \eta)}. \end{aligned}$$

When $|\eta| \leq r$ the last term is arbitrarily small provided that n is large enough. It follows that on $b\Delta \times \overline{\Delta}$, $G(\zeta, \eta)$ can be uniformly approximated by polynomials $B_1(\zeta)\eta + \cdots + B_n(\zeta)\eta^n$ with B_j continuous on $b\Delta$. For each j , $1 \leq j \leq n$, one can approximate, uniformly on $b\Delta$, $B_j(\zeta)$ arbitrarily well by $A_j(\zeta)/\zeta^m$ where A_j is a polynomial and where m can be chosen the same for all j . This completes the proof.

Lemma 5.2 *Let $t_1 < t_2$ and assume that every $t \in [t_1, t_2]$ is a regular value of ρ . There is a $\mu_0 > 0$ with the following property:*

Suppose that $f: \overline{\Delta} \rightarrow M$ is a continuous map, holomorphic in a neighbourhood of 0, such that $t_1 \leq \rho(f(\zeta)) \leq t_2$ ($\zeta \in b\Delta$) and such that $|f(\zeta) - H(\zeta)| \leq \tau$ ($\zeta \in \overline{\Delta}$) where $H: \overline{\Delta} \rightarrow \mathbf{C}^N$ is a continuous map, holomorphic on Δ which satisfies $H(0) = f(0)$ and $H'(0) = f'(0)$. Given R , $0 < R < 1$, $\varepsilon > 0$ and a positive continuous function μ on $b\Delta$, $\mu(\zeta) < \mu_0$ ($\zeta \in b\Delta$), there are r , $R < r < 1$, and a continuous map $g: \overline{\Delta} \rightarrow M$, holomorphic in a neighbourhood of 0, such that

- (i) $|\rho(g(\zeta)) - [\rho(f(\zeta)) + \mu(\zeta)]| < \varepsilon$ ($\zeta \in b\Delta$)
- (ii) $\rho(g(t\zeta)) \geq \rho(f(\zeta)) - \varepsilon$ ($r \leq t \leq 1$, $\zeta \in b\Delta$)
- (iii) $|g(\zeta) - f(\zeta)| < \varepsilon$ ($|\zeta| \leq r$)
- (iv) $g(0) = f(0)$,
- (v) $g'(0) = f'(0)$

and such that $|g(\zeta) - \Phi(\zeta)| < \tau + \varepsilon$ ($\zeta \in \overline{\Delta}$) where $\Phi: \overline{\Delta} \rightarrow \mathbf{C}^N$ is a continuous map, holomorphic on Δ which satisfies $\Phi(0) = g(0)$ and $\Phi'(0) = g'(0)$. If f is holomorphic on Δ then g can be chosen to be holomorphic on Δ .

Proof. Let $Q = \{z \in M: t_1 \leq \rho(z) \leq t_2\}$. The set Q is compact and consists of regular points of ρ . Let μ_0 be as in Lemma 4.1. Suppose that $f: \overline{\Delta} \rightarrow M$ is a continuous map, holomorphic in a neighbourhood of 0, such that $f(b\Delta) \subset Q$ and such that $|f(\zeta) - H(\zeta)| \leq \tau$ ($\zeta \in \overline{\Delta}$) where $H: \overline{\Delta} \rightarrow \mathbf{C}^N$ is a continuous map, holomorphic on Δ and such that $H(0) = f(0)$ and $H'(0) = f'(0)$. Let $\varepsilon > 0$ and $0 < R < 1$ and let μ be a positive continuous function on $b\Delta$, $\mu(\zeta) < \mu_0$ ($\zeta \in b\Delta$).

By Lemma 4.1 there is a continuous map $F: b\Delta \times \overline{\Delta} \rightarrow M$ such that

- (a) for each $\zeta \in b\Delta$ the map $\eta \mapsto F(\zeta, \eta)$ is holomorphic on Δ
- (b) $F(\zeta, 0) = f(\zeta)$ ($\zeta \in b\Delta$)
- (c) $\rho(F(\zeta, \eta)) \geq \rho(f(\zeta))$ ($\zeta \in b\Delta, \eta \in \overline{\Delta}$)
- (d) $\rho(F(\zeta, \eta)) = \rho(f(\zeta)) + \mu(\zeta)$ ($\zeta \in b\Delta, \eta \in b\Delta$).

Choose an open neighbourhood $W \subset M$ of $F(b\Delta \times \overline{\Delta}) \cup f(\overline{\Delta})$ with compact closure. One can choose δ , $0 < \delta < \varepsilon$, so small that

$$\rho(z) - \varepsilon \leq \rho(w) \leq \rho(z) + \varepsilon \quad \text{whenever} \quad z \in W, w \in M, |w - z| < 2\delta \quad (5.1)$$

Shrinking Ω if necessary we may assume that

$$|\pi(w) - w| < \delta \quad (w \in \Omega). \quad (5.2)$$

Choose γ , $0 < \gamma < \delta$, so small that

$$\left. \begin{array}{l} \text{if } z \in F(b\Delta \times \overline{\Delta}) \cup f(\overline{\Delta}), \quad w \in \mathbb{C}^N, \quad |w - z| < 2\gamma \\ \text{then } w \in \Omega, \quad \pi(w) \in W, \quad |\pi(w) - z| < \delta. \end{array} \right\} \quad (5.3)$$

Choose r , $R < r < 1$, so close to 1 that

$$|f(t\zeta) - f(\zeta)| < \gamma \quad (\zeta \in b\Delta, \quad r \leq t \leq 1). \quad (5.4)$$

Put $G(\zeta, \eta) = -f(\zeta) + F(\zeta, \eta)$ ($\zeta \in b\Delta$, $\eta \in \overline{\Delta}$). The map G satisfies the assumptions in Lemma 5.1 which provides a polynomial $P: \mathbb{C} \rightarrow \mathbb{C}^N$ which satisfies (i)-(v) in Lemma 5.1.

Put $\tilde{f} = f + P$. We have

$$(a') \quad \tilde{f}(\zeta) \in F(\zeta, b\Delta) + \gamma\mathbb{B} \quad (\zeta \in b\Delta).$$

By (5.4), $\zeta \in b\Delta$ and $r \leq t \leq 1$ imply that $f(t\zeta) + P(t\zeta) \in f(\zeta) + P(t\zeta) + \gamma\mathbb{B}$ which, by (ii) in Lemma 5.1, implies that

$$(b') \quad \tilde{f}(t\zeta) \in F(\zeta, \overline{\Delta}) + 2\gamma\mathbb{B} \quad (\zeta \in b\Delta, \quad r \leq t \leq 1).$$

Further, (iii)-(v) in Lemma 5.1 imply that

$$(c') \quad |\tilde{f}(\zeta) - f(\zeta)| < \gamma \quad (|\zeta| \leq r)$$

$$(d') \quad \tilde{f}(0) = f(0)$$

$$(e') \quad \tilde{f}'(0) = f'(0).$$

Now, (a'), (b'), and (c') together with (5.3) imply that $\tilde{f}(\overline{\Delta}) \subset \Omega$ and $\pi(\tilde{f}(\overline{\Delta})) \subset W$.

