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COMMON INVARIANT SUBSPACES FOR COLLECTIONS OF OPERATORS

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Common invariant subspaces for collections of operators

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Abstract

Let $\mathcal{C}$ be a collection of bounded operators on a Banach space X of dimension at least two. We say that $\mathcal{C}$ is finitely quasinilpotent at a vector $x_0 \in X$ whenever for any finite subset $\mathcal{F}$ of $\mathcal{C}$ the joint spectral radius of $\mathcal{F}$ at x_0 is equal 0. If such collection $\mathcal{C}$ contains a non-zero compact operator, then $\mathcal{C}$ and its commutant $\mathcal{C}'$ have a common non-trivial invariant subspace. If, in addition, $\mathcal{C}$ is a collection of positive operators on a Banach lattice, then $\mathcal{C}$ and $\mathcal{C}'$ have a common non-trivial closed ideal. This result and a recent remarkable result of Turovskii imply the following extension of the famous result of de Pagter. Let $\mathcal{S}$ be a non-zero multiplicative semigroup of quasinilpotent compact positive operators on a Banach lattice of dimension at least two. Then $\mathcal{S}$ and its commutant $\mathcal{S}'$ have a common non-trivial invariant closed ideal.

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1. Introduction

Throughout the paper, let X be a real or complex Banach space of dimension at least 2. A *subspace* of X means a closed linear manifold of X. By an *operator* on X we mean a bounded linear transformation from X into itself. We denote by $\mathcal{B}(X)$ the Banach algebra of all operators on X. Let $\mathcal{C}$ be a collection of operators of $\mathcal{B}(X)$. We say that $\mathcal{C}$ is *reducible* if there exists a subspace of X, other than $\{0\}$ and X, which is invariant under every member of $\mathcal{C}$. If there exists even a maximal subspace chain (i.e., a maximal totally ordered set of subspaces) whose elements are invariant under every member of $\mathcal{C}$, then $\mathcal{C}$ is said to be *triangularizable*. The commutant of $\mathcal{C}$ is denoted by $\mathcal{C}'$.

Let B and $\mathcal{C}$ be subsets of X and $\mathcal{B}(X)$ respectively. Define

$$\|B\| := \sup\{\|x\| : x \in B\} \quad \text{and} \quad \|\mathcal{C}\| := \sup\{\|T\| : T \in \mathcal{C}\} .$$

Denote also $\mathcal{C}B := \{Tx : T \in \mathcal{C}, x \in B\}$. If B is a singleton $\{x\}$, we shall write $\mathcal{C}x$ instead of $\mathcal{C}\{x\}$. If $\mathcal{D}$ is another subset of $\mathcal{B}(X)$, we will write $\mathcal{C}\mathcal{D} := \{TS : T \in \mathcal{C}, S \in \mathcal{D}\}$. Similarly, if $\mathcal{D} = \{S\}$, we write $\mathcal{C}S$ instead of $\mathcal{C}\{S\}$. The powers $\mathcal{C}^n$ are defined inductively by $\mathcal{C}^1 := \mathcal{C}$, $\mathcal{C}^n := \mathcal{C}\mathcal{C}^{n-1}$ for $n \geq 2$.

Let $\mathcal{C}$ be a collection of operators on X. *The Rota-Strang spectral radius* or *the joint spectral radius* $\hat{r}(\mathcal{C})$ of $\mathcal{C}$ is defined by

$$\hat{r}(\mathcal{C}) = \inf_{n \in \mathbb{N}} \|\mathcal{C}^n\|^{1/n} .$$

If $\mathcal{C}$ is bounded, then the infimum is actually the limit (see [19]). Note that Yu. Turovskii [23] also studied the above notion. Here we introduce *the joint spectral radius of* $\mathcal{C}$ *at a vector* $x \in X$ by the formula

$$\hat{r}(\mathcal{C}, x) = \limsup_{n \to \infty} \|\mathcal{C}^n x\|^{1/n} .$$

For finite subsets $\mathcal{C}$ this notion has already been introduced in [8]. Following [21] a collection $\mathcal{C}$ is said to be *finitely quasinilpotent* if $\hat{r}(\mathcal{F}) = 0$ for any finite subset $\mathcal{F}$ of $\mathcal{C}$. In a similar manner we introduce the following concept.

Definition 1.1 A collection $\mathcal{C}$ of operators on X is *finitely quasinilpotent at a vector* $x_0 \in X$ whenever $\hat{r}(\mathcal{F}, x_0) = 0$ for any finite subset $\mathcal{F}$ of $\mathcal{C}$.

The following remarkable result has been shown very recently by Yu. Turovskii [22]. This theorem answers the long standing open question.