Define $g(\zeta) = \pi(\tilde{f}(\zeta))$ ($\zeta \in \overline{\Delta}$). It is clear that g is continuous on $\overline{\Delta}$, holomorphic in a neighbourhood of 0 and maps $\overline{\Delta}$ to M .

Let $\zeta \in b\Delta$. By (a') there is an $\eta \in b\Delta$ such that $|\tilde{f}(\zeta) - F(\zeta, \eta)| < \gamma$ so by (5.3), $|g(\zeta) - F(\zeta, \eta)| < \delta$. Now (d) and (5.1) imply (i). Given t , $r \leq t \leq 1$, and $\zeta \in b\Delta$, (b') implies that there is an $s \in \overline{\Delta}$ such that $|\tilde{f}(t\zeta) - F(\zeta, s)| < 2\gamma$ so by (5.3) it follows that $|g(t\zeta) - F(\zeta, s)| < \delta$. Now (c) and (5.1) imply that $\rho(g(t\zeta)) \geq \rho(f(\zeta)) - \varepsilon$ which proves (ii). Further, (c') and (5.3) imply that $|g(\zeta) - f(\zeta)| < \delta < \varepsilon$ ($|\zeta| \leq r$) which proves (iii). (iv) and (v) follow from (d'),(e') and the fact that $\pi|_M = \text{id}$.

From the construction it is clear that g is holomorphic on Δ provided that f is holomorphic on Δ . Note that (5.2) implies that $|\pi(f(\zeta) + P(\zeta)) - (f(\zeta) + P(\zeta))| < \delta$ ($\zeta \in \overline{\Delta}$) so $|f(\zeta) - H(\zeta)| \leq \tau$ ($\zeta \in \overline{\Delta}$) implies that $|\pi(f(\zeta) + P(\zeta)) - (H(\zeta) + P(\zeta))| \leq |\pi(f(\zeta) + P(\zeta)) - (f(\zeta) + P(\zeta))| + |f(\zeta) - H(\zeta)| < \delta + \tau < \varepsilon + \tau$ ($\zeta \in \overline{\Delta}$) where $\Phi = H + P$ is continuous on $\overline{\Delta}$ and holomorphic on Δ and satisfies $\Phi(0) = g(0)$ and $\Phi'(0) = g'(0)$. This completes the proof.

We will use the following consequence of Lemma 5.2.

Lemma 5.3 *Let $t_1 < t_2$ and assume that every $t \in [t_1, t_2]$ is a regular value of ρ .*

Suppose that $f: \overline{\Delta} \rightarrow M$ is a continuous map, holomorphic in a neighbourhood of 0, such that $t_1 < \rho(f(\zeta)) < t_2$ ($\zeta \in b\Delta$) and such that $|f(\zeta) - H(\zeta)| < \tau$ ($\zeta \in \overline{\Delta}$) where $H: \overline{\Delta} \rightarrow \mathbb{C}^N$

is a continuous map, holomorphic on Δ which satisfies $H(0) = f(0)$ and $H'(0) = f'(0)$. Given R , $0 < R < 1$, and $\varepsilon > 0$ there are r , $R < r < 1$, and a continuous map $g: \overline{\Delta} \rightarrow M$, holomorphic in a neighbourhood of 0 , such that

- (i) $t_2 - \varepsilon \leq \rho(g(\zeta)) \leq t_2 + \varepsilon$ ($\zeta \in b\Delta$)
- (ii) $\rho(g(t\zeta)) \geq \rho(f(\zeta)) - \varepsilon$ ($\zeta \in b\Delta$, $r \leq t \leq 1$)
- (iii) $|g(\zeta) - f(\zeta)| < \varepsilon$ ($|\zeta| \leq r$)
- (iv) $g(0) = f(0)$
- (v) $g'(0) = f'(0)$,

and such that $|g(\zeta) - G(\zeta)| < \tau + \varepsilon$ ($\zeta \in \overline{\Delta}$) where $G: \overline{\Delta} \rightarrow \mathbb{C}^N$ is a continuous map which is holomorphic on Δ and satisfies $G(0) = g(0)$ and $G'(0) = g'(0)$. If f is holomorphic on Δ then g can be chosen to be holomorphic on Δ .

Proof. Let μ_0 be the constant from Lemma 5.2. Choose $n \in \mathbb{N}$ so large that $n\mu_0 > t_2 - t_1$ and let $\mu(\zeta) = [t_2 - \rho(f(\zeta))]/n$ ($\zeta \in \overline{\Delta}$). Then μ is a positive continuous function on $b\Delta$ which satisfies $\mu(\zeta) < \mu_0$ ($\zeta \in b\Delta$).

Choose $\delta > 0$ so small that

$$n\delta < \varepsilon/2 \tag{5.5}$$

and that

$$\left. \begin{array}{l} |\rho(x) - \rho(z)| < \varepsilon/2 \text{ whenever} \\ z \in M, \rho(z) < [2(\tau + \varepsilon) + (t_2 + \varepsilon)^{1/2}]^2, x \in M, |x - z| < n\delta, \end{array} \right\} \tag{5.6}$$

that

$$\rho(f(\zeta)) + \frac{n-1}{n} [t_2 - \rho(f(\zeta))] + (n-1)\delta < t_2 \quad (\zeta \in b\Delta), \tag{5.7}$$

and that

$$\delta \leq \mu(\zeta) \quad (\zeta \in b\Delta). \tag{5.8}$$

Let $g_0 = f$, $G_0 = H$ and $r_0 = r$. Using Lemma 5.2 repeatedly with δ in place of ε one constructs inductively r_k , $1 \leq k \leq n$, $R < r_1 < \dots < r_n < 1$, continuous maps $g_k: \overline{\Delta} \rightarrow M$, holomorphic in a neighbourhood of 0 , and continuous maps $G_k: \overline{\Delta} \rightarrow \mathbb{C}^N$, holomorphic on Δ , such that for each k , $1 \leq k \leq n$,

- (i') $|\rho(g_k(\zeta)) - [\rho(g_{k-1}(\zeta)) + \mu(\zeta)]| < \delta$ ($\zeta \in b\Delta$)
- (ii') $\rho(g_k(t\zeta)) \geq \rho(g_{k-1}(\zeta)) - \delta$ ($\zeta \in b\Delta$, $r_k \leq t \leq 1$)
- (iii') $|g_k(\zeta) - g_{k-1}(\zeta)| < \delta$ ($|\zeta| \leq r_k$)
- (iv') $g_k(0) = g_{k-1}(0)$, $g'_k(0) = g'_{k-1}(0)$
- (v') $|g_k(\zeta) - G_k(\zeta)| < \tau + k\delta$ ($\zeta \in \overline{\Delta}$)
- (vi') $g_k(0) = G_k(0)$, $g'_k(0) = G'_k(0)$.

Indeed, suppose that $0 \leq m \leq n-1$ and that we have already constructed r_k, g_k, G_k for $0 \leq k \leq m$. Now (i') for $1 \leq k \leq m$ implies that $|\rho(g_m(\zeta)) - [\rho(f(\zeta)) + m\mu(\zeta)]| < m\delta$ ($\zeta \in b\Delta$), which, by (5.7), gives $\rho(g_m(\zeta)) < \rho(f(\zeta)) + m\mu(\zeta) + m\delta < t_2$ ($\zeta \in b\Delta$) and which, by (5.8), gives $\rho(g_m(\zeta)) > \rho(f(\zeta)) + m\mu(\zeta) - m\delta > t_1$ ($\zeta \in b\Delta$). Now we apply Lemma 5.2 to get r_{m+1}, g_{m+1} and G_{m+1} with the required properties.

Put $r = r_1$, $g = g_n$ and $G = G_n$. Notice that (i') implies that

$$|\rho(g_k(\zeta)) - [\rho(f(\zeta)) + k\mu(\zeta)]| < k\delta \quad (\zeta \in b\Delta, 1 \leq k \leq n). \quad (5.9)$$

In particular, for each k , $1 \leq k \leq n$ and for each $\zeta \in b\Delta$ we have $|g_k(\zeta)|^2 < |f(\zeta)|^2 + n\mu(\zeta) + n\delta = t_2 + n\delta$ which, by (v'), implies that $|G_k(\zeta)| < (t_2 + n\delta)^{1/2} + \tau + n\delta$ ($\zeta \in b\Delta$). Since G_k is holomorphic on Δ the maximum principle gives $|G_k(\zeta)| < (t_2 + n\delta)^{1/2} + \tau + n\delta$ ($\zeta \in \overline{\Delta}$) which, by (v') and by (5.5) implies that $|g_k(\zeta)| < 2(\tau + n\delta) + (t_2 + n\delta)^{1/2} < 2(\tau + \varepsilon) + (t_2 + \varepsilon)^{1/2}$ so

$$\rho(g_k(\zeta)) < [2(\tau + \varepsilon) + (t_2 + \varepsilon)^{1/2}]^2 \quad (\zeta \in \overline{\Delta}, 1 \leq k \leq n). \quad (5.10)$$

If $\zeta \in b\Delta$ then (5.9) implies that $|\rho(g(\zeta)) - t_2| = |\rho(g_n(\zeta)) - [\rho(f(\zeta)) + n\mu(\zeta)]| < n\delta < \varepsilon$ which proves (i). (iv) and (v) follow from (iv'). If $|\zeta| \leq r$ then $|\zeta| \leq r_k$ for all k , $1 \leq k \leq n$, so (iii') implies that $|g(\zeta) - f(\zeta)| < n\delta < \varepsilon$ which proves (iii). Further, (v') and (5.5) imply that $|g(\zeta) - G(\zeta)| < \tau + \varepsilon$ ($\zeta \in \overline{\Delta}$). Moreover, (vi') implies that $G(0) = g(0)$ and $G'(0) = g'(0)$.

To prove (ii), let $\zeta \in b\Delta$ and $r \leq t \leq 1$. If $t \geq r_n$ then by (ii'), $\rho(g(t\zeta)) \geq \rho(f(\zeta)) - n\delta > \rho(f(\zeta)) - \varepsilon$. Let $r_k \leq t \leq r_{k+1}$ for some k , $1 \leq k \leq r_{k+1}$. Since $t \geq r_k$, (ii') implies that $\rho(g_k(t\zeta)) \geq \rho(f(\zeta)) - k\delta$ and by (5.5) it follows that $\rho(g_k(t\zeta)) \geq \rho(f(\zeta)) - \varepsilon/2$. On the other hand, since $t \leq r_{k+1}$, (iii') implies that $|g(t\zeta) - g_k(t\zeta)| < (n - k)\delta < n\delta$. By (5.10) and (5.6) it follows that $|\rho(g_k(t\zeta)) - \rho(g(t\zeta))| < \varepsilon/2$ and the preceding discussion implies that $\rho(G(t\zeta)) > \rho(f(\zeta)) - \varepsilon$. This proves (ii).

By Lemma 5.2 it is clear that if f is holomorphic on Δ then each g_k , $1 \leq k \leq n$ can be chosen holomorphic on Δ . In particular, this holds for $g = g_n$. This completes the proof.

6. Pushing to increase the values of ρ near a critical point

The following lemma shows how to push to higher levels of ρ those points near a critical point that are not contained in a small exceptional set.

Lemma 6.1 *Let q be a critical point of ρ . There are arbitrarily small neighbourhoods $U \subset\subset V \subset\subset W$ of q in M and a set $E \subset V$ such that*

(A) *Given $\varepsilon > 0$ there are $\tau > 0$ and a continuous map $\Phi: \overline{W} \setminus E \rightarrow \overline{W} \setminus E$ such that*

- (i) $|\Phi(\zeta) - z| < \varepsilon$ ($z \in \overline{W} \setminus E$)
- (ii) $\Phi|_{bW} = id$
- (iii) $\Phi(z) \in U$ implies that $z \in U$
- (iv) $\rho(\Phi(z)) \geq \rho(z)$ ($z \in \overline{W} \setminus E$)
- (v) $\rho(\Phi(z)) \geq \rho(z) + \tau$ ($z \in U \setminus E$)

(B) *Given S , a closed subset of W of two dimensional Hausdorff measure zero and $\varepsilon > 0$ there is a continuous map $\Phi: \overline{W} \rightarrow \overline{W}$ such that*

- (i') $|\Phi(z) - z| < \varepsilon$ ($z \in \overline{W}$)
- (ii') $\Phi|_{bW} = id$
- (iii') $\Phi(S)$ misses E

Proof, Part 1. We look first at the model case. Write points in \mathbb{R}^4 in the form $(x_1, x_2, y_1, y_2) = (x, y)$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Let D be the open unit disc in \mathbb{R}^2 .

Suppose that $0 < r < R < \tilde{R}$ and let $0 < \eta < R - r$. Choose $r' > r$ such that $r' + \eta < R$ and let φ be a smooth decreasing function on $[0, \infty)$ such that $\varphi(t) \equiv 1$ ($0 \leq t \leq r$), $\varphi(t) \equiv 0$ ($t \geq r'$) and define

$$h(x, y) = (x + \eta\varphi(|x|)\varphi(|y|)\frac{x}{|x|}, y).$$

Write $A = (\tilde{R}\overline{D})^2 \setminus (\{0\} \times RD)$. Then $h: A \rightarrow A$ is a continuous map such that

- (a) $|h(x, y) - (x, y)| \leq \eta$ ($(x, y) \in A$)
- (b) $h|_{[(\tilde{R}\overline{D})^2 \setminus (RD)^2]} = \text{id}$
- (c) $h(x, y) = (x + \eta\frac{x}{|x|}, y)$ ($(x, y) \in (rD)^2 \setminus [\{0\} \times RD]$)
- (d) if $h(x, y) \in (rD)^2$ then $(x, y) \in (rD)^2$.

Thus h keeps y fixed and for $|y| < r$ pushes a point (x, y) , $x \neq 0$, away from the two dimensional subspace $x_1 = x_2 = 0$. The exceptional set where h is not continuous (and not even defined) is contained in $\{0\} \times RD$.