Theorem 1.2 *Let $\mathcal{S}$ be a non-zero multiplicative semigroup of quasinilpotent compact operators on X. Then $\mathcal{S}$ and $\mathcal{S}'$ have a common non-trivial invariant subspace.*

As a corollary Yu. Turovskii has proved the following

Theorem 1.3 *Every multiplicative semigroup of quasinilpotent compact operators on X is finitely quasinilpotent.*

In 1986 B. de Pagter [16] proved the long standing conjecture that every positive quasinilpotent compact operator on a Banach lattice (of dimension of at least 2) has a non-trivial invariant closed ideal. His proof employed a clever modification of Hilden's proof of the famous Lomonosov's invariant subspace theorem. In this paper we extend his result to multiplicative semigroups of positive operators. We first generalized some results of the paper [2] to a collection of positive operators on a Banach lattice that is finitely quasinilpotent at a non-zero positive vector. Note that the results of [1] and [2] have been slightly improved in the survey paper [4]. We then apply Theorem 1.3 and one of our result to prove that every multiplicative semigroup of quasinilpotent compact positive operators have a common non-trivial invariant closed ideal. These results are given in Section 4, while in Section 3 we prove some similar results for a collection of operators on a Banach space that is finitely quasinilpotent at a non-zero vector. These results can be added to a number of results on reducibility of collections (in particular, semigroups) of operators on Banach spaces; see e.g. [6], [17], [10], [12], [18], and [7].

2. Auxiliary results

Throughout of this paper, let E be a real or complex Banach lattice of dimension at least two. The cone of positive elements of E is denoted by E^+. A linear manifold I of E is said to be an *(order) ideal* whenever $|x| \leq |y|$ and $y \in I$ imply that $x \in I$. For the terminology not explained in the text we refer to the standard books [13], [20], [24], [5], and [14].

In the proof of one of our results we need the concept of a quasi-interior set of a normed Riesz space. Our definition is inspired by the definition of a quasi-interior point.

Definition 2.1 An additive set D of positive vectors in a normed Riesz space L is said to be a *quasi-interior set* whenever the ideal generated by D is norm dense in L.

Evidently, a positive vector $u \in L$ is a quasi-interior point if and only if the set $\{nu : n \in \mathbb{N}\}$ is a quasi-interior set. The quasi-interior sets can be characterized in a similar manner as quasi-interior points (see [5, Theorem 15.13]). Although the proof of this result follows similar lines as the proof of [5, Theorem 15.13], we give here for the sake of completeness.

Proposition 2.2 *For an additive set D of positive vectors in a normed Riesz space L the following assertions are equivalent:*

(a) D is a quasi-interior set.

(b) For each positive vector $x \in L$ the net $\{x - x \wedge u\}_{u \in D}$ norm converges to 0.

(c) For each non-zero positive continuous linear functional f on L there is some $u \in D$ such that $f(u) > 0$.

Proof. (a) $\Rightarrow$ (b) Let I be the ideal generated by D. Since D is additive, we have $I = \{y \in L : \exists u \in D \text{ such that } |y| \leq u\}$. Choose a positive vector $x \in L$ and $\epsilon > 0$, and pick a vector $y \in I$ with $\|x - y\| < \epsilon$. Let $z := x \wedge y^+ \in I$. Then $0 \leq z \leq x$ and $z \geq x \wedge y$. Hence

$$0 \leq x - z \leq x - x \wedge y \leq x \vee y - x \wedge y = |x - y| \ ,$$

where we have used the equality [5, Theorem 1.4.(2)]. It follows that $\|x - z\| \leq \|x - y\| < \epsilon$. Since D is additive and $z \in I$, there exists $u \in D$ such that $z \leq u$. Then for all $v \in D$ with $v \geq u$ we have

$$0 \leq x - x \wedge v \leq x - x \wedge u \leq x - z \ ,$$

and so $\|x - x \wedge v\| < \epsilon$. This shows that the net $\{x - x \wedge u\}_{u \in D}$ norm converges to 0.

(b) $\Rightarrow$ (c) Assume on the contrary that $f(u) = 0$ for all $u \in D$, and pick $x \in L^+$. Since $0 \leq x \wedge u \leq u$ for any $u \in D$, we conclude that $f(x \wedge u) = 0$. Since the net $\{x - x \wedge u\}_{u \in D}$ norm converges to 0, this implies that $f(x) = 0$, and so $f = 0$ which is a contradiction.

(c) $\Rightarrow$ (a) If the ideal I generated by D is not dense in E, the Hahn-Banach theorem gives a non-zero continuous linear functional f which vanishes on I.

With no loss of generality we can assume that the positive part f^+ of f is non-zero. Then for all $u \in D$ it holds

$$f^+(u) = \sup\{f(x) : x \in L, 0 \leq x \leq u\} = \sup\{f(x) : x \in I, 0 \leq x \leq u\} = 0 \ ,$$

where we have used the fact that $x \in L$ and $0 \leq x \leq u$ imply that $x \in I$. This contradiction completes the proof. □

We will now show the Banach lattice analog of the Triangularization Lemma that was implicitly used in many results, although it has been explicitly stated only in [18].