Let a be a function of the form $a(x, y) = x_1^2 + x_2^2 \pm y_1^2 \pm y_2^2$. Applying h increases the values of $a(x, y)$:

- (e) $a(h(x, y)) \geq a(x, y)$ ($(x, y) \in A$)
- (f) $a(h(x, y)) \geq a(x, y) + \eta^2$ ($(x, y) \in (rD)^2, x \neq 0$).

Part 2. We now use the fact that the set $\{0\} \times RD$ is small. Suppose that \mathcal{T} is a closed subset of $(\tilde{R}\overline{D})^2$ of two dimensional Hausdorff measure zero. Then $\mathcal{T} \cap (\tilde{R}\overline{D})^2$ is a compact set of two dimensional Hausdorff measure zero whose image under the projection $(x, y) \mapsto x$ is a compact subset of $\tilde{R}\overline{D}$ of two dimensional Hausdorff measure zero which implies that arbitrarily close to 0 there is an x such that $\{x\} \times \tilde{R}\overline{D}$ misses \mathcal{T} . Given $\eta > 0$ choose such an x which satisfies $|x| < \eta$. It is easy to construct a homeomorphism $h: (\tilde{R}\overline{D})^2 \rightarrow (\tilde{R}\overline{D})^2$ such that

- (a') $|h(x, y) - (x, y)| < \eta$ ($(x, y) \in (\tilde{R}\overline{D})^2$)
- (b') $h|_{b(\tilde{R}\overline{D})^2} = \text{id}$

and which maps $\{x\} \times \tilde{R}\overline{D}$ onto $\{0\} \times \tilde{R}\overline{D}$. Clearly $h(\mathcal{T})$ misses $\{0\} \times \tilde{R}\overline{D}$.

Part 3. We now consider the general case. Let q be a critical point of ρ . Since ρ is strictly plurisubharmonic there are a neighbourhood $W \subset M$ of q , a neighbourhood P of 0 in \mathbb{C}^2 and a biholomorphic map $g: P \rightarrow W$, $g(0) = q$, such that $(\rho \circ g)(w) - \rho(q) = (1 + \lambda_1)x_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_1)y_1^2 + (1 - \lambda_2)y_2^2 + o(|w|^2)$ ($w \in P$) where $\lambda_i \geq 0$, $i = 1, 2$, and where $w = (x_1 + y_1, x_2 + y_2)$ [HL, p. 29]. Using the Morse lemma and passing to smaller P and W if necessary we may assume that $g: P \rightarrow W$ is a smooth change of coordinates such that $g(0) = q$ and such that $(\rho \circ g)(w) - \rho(q) = a(x, y)$ where a is as above.

There is an $\tilde{R} > 0$ such that $(\tilde{R}D)^2 \subset\subset P$. Replace P by $(\tilde{R}D)^2$ and let $W = g(P)$. Let $\lambda < \infty$ be a Lipschitz constant of g on $(\tilde{R}D)^2$.

Let $\varepsilon > 0$. Choose $r, R, 0 < r < R < \tilde{R}$, and choose $\eta, 0 < \eta < R - r$, so small that $\lambda\eta < \varepsilon$. Let $U = g((rD)^2)$, $V = g((R\overline{D})^2)$. Let $E = g(\{0\} \times RD)$ and let h

and A be as in Part 1. Define $\Phi = g \circ h \circ g^{-1}$. The map Φ is a continuous map from $\overline{W} \setminus E$ to $\overline{W} \setminus E$. If $z \in \overline{W} \setminus E$ then $g^{-1}(z) \in A$ so by (a), $|h(g^{-1}(z)) - g^{-1}(z)| \leq \eta$ and consequently $|\Phi(z) - z| = |g(h(g^{-1}(z))) - g(g^{-1}(z))| \leq \lambda\eta < \varepsilon$. So Φ satisfies (i). Since $g(bP) = bW$, (b) implies (ii). If $\Phi(z) \in U$ then $h(g^{-1}(z)) \in (rD)^2$ so by (d), $g^{-1}(z) \in (rD)^2$ and consequently $z \in g((rD)^2) = U$ which implies (iii). Let $z \in \overline{W} \setminus E$. Then $g^{-1}(z) \in A$ so by (e), $\rho(\Phi(z)) - \rho(q) = \rho(g(h(g^{-1}(z)))) - \rho(q) = a(h(g^{-1}(z))) \geq a(g^{-1}(z)) = (\rho \circ g)(g^{-1}(z)) - \rho(q) = \rho(z) - \rho(q)$ so Φ satisfies (iv). Finally, if $z \in U$ then $g^{-1}(z) \in (rD)^2$ so by (f), $\rho(\Phi(z)) - \rho(q) = a(h(g^{-1}(z))) \geq a(g^{-1}(z)) + \eta^2 = \rho(z) - \rho(q) + \eta^2$ so Φ satisfies (v) with $\tau = \eta^2$. This completes the proof of (A).

Part 4. To prove (B), let g , λ and η be as above. Let S be a closed subset of W of two dimensional Hausdorff measure zero. Then $\mathcal{T} = g^{-1}(S)$ is a closed subset of $(\tilde{R}D)^2$ of two dimensional Hausdorff measure zero. By Part 2 there is a homeomorphism $h: (\tilde{R}D)^2 \rightarrow (\tilde{R}D)^2$ which satisfies (a') and (b') and is such that $h(\mathcal{T})$ misses $\{0\} \times R\tilde{D}$. Let $\Phi = g \circ h \circ g^{-1}$. Then $\Phi: \overline{W} \rightarrow \overline{W}$ is a homeomorphism. Clearly $\Phi(S) = g(h(\mathcal{T}))$ misses $g(\{0\} \times R\tilde{D}) = E$ so Φ satisfies (iii'). As in Part 3, (a') implies that Φ satisfies (i'). By (b'), Φ satisfies (ii'). This completes the proof.

7. Pushing to increase the values of ρ away from critical points

The following lemma shows how to push to higher levels of ρ the points which are away from critical points.

Lemma 7.1 *Let $a < c < b$ and assume that c is the only critical point of ρ on $[a, b]$. Write $P = \{z \in M: a \leq \rho(z) \leq b\}$. Let z_1, z_2, \dots, z_m be the critical points of ρ contained in $\{z \in M: \rho(z) = c\}$. Let U_j , $1 \leq j \leq m$, be open neighbourhoods of z_j in M , with pairwise disjoint closures contained in $\{z \in M: a < \rho(z) < b\}$. For each j , $1 \leq j \leq m$, let $B_j \subset\subset U_j$ be an open neighbourhood of z_j .*

Given $\varepsilon > 0$ there are $\tau > 0$ and a continuous map $H: P \rightarrow M$ such that

- (i) $|H(z) - z| < \varepsilon$ ($z \in P$)
- (ii) $H|_{\cup_{j=1}^m B_j} = id$
- (iii) $\rho(H(z)) \geq \rho(z)$ ($z \in P$)
- (iv) $\rho(H(z)) \geq \rho(z) + \tau$ ($z \in P \setminus \cup_{j=1}^m U_j$).