The following notion has been introduced in [11]. A collection $\mathcal{C}$ of operators on E is *ideal-triangularizable* if there is a chain of closed ideals which is maximal in the lattice of all closed ideals of E invariant under $\mathcal{C}$. It has been shown in [9, Proposition 1.2] that such chain is also maximal in the lattice of all closed subspaces of E. So, a collection $\mathcal{C}$ of operators on E is ideal-triangularizable if and only if it is triangularizable with respect to the subspace chain consisting from closed ideals.

Let $\mathcal{C}$ be a collection of operators on E, let I and J be closed ideals invariant under $\mathcal{C}$ satisfying $J \subseteq I$. Then $\mathcal{C}$ induces a collection $\hat{\mathcal{C}}$ of operators on the quotient Banach lattice I/J as follows. For each $T \in \mathcal{C}$ the operator $\hat{T}$ is defined on I/J by

$$\hat{T}(x + J) = Tx + J \ .$$

Because of the invariance I and J the operator $\hat{T}$ is a well-defined operator on I/J. Any such $\hat{\mathcal{C}}$ is called a *collection of ideal-quotients* of the collection $\mathcal{C}$. A set of properties $\mathcal{P}$ is said to be *inherited by ideal-quotients* if every collection of ideal-qoutients of a collection of operators satisfying properties $\mathcal{P}$ also satisfies the same properties.

Lemma 2.3 *(The Triangularization Lemma for Banach lattices) Let $\mathcal{P}$ be a set of properties inherited by ideal-quotients. If every collection of operators on a Banach lattice of dimension greater than one which satisfies $\mathcal{P}$ has a common non-trivial invariant closed ideal, then every such collection is ideal-triangularizable.*

Proof. Let $\mathcal{C}$ be a collection of operators on E satisfying $\mathcal{P}$. An application of Zorn's Lemma gives a chain $\mathcal{F}$ of closed ideals of E that is maximal in the lattice of all closed ideals invariant under $\mathcal{C}$. Let I and J be closed ideals of $\mathcal{F}$ with $J \subseteq I$. If the dimension of I/J is greater than one, then the set $\hat{\mathcal{C}}$ of quotients of $\mathcal{C}$ with respect to I/J has a common non-trivial invariant closed ideal $\hat{K}$. Since the natural projection p from I to I/J is an onto continuous lattice homomorphism, the set $K = \{x \in I : p(x) \in \hat{K}\}$ is a closed ideal of E that is invariant under $\mathcal{C}$ and that is properly between I and J. This shows that the chain $\mathcal{F}$ is maximal in the lattice of all closed ideals of E. □

We will also need the following.

Lemma 2.4 *Let K be a compact operator on a Banach lattice E, and let $x_0 \in E$ be such that $Kx_0 \neq 0$. Denote by L the ideal generated by the range of K. Then there exists an operator V from the closure $\overline{L}$ to E such that VKx_0 is a non-zero positive vector and $|VKx| \leq |Kx|$ for all $x \in E$.*

Proof. By the definition,

$$L = \{y \in E : \exists x_1, x_2, \ldots, x_n \in E \ \text{ such that } \ |y| \leq \sum_{m=1}^{n} |Kx_m|\} \, .$$

The compactness of K implies that the range of K is separable. So, there exists a countable subset $\{Kx_n\}_{n \in \mathbb{N}}$ of non-zero vectors that is norm dense in the range of K. Put

$$u := \sum_{n=1}^{\infty} \frac{|Kx_n|}{2^n \|Kx_n\|} \in \overline{L} ,$$

where the series norm and order converges (see [24, Lemma 101.1]). Then the ideal I_u generated by u is contained in $\overline{L}$, and so $\overline{I_u} \subseteq \overline{L}$. For the proof of the reverse inclusion note that $Kx_n \in I_u$ for each n, and so the range of K is contained in $\overline{I_u}$. This implies that $\overline{L} \subseteq \overline{I_u}$, and so $\overline{L} = \overline{I_u}$.

Since u is a quasi-interior set of $\overline{L}$, the existence of the operator V now follows from [2, Lemma 2.3]. □

3. Invariant subspaces for collections of operators on a Banach space

Let $\mathcal{C}$ be a collection of operators on X. By $Q_f(\mathcal{C})$ we denote the set of all vectors of X at which $\mathcal{C}$ is finitely quasinilpotent.