Proof. The set $Q = P \setminus \cup_{j=1}^m B_j$ is compact and consists of regular points of ρ . Given $q \in M$ and $r > 0$ write $D(q, r) = \{z \in T(q): |z - q| < r\}$. Further, let $A(q)$ be the orthogonal complement of $-q + T(q)$. $A(q)$ is a complex subspace of \mathbb{C}^N of dimension $N - 2$. For each $q \in M$ there are an $r > 0$ and a holomorphic map $\varphi_q: D(q, r) \rightarrow A(q)$ such that $\varphi_q(q) = (D\varphi_q)(q) = 0$ and such that $\{z + \varphi_q(z): z \in D(q, r)\} \subset M$. Since Q is compact there is an $r > 0$ that works for all $q \in Q$ and there is a constant $\nu < \infty$ such that for each $q \in Q$, the function φ_q and its first and second derivatives are bounded by ν on $D(q, r)$.

For each $q \in Q$ consider the function $s_q: D(q, r) \rightarrow \mathbb{R}$ defined by $s_q(z) = \rho(z + \varphi_q(z))$. Passing to a smaller r and to a larger ν if necessary we may assume that $1/\nu < |(\text{grad } s_q)(z)| < \nu$ ($z \in D(q, r)$) and that the second derivative of s_q is also bounded by ν on $D(q, r)$, $q \in Q$. Elementary analysis now shows that there is a constant $t_0 > 0$

such that for every $q \in Q$ and every t , $0 < t \leq t_0$, we have $q + t(\text{grad } s_q)(q) \in D(q, r)$ and

$$s_q(q + t(\text{grad } s_q)(q)) \geq s_q(q) + t \frac{|(\text{grad } s_q)(q)|^2}{2} \quad (7.1)$$

Passing to a smaller t_0 if necessary we may assume that $\nu t_0 + \nu < \varepsilon$.

Let α be a continuous real function on P such that $0 \leq \alpha \leq 1$, such that $\alpha \equiv 0$ on $\cup_{j=1}^m B_j$ and $\alpha \equiv 1$ on $P \setminus \cup_{j=1}^m U_j$. Define the map $H: P \rightarrow M$ by

$$H(q) = q + t_0 \alpha(q) (\text{grad } s_q)(q) + \varphi_q [q + t_0 \alpha(q) (\text{grad } s_q)(q)] \quad (q \in Q)$$

and $H(q) = q$ ($q \in \cup_{j=1}^m B_j$). Since all the quantities depend continuously on q and since $\alpha \equiv 0$ on $\cup_{j=1}^m B_j$ it follows that $H: P \rightarrow M$ is a continuous map. Since $|(\text{grad } s_q)(q)| < \nu$ and $|\varphi_q| < \nu$ and since $\nu t_0 + \nu < \varepsilon$, (i) follows. By definition, H satisfies (ii). By (7.1), H satisfies (iii), and since $\alpha \equiv 1$ on $P \setminus \cup_{j=1}^m U_j$, (7.1) implies that $\rho(H(z)) > \rho(z)$ ($z \in P \setminus \cup_{j=1}^m U_j$) and since $P \setminus \cup_{j=1}^m U_j$ is compact it follows that there is a $\tau > 0$ such that (iv) holds. This completes the proof.

8. Pushing through a critical level in a nonholomorphic way

Lemma 8.1 *Let $c > \rho(p)$ be a critical value of ρ and let $\varepsilon > 0$. There is a $\lambda < c$ with the following property:*

Let $\varphi: \overline{\Delta} \rightarrow M$ be a continuous map, holomorphic on Δ and such that $\lambda < \rho(\varphi(\zeta)) < c$ ($\zeta \in bD$) and $\varphi(0) = p$. There is a continuous map $\psi: \overline{\Delta} \rightarrow M$, holomorphic in a neighbourhood of 0, such that $|\psi(\zeta) - \varphi(z)| < \varepsilon$ ($\zeta \in \overline{\Delta}$), such that $\psi(0) = p$, $\psi'(0) = \varphi'(0)$ and such that $\rho(\psi(\zeta)) > c$ ($\zeta \in b\Delta$).

Proof. We first replace $\varphi|_{b\Delta}$ by a smooth map close to $\varphi|_{b\Delta}$. Then we use Lemma 6.1 to push through the critical level those points $\varphi(\zeta)$, ($\zeta \in b\Delta$), that are close to the critical points. After that we use Lemma 7.1 to push through the critical level the points $\varphi(\zeta)$, ($\zeta \in b\Delta$), that are away from critical points. Finally, we extend the change continuously to all $\overline{\Delta}$ so that the extension vanishes identically in a neighbourhood of 0. We now pass to the details.

Part 1. There are a, b such that $a < c < b$ and such that c is the only critical point on $[a, b]$. Denote $P = \{z \in M: a \leq \rho(z) \leq b\}$. Let z_1, z_2, \dots, z_m be the critical points of ρ contained in $\{z \in M: \rho(z) = c\}$. By Lemma 6.1 there are open neighbourhoods W_j of z_j in M , with pairwise disjoint closures contained in $\{z \in M: a < \rho(z) < b\}$, and for each j , $1 \leq j \leq m$, there are open neighbourhoods $U_j \subset\subset V_j \subset\subset W_j$ of z_j and a set $E_j \subset V_j$ such that given $\varepsilon > 0$ there are a $\tau_j > 0$ and a continuous map $\Phi_j: \overline{W}_j \setminus E_j \rightarrow \overline{W}_j \setminus E_j$ such that

- (i) $|\Phi_j(z) - z| < \varepsilon$ ($z \in \overline{W}_j \setminus E_j$)
- (ii) $\Phi_j|_{bW_j} = \text{id}$
- (iii) $\Phi_j(z) \in U_j$ implies that $z \in U_j$
- (iv) $\rho(\Phi_j(z)) \geq \rho(z)$ ($z \in \overline{W}_j \setminus E_j$)
- (v) $\rho(\Phi_j(z)) \geq \rho(z) + \tau_j$ ($z \in U_j \setminus E_j$).

Moreover, given a closed subset S_j of W_j of two dimensional Hausdorff measure zero and $\eta > 0$ there is a continuous map $\Psi_j: \overline{W_j} \rightarrow \overline{W_j}$ such that

- (i') $|\Psi_j(z) - z| < \eta$ ($z \in \overline{W_j}$)
- (ii') $\Psi_j|_{bW_j} = \text{id}$
- (iii') $\Psi_j(S_j)$ misses E_j .

We will show that

$$\left. \begin{array}{l} \text{given } \varepsilon > 0 \text{ there are } \tau > 0 \text{ and a continuous map } F: P \setminus \cup_{j=1}^m E_j \rightarrow M \\ \text{such that } |F(z) - z| < 2\varepsilon \text{ (} z \in P \setminus \cup_{j=1}^m E_j \text{) and such that} \\ \rho(F(z)) \geq \rho(z) + \tau \text{ (} z \in P \setminus \cup_{j=1}^m E_j \text{).} \end{array} \right\} \quad (8.1)$$

Let $\varepsilon > 0$. By Lemma 7.1 there are $\tau_0 > 0$ and a continuous map $H: P \rightarrow M$ such that

- (i'') $|H(z) - z| < \varepsilon$ ($z \in P$)
- (ii'') $H|_{\cup_{j=1}^m B_j} = \text{id}$
- (iii'') $\rho(H(z)) \geq \rho(z)$ ($z \in P$)
- (iv'') $\rho(H(z)) \geq \rho(z) + \tau_0$ ($z \in P \setminus \cup_{j=1}^m U_j$).

Let Φ_j and τ_j , $1 \leq j \leq m$, be as above. Define the map $G: P \setminus \cup_{j=1}^m E_j \rightarrow P$ by

$$G(z) = \begin{cases} \Phi_j(z) & (z \in \overline{W_j} \setminus E_j, 1 \leq j \leq m) \\ z & (z \in P \setminus \cup_{j=1}^m \overline{W_j}) \end{cases}$$

By the properties of Φ_j the map G is continuous and satisfies $|G(z) - z| < \varepsilon$ ($z \in P \setminus \cup_{j=1}^m E_j$). Consequently the map $F = H \circ G$ is continuous, and by (i'') satisfies $|F(z) - z| < 2\varepsilon$ ($z \in P \setminus \cup_{j=1}^m E_j$).

Let $\tau = \min\{\tau_0, \tau_1, \dots, \tau_m\}$. Suppose that $z \in P \setminus \cup_{j=1}^m E_j$. If $z \in U_j \setminus E_j$ for some j , $1 \leq j \leq m$, then by (v), $\rho(G(z)) \geq \rho(z) + \tau$ so by (iii''), $\rho(F(z)) \geq \rho(G(z)) \geq \rho(z) + \tau$. If $z \in (\overline{W_j} \setminus E_j) \setminus U_j$ then by (iv), $\rho(G(z)) \geq \rho(z)$, and by (iii), $G(z) \in \overline{W_j} \setminus U_j$. Since $G(z) \in P \setminus \cup_{j=1}^m U_j$, (iv'') implies that $\rho(F(z)) \geq \rho(G(z)) + \tau \geq \rho(z) + \tau$. Finally, if $z \in P \setminus \cup_{j=1}^m \overline{W_j}$ then $G(z) = z$. In particular, $G(z) \in P \setminus \cup_{j=1}^m U_j$ so by (iv''), $\rho(F(z)) \geq \rho(z) + \tau$. This completes the proof of (8.1).

Part 2. Let $\varepsilon > 0$. Choose $\delta > 0$ so small that

$$\text{if } z \in M, \rho(z) \leq c \text{ and } w \in \mathbb{C}^N, |w - z| < 3\delta \text{ then } w \in \Omega \text{ and } |\pi(w) - z| < \varepsilon \quad (8.2)$$

By (8.1) there are τ , $0 < \tau < c - b$, and a continuous map $F: P \setminus \cup_{j=1}^m E_j \rightarrow M$ such that $|F(z) - z| < \delta$ ($z \in P \setminus \cup_{j=1}^m E_j$) and such that $\rho(F(z)) \geq \rho(z) + 2\tau$ ($z \in P \setminus \cup_{j=1}^m E_j$).

Let $\lambda = c - \tau$ and suppose that $\varphi: \overline{\Delta} \rightarrow M$ is a continuous map, holomorphic on Δ , $\varphi(0) = p$, and such that $\lambda < \rho(\varphi(z)) < c$ ($\zeta \in b\Delta$). Let $\varphi^* = \varphi|_{b\Delta}$ be the boundary map. If $r < 1$ is chosen sufficiently close to 1 then $\varphi_1: b\Delta \rightarrow M$ defined by $\varphi_1(\zeta) = \varphi(r\zeta)$ ($\zeta \in b\Delta$) is a smooth map such that $\lambda < \rho(\varphi_1(\zeta)) < c$ ($\zeta \in b\Delta$) and such that $|\varphi_1(\zeta) - \varphi^*(\zeta)| < \delta$ ($\zeta \in b\Delta$). Now $\varphi_1(b\Delta)$ is a compact set of two dimensional Hausdorff measure zero so for each j , $1 \leq j \leq m$, $S_j = \varphi_1(b\Delta) \cap W_j$ is a closed subset of W_j of two dimensional

Hausdorff measure zero. Given $\eta > 0$ there are continuous maps $\Psi_j: \overline{W_j} \rightarrow \overline{W_j}$, $1 \leq j \leq m$) which satisfy (i'), (ii') and (iii'). Define the map $\Theta: M \rightarrow M$ by

$$\Theta|_{\overline{W_j}} = \Psi_j \quad (1 \leq j \leq m), \quad \Theta(z) = z \quad (z \in M \setminus \cup_{j=1}^m \overline{W_j}).$$

Provided that η , $0 < \eta < \delta$, is small enough, $\varphi_2 = \Theta \circ \varphi_1$ is a continuous map from $b\Delta$ to M such that $|\varphi_2(\zeta) - \varphi_1(\zeta)| < \delta$ ($\zeta \in b\Delta$), $\lambda < \rho(\varphi_2(\zeta)) < c$ ($\zeta \in b\Delta$) and $(\cup_{j=1}^m E_j) \cap \varphi_2(b\Delta) = \emptyset$. Since $\lambda > b$ it follows that $\varphi_2(b\Delta) \subset P \setminus (\cup_{j=1}^m E_j)$. Thus $\varphi_3 = F \circ \varphi_2$ is a well defined continuous map from $b\Delta$ to M such that $|\varphi_3(\zeta) - \varphi_2(\zeta)| < \delta$ ($\zeta \in b\Delta$) and such that $\rho(\varphi_3(\zeta)) \geq \rho(\varphi_2(\zeta)) + 2\tau$ ($\zeta \in b\Delta$) $> \lambda + 2\tau$ which implies that $\rho(\varphi_3(\zeta)) > c$ ($\zeta \in b\Delta$). Clearly $|\varphi_3(\zeta) - \varphi^*(\zeta)| < 3\delta$ ($\zeta \in b\Delta$). Let $\omega: \overline{\Delta} \rightarrow \mathbb{C}^N$ be a continuous extension of $\zeta \mapsto \varphi_3(\zeta) - \varphi^*(\zeta)$ from bD to $\overline{\Delta}$ which satisfies $|\omega(\zeta)| < 3\delta$ ($\zeta \in \overline{\Delta}$) and which vanishes identically in a neighbourhood of 0. Then $\tilde{\varphi}_3 = \varphi + \omega: \overline{\Delta} \rightarrow \mathbb{C}^N$ is a continuous map which on bD coincides with φ_3 and in a neighbourhood of 0 coincides with φ . By construction, $|\tilde{\varphi}_3(\zeta) - \varphi(\zeta)| < 3\delta$ ($\zeta \in \overline{\Delta}$) so by (8.2), $\tilde{\varphi}_3(\overline{\Delta}) \subset \Omega$. Define $\psi = \pi \circ \tilde{\varphi}_3$. Then ψ is a continuous map from $\overline{\Delta}$ to M , holomorphic in a neighbourhood of 0, which satisfies $\psi(0) = p$ and $\psi'(0) = \varphi'(0)$. Further, (8.2) implies that $|\psi(\zeta) - \varphi(\zeta)| = |\pi(\tilde{\varphi}_3(\zeta)) - \varphi(\zeta)| < \varepsilon$ ($\zeta \in \overline{\Delta}$). This completes the proof.