Proposition 3.1 *The set $Q_f(\mathcal{C})$ is a linear manifold that is invariant under $\mathcal{C}$ and $\mathcal{C}'$.*

Proof. It is obvious that $Q_f(\mathcal{C})$ is closed under multiplication by scalars. Pick x, $y \in Q_f(\mathcal{C})$, and a finite subset $\mathcal{F}$ of $\mathcal{C}$. Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|\mathcal{F}^n x\| < \epsilon^n$ and $\|\mathcal{F}^n y\| < \epsilon^n$ for all $n \geq n_0$. Then $\|\mathcal{F}^n(x+y)\| \leq \|\mathcal{F}^n x\| + \|\mathcal{F}^n y\| < 2\epsilon^n$ for all $n \geq n_0$, and so $\hat{r}(\mathcal{F}, x+y) \leq \epsilon$. This shows that $x + y \in Q_f(\mathcal{C})$.

Now choose $T \in \mathcal{C}$, $x \in Q_f(\mathcal{C})$, and a finite subset $\mathcal{F}$ of $\mathcal{C}$. Put $\mathcal{F}_0 = \mathcal{F} \cup \{T\}$. Since $\|\mathcal{F}^n(Tx)\| \leq \|\mathcal{F}_0^{n+1} x\|$, we have $\hat{r}(\mathcal{F}, Tx) \leq \hat{r}(\mathcal{F}_0, x) = 0$, and so $Q_f(\mathcal{C})$ is invariant under $\mathcal{C}$. If $S \in \mathcal{C}'$, then $\|\mathcal{F}^n(Sx)\| \leq \|S\|\|\mathcal{F}^n x\|$ implies that $\hat{r}(\mathcal{F}, Sx) \leq \hat{r}(\mathcal{F}, x) = 0$, so that $Q_f(\mathcal{C})$ is invariant under $\mathcal{C}'$ as well. $\square$

Proposition 3.2 *Let $\mathcal{C}$ be a collection of operators on X that is finitely quasinilpotent at a vector $x_0 \in X$. Then the algebra generated by $\mathcal{C}\mathcal{C}'$ is also finitely quasinilpotent at x_0.*

Proof. Let $\mathcal{S}$ be the multiplicative semigroup generated by $\mathcal{C}$, and let $\mathcal{A}$ be the subalgebra of $\mathcal{B}(X)$ generated by $\mathcal{C}\mathcal{C}'$. We first show that the semigroup $\mathcal{S}$ is finitely quasinilpotent at x_0. To prove this, pick a finite subset $\mathcal{F}$ of $\mathcal{S}$. Then there exists an integer m and a finite subset $\mathcal{G}$ of $\mathcal{C}$ such that $\mathcal{F} \subseteq \cup_{k=1}^m \mathcal{G}^k$. Fix $0 < \epsilon < 1$. Since $\mathcal{C}$ is finitely quasinilpotent at a vector x_0, there exists n_0 such that $\|\mathcal{G}^n x_0\| < \epsilon^n$ for all $n \geq n_0$. Then

$$\|\mathcal{F}^n x_0\| \leq \| \left(\cup_{k=1}^m \mathcal{G}^k\right)^n x_0\| =$$

$$= \| \left(\cup_{k=n}^{mn} \mathcal{G}^k\right) x_0\| \leq \max\{\|\mathcal{G}^k x_0\| : n \leq k \leq mn\} < \epsilon^n$$

for all $n \geq n_0$. This shows that $\mathcal{S}$ is finitely quasinilpotent at x_0.

Now choose a finite subset $\mathcal{F} = \{A_1, A_2, \ldots, A_p\}$ of $\mathcal{A}$. Then there exist a finite subset $\mathcal{G} = \{S_1, \ldots, S_N\}$ of $\mathcal{S}$ and operators $C_{k,j} \in \mathcal{C}'$ ($k = 1, \ldots, p$, $j = 1, \ldots, N$) such that

$$A_k = \sum_{j=1}^{N} C_{k,j}\, S_j$$

for all $k = 1, \ldots, p$. Put $c = \max\{\|C_{k,j}\| : k = 1, \ldots, p, j = 1, \ldots, N\}$. Estimating the number $\|\mathcal{F}^n x_0\|$ in the obvious manner we obtain at the end that

$$\|\mathcal{F}^n x_0\| \leq c^n N^n \, \|\mathcal{G}^n x_0\|$$

for all n. Since the semigroup $\mathcal{S}$ is finitely quasinilpotent at x_0, it follows that the algebra $\mathcal{A}$ is also finitely quasinilpotent at x_0. This completes the proof. $\square$

The proof of our first result is based on the famous Lomonosov-Hilden technique (see e.g. [15]). A similar result for multiplicative semigroups of operators has been proved in [8], as an attempt to prove Theorem 1.2.

Theorem 3.3 *Let $\mathcal{C}$ be a collection in $\mathcal{B}(X)$ that is finitely quasinilpotent at a non-zero vector, and let $K \in \mathcal{C}$ be a non-zero compact operator. Then the algebra $\mathcal{A}$ generated by $\mathcal{C}$ and $\mathcal{C}'$ is reducible.*

Proof. It follows from Proposition 3.1 that the closure $\overline{Q_f(\mathcal{C})}$ is a subspace that is invariant under $\mathcal{A}$. Because of our assumption on $\mathcal{C}$ it is also non-zero. Thus, if $Q_f(\mathcal{C})$ is not dense in X, we are done.