9. The induction step

Lemma 9.1 *Let $\rho(p) < a_1 < a_2 < b_1 < b_2 < \infty$. Assume that ρ has at most one critical value on $[a_1, b_2]$ and if c is such a value then $a_2 < c < b_1$.*

Suppose that $f: \overline{\Delta} \rightarrow M$ is a continuous map, holomorphic on Δ and such that $a_1 < \rho(f(\zeta)) < a_2$ ($\zeta \in \overline{\Delta}$).

Given R , $0 < R < 1$, and $\varepsilon > 0$ there are r , $R < r < 1$, and a continuous map $g: \overline{\Delta} \rightarrow M$, holomorphic on Δ and such that

- (i) $b_1 < \rho(g(\zeta)) < b_2$ ($\zeta \in b\Delta$)
- (ii) $\rho(g(t\zeta)) > \rho(f(\zeta)) - \varepsilon$ ($\zeta \in b\Delta$, $r \leq t \leq 1$)
- (iii) $|g(\zeta) - f(\zeta)| < \varepsilon$ ($|\zeta| \leq r$)
- (iv) $g(0) = f(0)$
- (v) $g'(0) = f'(0)$.

Remark In particular, (ii) implies that $\rho(g(\zeta)) > a_1 - \varepsilon$ ($r \leq |\zeta| \leq 1$).

Proof. If ρ has no critical value on $[a_1, b_2]$ then the lemma follows from Lemma 5.3. Suppose that c , $a_2 < c < b_1$, is a critical value of ρ . Let $b_1 < \alpha < \beta < \gamma < b_2$.

Let $\varepsilon > 0$ and let $0 < R < 1$. Choose $\delta > 0$ so small that

$$\{z \in M: \rho(z) \leq \gamma\} + 2\delta\mathbf{B} \subset \Omega, \tag{9.1}$$

$$(\beta^{1/2} + 4\delta)^2 < \gamma \tag{9.2}$$

and that

$$\left. \begin{array}{l} x \in M, \rho(x) \leq \gamma, y \in \mathbb{C}^N, |y - x| < 2\delta \text{ implies that} \\ y \in \Omega \text{ and } |\rho(\pi(y)) - \rho(x)| < \min\{\alpha - b_1, b_2 - \beta, \varepsilon/4\} \end{array} \right\} \tag{9.3}$$

Passing to a smaller δ if necessary we may assume that $\delta < \varepsilon/4$.

By Lemma 8.1 there is a λ , $a_2 < \lambda < c$, such that

$$\left. \begin{aligned} &\text{if } \varphi: \overline{\Delta} \rightarrow M \text{ is a continuous map, holomorphic on } \Delta \text{ and such that} \\ &\lambda < \rho(\varphi(\zeta)) < c \ (\zeta \in b\Delta) \text{ then there is a continuous map } \psi: \overline{\Delta} \rightarrow M, \\ &\text{holomorphic in a neighbourhood of } 0 \text{ such that } |\psi(\zeta) - \varphi(\zeta)| < \delta \ (\zeta \in \overline{\Delta}), \\ &\psi(0) = \varphi(0), \ \psi'(0) = \varphi'(0) \text{ and } \rho(\psi(\zeta)) > c \ (\zeta \in b\Delta) \end{aligned} \right\} \quad (9.4)$$

By Lemma 5.3 there are r , $R < r < 1$, and a continuous map $f_1: \overline{\Delta} \rightarrow M$, holomorphic on Δ , such that

$$\lambda < \rho(f_1(\zeta)) < c \ (\zeta \in b\Delta), \quad (9.5)$$

$$\rho(f_1(t\zeta)) \geq \rho(f_1(\zeta)) - \delta \ (\zeta \in b\Delta, \ r \leq t \leq 1), \quad (9.6)$$

$$|f_1(\zeta) - f_1(\eta)| < \delta \ (|\zeta| \leq r) \quad (9.7)$$

and $f_1(0) = f_1(0)$, $f_1'(0) = f_1'(0)$.

By (9.4) there is a continuous map $f_2: \overline{\Delta} \rightarrow M$, holomorphic in a neighbourhood of 0, such that

$$\rho(f_2(\zeta)) > c, \quad (9.8)$$

$$|f_2(\zeta) - f_1(\zeta)| < \delta \ (\zeta \in \overline{\Delta}) \quad (9.9)$$

and $f_2(0) = f_1(0)$, $f_2'(0) = f_1'(0)$. Note that (9.9), (9.5) and (9.3) imply that $\rho(f_2(\zeta)) < \alpha$ ($\zeta \in b\Delta$). Further, (9.9) holds where f_1 is continuous on $\overline{\Delta}$ and holomorphic on Δ . Now Lemma 5.3 gives an r_1 , $r < r_1 < 1$, and a continuous map $f_3: \overline{\Delta} \rightarrow M$, holomorphic in a neighbourhood of 0, such that

$$\alpha < \rho(f_3(\zeta)) < \beta \ (\zeta \in b\Delta) \quad (9.10)$$

$$\rho(f_3(t\zeta)) \geq \rho(f_3(\zeta)) - \delta \ (\zeta \in b\Delta, \ r_1 \leq t \leq 1) \quad (9.11)$$

$$|f_3(\zeta) - f_3(\eta)| < \delta \ (|\zeta| \leq r_1) \quad (9.12)$$

and $f_3(0) = f_2(0)$, $f_3'(0) = f_2'(0)$, together with a continuous map $F: \overline{\Delta} \rightarrow \mathbb{C}^N$, holomorphic on $\overline{\Delta}$, such that $F(0) = f_3(0)$, $F'(0) = f_3'(0)$ and such that

$$|F(\zeta) - f_3(\zeta)| < 2\delta \ (\zeta \in \overline{\Delta}). \quad (9.13)$$

By (9.10), $|f_3(\zeta)| < \beta^{1/2}$ ($\zeta \in b\Delta$) so by (9.13), $|F(\zeta)| < \beta^{1/2} + 2\delta$ ($\zeta \in b\Delta$). Since F is holomorphic on Δ the maximum principle implies that $|F(\zeta)| < \beta^{1/2} + 2\delta$ ($\zeta \in \overline{\Delta}$) so by (9.13), $|f_3(\zeta)| < \beta^{1/2} + 4\delta$ ($\zeta \in \overline{\Delta}$). By (9.2) it follows that $\rho(f_3(\zeta)) < (\beta^{1/2} + 4\delta)^2 < \gamma$ ($\zeta \in \overline{\Delta}$). Now (9.13) and (9.1) imply that $F(\overline{\Delta}) \subset \Omega$.

Define $g = \pi \circ F$. The map g is a well defined continuous map from $\overline{\Delta}$ to M which is holomorphic on Δ . Now (9.13) and (9.3) imply that

$$|\rho(g(\zeta)) - \rho(f_3(\zeta))| < \min \{\alpha - b_1, b_2 - \beta, \varepsilon/4\} \ (\zeta \in \overline{\Delta}), \quad (9.14)$$

which, by (9.10) implies (i). (iv) is obvious and (v) follows from the fact that $F'(0) = f'(0)$ and that $\pi|_M = \text{id}$. Further, (9.7), (9.9) and (9.12) imply (iii) since $\delta < \varepsilon/4$.