Assume now that $\overline{Q_f(\mathcal{C})} = X$. Since K is a non-zero operator, there exists $x_0 \in X$ such that $Kx_0 \neq 0$ and $\mathcal{C}$ is finitely quasinilpotent at x_0. There is no loss of generality in assuming that $\|K\| = 1$, since we can assume that $\mathcal{C} = \mathbb{R}^+\mathcal{C}$. Replacing x_0 by λx_0 for an appropriate scalar $\lambda > 0$, we can also assume that $\|Kx_0\| > 1$, and so $\|x_0\| > 1$. Let $U = \{x \in X : \|x - x_0\| \leq 1\}$ be the closed unit ball centered at x_0. Obviously, 0 is not in U nor in the closure $\overline{K(U)}$. For each fixed $y \in X$ the set

$$\mathcal{A}[y] = \{Ay : \ A \in \mathcal{A}\}$$

is a linear manifold in X which is invariant under every member of $\mathcal{A}$. Since the identity operator on X is in $\mathcal{A}$, we have $\mathcal{A}[y] \neq \{0\}$ for all $y \neq 0$. Therefore, the proof will be finished after showing that there exists $y \neq 0$ such that $\mathcal{A}[y]$ is not dense in X. Assume on the contrary that $\mathcal{A}[y]$ is dense in X for all $y \neq 0$. Then for each $y \neq 0$ there is an $A \in \mathcal{A}$ such that $Ay \in \operatorname{int} U$, where

$\operatorname{int} U$ denotes the interior of U. We therefore have

$$\overline{K(U)} \subseteq X \setminus \{0\} \subseteq \bigcup_{A \in \mathcal{A}} A^{-1}(\operatorname{int} U) .$$

Since $\overline{K(U)}$ is a compact set in X, there exists a finite set $\{A_1, \ldots, A_p\} \subseteq \mathcal{A}$ such that

$$\overline{K(U)} \subseteq \bigcup_{i=1}^{p} A_i^{-1}(\operatorname{int} U) .$$

Now, it follows from $Kx_0 \in K(U)$ that $Kx_0 \in A_{i_1}^{-1}(U)$ for some $i_1 \in \{1, \ldots, p\}$, and hence $x_1 := A_{i_1} K x_0 \in U$. Then $KA_{i_1}Kx_0 \in K(U)$ implies that $KA_{i_1}Kx_0 \in A_{i_2}^{-1}(U)$ for some i_2, and so $x_2 := A_{i_2}KA_{i_1}Kx_0 \in U$. Proceeding with this "ping-pong" of Hilden's, we obtain after n steps an integer $i_n \in \{1, 2, \ldots, p\}$ such that

$$x_n := (A_{i_n}K)(A_{i_{n-1}}K) \cdots (A_{i_1}K)x_0 \in U .$$

Now observe that the set $\mathcal{A}K$ is contained in the algebra generated by $\mathcal{C}\mathcal{C}'$. Since this algebra is finitely quasinilpotent at x_0 by Proposition 3.2, $\mathcal{A}K$ is finitely quasinilpotent at x_0 as well. It follows that $\lim_{n\to\infty} \|x_n\|^{1/n} = 0$, so that $\lim_{m\to\infty} \|x_n\| = 0$. This implies that $0 \in U$, contradicting the above and completing the proof of the theorem. □

A subset $\mathcal{I}$ of a multiplicative semigroup $\mathcal{S}$ is said to be a *semigroup ideal* if ST and TS belong to $\mathcal{I}$ for all $S \in \mathcal{S}$ and $T \in \mathcal{I}$. The following result is a well-known tool for problems concerning reducibility of semigroups (see [17]).

Lemma 3.4 *Let $\mathcal{S}$ be a multiplicative semigroup of operators on X, and let $\mathcal{I}$ be a non-zero semigroup ideal of $\mathcal{S}$. If $\mathcal{I}$ is reducible, then $\mathcal{S}$ is reducible.*

Using Lemma 3.4 we now show the next companion of Theorem 3.3.

Theorem 3.5 *Let $\mathcal{C} \neq \{0\}$ be a collection in $\mathcal{B}(X)$ that is finitely quasinilpotent at a non-zero vector. Let $K \in \mathcal{C}'$ be a non-zero compact operator. Then the algebra $\mathcal{A}$ generated by $\mathcal{C}$ and $\mathcal{C}'$ is reducible.*

Proof. As in the proof of Theorem 3.3 we must consider only the case when $\overline{Q_f(\mathcal{C})} = X$. Since K is a non-zero operator, there exists a vector $x_0 \in X$ such that $Kx_0 \neq 0$ and $\mathcal{C}$ is finitely quasinilpotent at x_0. Let $\operatorname{Ker} \mathcal{C}$ be the intersection of all kernels of operators from $\mathcal{C}$. If $\operatorname{Ker} \mathcal{C}$ is non-zero, this is a non-trivial subspace invariant under $\mathcal{A}$. So assume that $\operatorname{Ker} \mathcal{C} = \{0\}$. Let $\mathcal{I}$ be the semigroup ideal of $\mathcal{A}$ generated by $\mathcal{C}K$. Since $\mathcal{I}$ is contained in the algebra generated by $\mathcal{C}\mathcal{C}'$ which is finitely quasinilpotent at x_0 by Proposition 3.2, it is finitely quasinilpotent at x_0 as well. Since there exists $T \in \mathcal{C}$ such that $TKx_0 \neq 0$, and hence $TK \in \mathcal{I}$ is a non-zero compact operator, the ideal $\mathcal{I}$ is reducible by Theorem 3.3. Now Lemma 3.4 implies that $\mathcal{A}$ is reducible.
$\square$

4. Invariant closed ideals for collections of operators on a Banach lattice

In this section we show the Banach lattice analogs of the results from the previous section. Our results extend some theorems from [2]; see also the survey paper [4].

Let T, B be operators on E with B positive. Following [2] we say that T is *dominated* by B whenever $|Tx| \leq B|x|$ for all $x \in E$. When E is a Dedekind complete Banach lattice, an operator T on E is dominated by a positive operator B if and only if T is regular and $|T| \leq B$. This observation has been already made in [2].

Let $\mathcal{C}$ be a collection of positive operators on E. We denote by $\mathcal{C}^\sharp$ the set of all positive operators C on E such that $TC \leq CT$ for all $T \in \mathcal{C}$. One can verify easily that $\mathcal{C}^\sharp$ is a multiplicative and an additive semigroup. By $\mathcal{Q}_f(\mathcal{C})$ we denote the set of all vectors $x \in E$ such that $\mathcal{C}$ is finitely quasinilpotent at $|x|$. The basic property of Banach lattices shows that $x \in \mathcal{Q}_f(\mathcal{C})$ and $|y| \leq |x|$ imply that $y \in \mathcal{Q}_f(\mathcal{C})$. Moreover, the following analog of Proposition 3.1 holds.

Proposition 4.1 *The set $\mathcal{Q}_f(\mathcal{C})$ is an order ideal of E that is invariant under $\mathcal{C}$ and $\mathcal{C}^\sharp$.*

The proof of the next result is only a slight modification of the proof of Proposition 3.2, so that we omit it.

Proposition 4.2 *Let $\mathcal{C}$ be a collection of positive operators on E that is finitely quasinilpotent at a non-zero vector $x_0 \in E^+$. Let $\mathcal{S}$ be the multiplicative semigroup generated by $\mathcal{C}$, and let $\mathcal{A}$ be the algebra of all operators on E such that each $T \in \mathcal{A}$ is dominated by some operator of the form $\sum_{j=1}^n C_j S_j$ with $C_j \in \mathcal{C}^\sharp$ and $S_j \in \mathcal{S}$. Then $\mathcal{A}$ is also finitely quasinilpotent at x_0.*

We are now in position to show the analog of Theorem 3.3 in the case of Banach lattices. We thus extend [2, Theorem 4.1]. As already observed in [2], the order structure allows us to relax the compactness assumption. Recall that an operator T on a Banach lattice E is said to be AM-compact whenever T maps order intervals of E onto norm precompact subsets of E. Evidently, every compact operator is also AM-compact.

Theorem 4.3 *Let $\mathcal{C}$ be a collection of positive operators on E that is finitely quasinilpotent at a non-zero positive vector. Assume that there is some*

$T_0 \in \mathcal{C}$ that dominates a non-zero AM-compact operator. Then $\mathcal{C}$ and $\mathcal{C}^\sharp$ have a common non-trivial invariant closed ideal.

Proof. By Proposition 4.1, the closure $\overline{\mathcal{Q}_f(\mathcal{C})}$ is a closed ideal that is invariant under $\mathcal{C}$ and $\mathcal{C}^\sharp$. It is also non-zero by the assumption on $\mathcal{C}$. So, if $\overline{\mathcal{Q}_f(\mathcal{C})} \neq E$, the proof is finished.

Assume now that $\overline{\mathcal{Q}_f(\mathcal{C})} = E$. Let K be a non-zero compact operator on E that is dominated by T_0, i.e., $|Kx| \leq T_0|x|$ for all $x \in E$. Since K is a non-zero operator and $\overline{\mathcal{Q}_f(\mathcal{C})} = E$, there exists $x_0 \in E^+$ such that $Kx_0 \neq 0$ and $\mathcal{C}$ is finitely quasinilpotent at x_0. As in the proof of Theorem 3.3 there is no loss of generality in assuming that $\|K\| = 1$, $\|Kx_0\| > 1$, and $\|x_0\| > 1$. Let $U = \{x \in X : \|x - x_0\| \leq 1, 0 \leq x \leq x_0\}$.