Since f_1 is holomorphic on Δ , (9.5) implies that $|f_1(\zeta)| < c^{1/2}$ ($\zeta \in \overline{\Delta}$) so (9.9) and (9.2) imply that $|f_2(\zeta)|^2 < (c^{1/2} + \delta)^2 < (\beta^{1/2} + 4\delta)^2 < \gamma$ ($\zeta \in \overline{\Delta}$), that is,

$$\rho(f_2(\zeta)) < \gamma \quad (\zeta \in \overline{\Delta}). \quad (9.15)$$

Let $\zeta \in b\Delta$ and let $r \leq t \leq 1$. Assume first that $t \geq r_1$. By (9.14), $\rho(g(t\zeta)) \geq \rho(f_3(t\zeta)) - \varepsilon/4$. Since $t \geq r_1$, (9.11) implies that $\rho(g(t\zeta)) \geq \rho(f_2(\zeta)) - 2\varepsilon/4$. By (9.15), (9.9) and (9.3), $\rho(f_2(\zeta)) > \rho(f_1(\zeta)) - \varepsilon/4$ so $\rho(g(t\zeta)) \geq \rho(f_1(\zeta)) - 3\varepsilon/4$ and (9.6) implies that $\rho(g(t\zeta)) \geq \rho(f(\zeta)) - \varepsilon$.

Now let $r \leq t \leq r_1$. By (9.12) and (9.9), $|f_3(t\zeta) - f_1(t\zeta)| < 2\delta$. Since $\rho(f_1(\zeta)) < \gamma$ ($\zeta \in \overline{\Delta}$), (9.6) and (9.3) imply that $\rho(f_3(t\zeta)) \geq \rho(f(\zeta)) - \varepsilon/2$ and (9.14) implies that $\rho(g(t\zeta)) \geq \rho(f(\zeta)) - 3\varepsilon/4 > \rho(f(\zeta)) - \varepsilon$. This proves (ii). The proof is complete.

10. The completion of the proof

Let X be a vector tangent to M at p .

Choose $a_j, b_j, j \in \mathbb{N} \cup \{0\}$, such that $a_0 < \rho(p) < b_0 < a_1 < b_1 < \dots < a_n < b_n < a_{n+1} < \dots, \lim a_n = +\infty$, such that for each $j \in \mathbb{N} \cup \{0\}$ there is no critical point of ρ on $[a_j, b_j]$, and such that for each $j \in \mathbb{N}$ there is at most one critical point on (b_{j-1}, a_j) .

Choose a decreasing sequence $\delta_n > 0$ such that for each $n \in \mathbb{N}$

$$z \in M, \rho(z) \leq b_n, w \in M, |z - w| < \delta_n \text{ imply that } |\rho(z) - \rho(w)| < 1. \quad (10.1)$$

Choose $\lambda > 0$ so small that $p + \lambda\zeta X \in \Omega$ ($\zeta \in \overline{\Delta}$) and that $a_0 < \rho(\pi(p + \lambda\zeta X)) < b_0$ ($\zeta \in \overline{\Delta}$) and define $f_0(\zeta) = \pi(p + \lambda\zeta X)$ ($\zeta \in \overline{\Delta}$). Then $f_0: \overline{\Delta} \rightarrow M$ is a continuous map, holomorphic on Δ which satisfies $f_0(0) = p, f'_0(0) = \lambda X$. Using Lemma 9.1 one can construct inductively a sequence of continuous maps $f_n: \overline{\Delta} \rightarrow M$, holomorphic on Δ , and an increasing sequence r_n of positive numbers, converging to 1, such that for each $n \in \mathbb{N}$,

- (i) $a_n < \rho(f_n(\zeta)) < b_n$ ($\zeta \in b\Delta$)
- (ii) $\rho(f_n(\zeta)) > a_{n-1} - 1$ ($r_n \leq |\zeta| \leq 1$)
- (iii) $|f_n(\zeta) - f_{n-1}(\zeta)| < \delta_n/2^n$ ($|\zeta| \leq r_n$)
- (iv) $f_n(0) = p$
- (v) $f'_n(0) = \lambda X$.

By (iii), f_n converges uniformly on compacta in Δ , so $f = \lim f_n$ is holomorphic on Δ .

Since $f_n(\overline{\Delta}) \subset M$ and since M is closed it follows that $f(\Delta) \subset M$. By (iv), $f(0) = p$ and by (v), $f'(0) = \lambda X$.

Suppose that $n \in \mathbb{N}$ and that $r_n \leq |\zeta| \leq r_{n+1}$. Since $|\zeta| \leq r_{n+1}$, (iii) gives $|f(\zeta) - f_n(\zeta)| \leq |f_{n+1}(\zeta) - f_n(\zeta)| + |f_{n+2}(\zeta) - f_{n+1}(\zeta)| + \dots \leq \delta_{n+1}/2^{n+1} + \delta_{n+2}/2^{n+2} + \dots \leq \delta_n$. Note that (i) and the fact that f is holomorphic on Δ imply that $\rho(f_n(\zeta)) < b_n$ ($\zeta \in \overline{\Delta}$) so the preceding discussion together with (10.1) implies that $|\rho(f(\zeta)) - \rho(f_n(\zeta))| < 1$ ($r_n \leq |\zeta| \leq r_{n+1}$). Since $\lim_{n \rightarrow \infty} r_n = 1$ and since $\lim_{n \rightarrow \infty} a_n = +\infty$ it follows that $f: \Delta \rightarrow M$ is a proper map. This completes the proof.

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References

- [FG] F. Forstnerič, J. Globevnik: Discs in pseudoconvex domains.
Comment. Math. Helv. 67 (1992) 129-145
- [G1] J. Globevnik: Boundary interpolation and proper holomorphic maps from the disc to the ball.
Math. Z. 198 (1988)143-150
- [G2] J. Globevnik: Discs in the ball containing given discrete sets.
Math. Ann. 281 (1988) 87-96
- [G3] J. Globevnik: Relative embeddings of discs into convex domains.
Invent. Math. 98 (1989)331-350
- [G4] J. Globevnik: On moduli of boundary values of holomorphic functions.
Math. Z. 208 (1991) 627-633
- [GR] R. C. Gunning, H. Rossi: Analytic functions of several complex variables.
Prentice-Hall, Englewood Cliffs, 1965
- [Hi] M. W. Hirsch: Differential topology.
Springer-Verlag, New York - Heidelberg, 1976
- [Ho] Lars Hörmander: An introduction to complex analysis in several variables.
Van Nostrand, Princeton, 1966
- [HL] G. M. Henkin, J. Leiterer: Theory of functions on complex manifolds.
Akademie-Verlag, Berlin 1984
- [Mi] J. Milnor: Morse theory.
Princeton University Press, Princeton 1963
- [Po] Ch. Pommerenke: Boundary behaviour of conformal maps.
Grundle. der Math. Wiss. 299. Springer-Verlag, Heidelberg-New York 1992

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