Let $\mathcal{S}$ be the multiplicative semigroup generated by $\mathcal{C}$, and let $\mathcal{P}$ be the multiplicative and the additive semigroup of all positive operators on E such that each $T \in \mathcal{P}$ is dominated by some operator of the form $\sum_{j=1}^n C_j S_j$ with $C_j \in \mathcal{C}^\sharp$ and $S_j \in \mathcal{S}$. It follows from Proposition 4.2 that $\mathcal{P}$ is finitely quasinilpotent at x_0.

For each non-zero vector $x \in E^+$ we denote by $J[x]$ the order ideal generated by the set $\{Ax : A \in \mathcal{P}\}$, that is,

$$J[x] = \{y \in E : |y| \leq Ax \ \text{ for some } \ A \in \mathcal{P}\} \ .$$

We claim that $J[x]$ is invariant under $\mathcal{C}$ and $\mathcal{C}^\sharp$. To this end, pick $y \in J[x]$, $T \in \mathcal{C}$, and $C \in \mathcal{C}^\sharp$. Then there exist $S_1, \ldots, S_n \in \mathcal{S}$ and $C_1, \ldots, C_n \in \mathcal{S}^\sharp$ such that

$$|y| \leq (C_1S_1 + \ldots + C_nS_n)x \ .$$

Hence

$$|Ty| \leq T|y| \leq T(C_1S_1 + \ldots + C_nS_n)x \leq (C_1TS_1 + \ldots + C_nTS_n)x \ .$$

Since $TS_i \in \mathcal{S}$, we conclude that $Ty \in J[x]$. In a similar manner we show that $Cy \in J[x]$ using the observation that $\mathcal{C}^\sharp$ is a multiplicative semigroup. If $J[x] = 0$ for some non-zero $x \in E^+$, the closed ideal generated by x is a non-trivial closed ideal invariant under $\mathcal{C}$ and $\mathcal{C}^\sharp$. Thus, the theorem will be proved if we show that $\overline{J[x]} \neq E$ for some non-zero $x \in E^+$. Assume on the contrary that $\overline{J[x]} = E$ for each non-zero vector $x \in E^+$. Then the set $\mathcal{P}[x] := \{Ax : A \in \mathcal{P}\}$ is a quasi-interior set. By Proposition 2.2, the net $\{x_0 - x_0 \wedge u\}_{u \in \mathcal{P}[x]}$ norm converges to 0. This means that for each $y \neq 0$ there exists some $A \in \mathcal{P}$ such that $\|x_0 - x_0 \wedge A|y|\| < 1$. It follows that

$$\overline{K(U)} \subseteq \bigcup_{A \in \mathcal{P}} \{y \in E : \|x_0 - x_0 \wedge A|y|\| < 1\} .$$

Since the function $y \to x_0 \wedge A|y|$ is continuous, the set $\{y \in E : \|x_0 - x_0 \wedge A|y|\| < 1\}$ is open for each $A \in \mathcal{P}$. Since $\overline{K(U)}$ is a compact set, there exist A_1, A_2, ..., $A_p \in \mathcal{P}$ such that

$$\overline{K(U)} \subseteq \bigcup_{j=1}^{p} \{y \in E : \|x_0 - x_0 \wedge A_j|y|\| < 1\} .$$

Put $A := A_1 + A_2 + \ldots + A_p + T_0 \in \mathcal{P}$. Since $A_j|y| \leq A|y|$ for all $y \in E$ and for all $j = 1, \ldots, p$, we conclude that $\|x_0 - x_0 \wedge A_j|y|\| \geq \|x_0 - x_0 \wedge A|y|\|$, so that

$$\overline{K(U)} \subseteq \{y \in E : \|x_0 - x_0 \wedge A|y|\| < 1\} .$$

This means that $y \in \overline{K(U)}$ implies that $x_0 \wedge A|y| \in U$. In particular, we obtain that $x_1 := x_0 \wedge A|Kx_0| \in U$. Now, it follows from $Kx_1 \in K(U)$ that $x_2 := x_0 \wedge A|Kx_1| \in U$. Proceeding with this modified "ping-pong" of Hilden's, we define a sequence $\{x_n\}$ of positive vectors in U by $x_{n+1} := x_0 \wedge A|Kx_n|$. Then we have $x_{n+1} \leq A|Kx_n| \leq AT_0x_n \leq A^2x_n$ for all n. An easy induction now gives that $x_n \leq A^{2n}x_0$ for all n. Since $\mathcal{P}$ is finitely quasinilpotent at x_0, we have $\lim_{n\to\infty} \|x_n\|^{1/n} = 0$, and so $\lim_{m\to\infty} \|x_n\| = 0$. Since $x_n \in U$ for all n, we obtain that $0 \in U$, which is a contradiction. The proof is finished. □

An application of Theorem 1.3 now gives the following extension of the famous result of de Pagter [16].

Theorem 4.4 *Let $\mathcal{S}$ be a non-zero multiplicative semigroup of quasinilpotent compact positive operators on E. Then $\mathcal{S}$ and $\mathcal{S}^\sharp$ have a common non-trivial invariant closed ideal.*

Proof. By Theorem 1.3, the semigroup $\mathcal{S}$ is finitely quasinilpotent. Then apply Theorem 4.3. □

Theorem 4.4 and Lemma 2.3 now imply the following.

Theorem 4.5 *Every multiplicative semigroup $\mathcal{S}$ of quasinilpotent compact positive operators on E is ideal-triangularizable, that is, there exists a maximal subspace chain consisting of closed ideals invariant under $\mathcal{S}$.*

The proof of the following analog of Lemma 3.4 can be found in [9].

Lemma 4.6 *Let $\mathcal{S}$ be a multiplicative semigroup of positive operators on a normed Riesz space L. If a non-zero semigroup ideal $\mathcal{I}$ of $\mathcal{S}$ has a non-trivial invariant closed ideal, then the semigroup $\mathcal{S}$ has a non-trivial invariant closed ideal as well.*

Using Lemma 4.6 we can prove the next companion of Theorem 4.3.

Theorem 4.7 *Let $\mathcal{C} \neq \{0\}$ be a collection of positive operators on E that is finitely quasinilpotent at a non-zero positive vector. Let $T_0 \in \mathcal{C}'$ be an operator that dominates a non-zero compact operator. Then $\mathcal{C}$ and $\mathcal{C}^\sharp$ have a common non-trivial invariant closed ideal.*

Proof. As in the proof of Theorem 4.3 we must consider only the case when $\overline{\mathcal{Q}_f(\mathcal{C})} = E$. Let $N_{\mathcal{C}}$ be the intersection of all null ideals of operators from $\mathcal{C}$. Then it is clearly invariant under $\mathcal{C}$. To prove that it is also invariant under $\mathcal{C}^\sharp$, we pick $C \in \mathcal{C}^\sharp$ and $x \in N_{\mathcal{C}}$. Then for each $T \in \mathcal{C}$ we have

$$0 \le T|Cx| \le TC|x| \le CT|x| = 0 \ ,$$

so that $Cx \in N_{\mathcal{C}}$. Thus, if $N_{\mathcal{C}}$ is non-zero, it is a non-trivial invariant closed ideal invariant under both $\mathcal{C}$ and $\mathcal{C}^\sharp$.

So assume that $N_{\mathcal{C}} = \{0\}$. Let K be a non-zero compact operator on E that is dominated by T_0. Since K is a non-zero operator and $\overline{\mathcal{Q}_f(\mathcal{C})} = E$, there exists a vector $x_0 \in E^+$ such that $Kx_0 \neq 0$ and $\mathcal{C}$ is finitely quasinilpotent at x_0. Let $\mathcal{S}$ be the multiplicative semigroup generated by $\mathcal{C}$, and let $\mathcal{P}$ be the multiplicative semigroup of all operators on E such that each $T \in \mathcal{P}$ is dominated by some operator of the form CS with $C \in \mathcal{C}^\sharp$ and $S \in \mathcal{S}$. By $\mathcal{T}$ we denote the multiplicative semigroup generated by $\mathcal{C}$ and $\mathcal{C}^\sharp$. Furthermore, let $\mathcal{I}$ be the semigroup ideal of $\mathcal{T}$ generated by $\mathcal{C}T_0$. Since $\mathcal{I} \subseteq \mathcal{P}$, it follows from Proposition 4.2 that $\mathcal{I}$ is finitely quasinilpotent at x_0. We now claim that $\mathcal{I}$ contains some operator that dominates a non-zero compact operator.

By L we denote the ideal generated by the range of K. By Lemma 2.4, there exists an operator V from $\overline{L}$ to E such that VKx_0 is a non-zero positive vector and $|VKx| \le |Kx|$ for all $x \in E$. Since $N_{\mathcal{C}} = \{0\}$, there is some $T \in \mathcal{C}$ such that $TVKx_0$ is a non-zero positive vector. Hence, TVK is a non-zero compact operator on E. Furthermore, for each $x \in E$ it holds

$$|TVKx| \le T|VKx| \le T|Kx| \le TT_0|x| \ ,$$

and so $TT_0 \in \mathcal{I}$ dominates the operator TVK. This proves what we have claimed.

By Theorem 4.3, the ideal $\mathcal{I}$ has a non-trivial invariant closed ideal. Now Lemma 4.6 implies that $\mathcal{T}$ has a non-trivial invariant closed ideal. This completes the proof of the theorem. □

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