University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 36 (1998), 637

OKA'S PRINCIPLE FOR HOLOMORPHIC FIBER BUNDLES WITH SPRAYS

Franc Forstnerič Jasna Prezelj

ISSN 1318-4865

November 13, 1998

Ljubljana, November 13, 1998

OKA'S PRINCIPLE FOR HOLOMORPHIC FIBER BUNDLES WITH SPRAYS

Franc Forstnerič and Jasna Prezelj

&1. The h-principle of Oka-Grauert-Gromov.

In this paper we give a complete and self contained proof of the results of M. Gromov [Gro] (1989) on the h-principle (homotopy principle) for sections of certain holomorphic bundles over Stein manifolds.

1.1 Definition. *Let $h: Z \to X$ be a holomorphic mapping of complex manifolds. A section of h is any map $f: X \to Z$ such that $h \circ f$ is the identity on X. Sections of h satisfy the* **h-principle** *(or the* **Oka's principle***) if each continuous section $f_0: X \to Z$ of $h: Z \to X$ can be deformed to a holomorphic section $f_1: X \to Z$ through a continuous one parameter family (a homotopy) of continuous sections $f_t: X \to Z$ $(0 \le t \le 1)$, and any two holomorphic sections $f_0, f_1: X \to Z$ which are homotopic through continuous sections of Z are also homotopic through holomorphic sections. If this holds for a trivial bundle $Z = X \times F \to X$ then we say that maps $X \to F$ satisfy the h-principle.*

For the basic examples when the h-principle holds (or fails) we refer the reader to [Gro]. One is interested in complex manifolds F which satisfy the h-principle for maps $X \to F$ from any Stein manifold X. With this in mind we recall

1.2 Definition. (Gromov [Gro]) *A* **spray** *on a complex manifold F is a holomorphic vector bundle $p: E \to F$ together with a holomorphic map $s: E \to F$ such that s is the identity on the zero section $F \subset E$, and for each $x \in F$ the derivative $Ds(x)$ maps E_x (which is a linear subspace of $T_x E$) surjectively onto $T_x F$. (Gromov calls such maps dominating sprays).*

1.3 Theorem. (Gromov [Gro], sec. 2.9) *If F is a complex manifold which admits a spray, then the sections of any locally trivial holomorphic fiber bundle with fiber F over any Stein manifold satisfy the h-principle. In particular, mappings from Stein manifolds into F satisfy the h-principle.*

For stronger results see theorem 1.4 and corollary 1.5 below. In a sequel to this paper we shall use the tools developed here to supply the details to another approach of Gromov (sect. 4 in [Gro]) to prove the h-principle for sections of holomorphic submersions $h: Z \to X$, where the base X is Stein and each point $x \in X$ has a neighborhood $U \subset X$ such that $Z|U = h^{-1}(U)$ admits a fiber-spray (see definition 3.1 below).

For the non-specialists we recall that a complex manifold is called *Stein* if it has 'plenty' of global holomorphic functions. For the precise definition and properties of such manifolds we refer to the monographs of Gunning and Rossi [GRo], Hörmander [Hö2], and Grauert and Remmert [GRe]. The most commonly used characterizations of Stein manifolds are the following. A complex manifold X is Stein if and only if any of the following two conditions holds:

- X can be represented as a closed complex submanifold of some complex euclidean space (this is the embedding theorem of Remmert, Bishop and Narasimhan [GRo, p. 224]);
- X admits a smooth strongly plurisubharmonic exhaustion function (Grauert [Gr1]).

By a *locally trivial holomorphic fiber bundle* with fiber F over a complex manifold X we mean a bundle obtained by patching the trivial bundles $U_\alpha \times F$ over an open cover $\{U_\alpha\}$ of X by transition functions of the form

$$\phi_{\alpha,\beta}(x,\xi) = \big(x, \psi_{\alpha,\beta}(x,\xi)\big) \quad (x \in U_\alpha \cap U_\beta,\ \xi \in F),$$

where $\psi_{\alpha,\beta}\colon (U_\alpha \cap U_\beta) \times F \to F$ is holomorphic and $\psi_{\alpha,\beta}(x,\cdot\,) \in \mathrm{Aut}F$ is an automorpism of F for each fixed $x \in U_\alpha \cap U_\beta$. Of course these transition function must satisfy the usual compatibility conditions, see Steenrod [Ste].

Theorem 1.3 is an extension of the *Oka's principle* which can vaguely be stated by saying that, *on a Stein manifold or a reduced Stein space, analytic problems which can be cohomologically formulated have only topological obstructions* ([GRe], p. 145). Early examples of Oka's principle include solvability of Cousin problems and Oka's theorem to the effect that holomorphic line bundles over a Stein manifold which are topologically isomorphic are also holomorphically isomorphic. This was extended by H. Grauert [Gr2, Gr3] (1957) to holomorphic vector bundles and, more generally, to holomorphic fiber bundles whose structure group is a complex Lie group. The problem of constructing a holomorphic isomorphism between two such bundles can be changed to the problem of constructing a holomorphic section of an associated bundle. Grauert's result was improved and generalized by Cartan [Car], Ramspott [Ram], Forster and Ramspott [FR1, FR2], Forster [Fr1, Fr2], Heinzner and Kutzschebauch [HKu], and others. A proof of Grauert's theorem without induction on the basis dimension was given by Henkin and Leiterer [HL2]. Note that the exponential map $\exp\colon \mathbf{g} \to G$ from the Lie algebra $\mathbf{g}$ of a Lie group G induces a spray on G (and on the associated G-homogeneous spaces), so theorem 1.3 includes Grauert's theorem. Even when the fiber is a Lie group, the bundles in theorem 1.3 are more general than the principal G-bundles in which the transition maps are left and right multiplications (translations) by elements of G.

Example 1. (Demailly [Dem]) There exists a locally trivial holomorphic fiber bundle $h\colon Z \to X = \mathbf{C}\backslash\{0,1\}$ with fiber $\mathbf{C}^2$ such that Z has no holomorphic functions other than those of the form $g \circ h$, where g is holomorphic on the base X. So Z is not Stein. In particular, Z admits no holomorphic vector bundle structure which gives a negative answer to the question of Gromov [Gro, 2.5.B]. The transition functions used in the construction of Z are (nonlinear) holomorphic automorphisms of the fiber $\mathbf{C}^2$. Nevertheless the sections of such bundles satisfy the h-principle according to theorem 1.3. ♠

Here are two other interesting cases of spaces with sprays (Gromov [Gro]):

(a) A complex manifold F which admits finitely many complete holomorphic vector fields V_j $(1 \le j \le J)$ such that the vectors $V_j(x)$ span T_xF for each $x \in F$.

(b) $F = \mathbf{C}^n \backslash \Sigma$, where Σ is an algebraic subvariety of complex codimension at least two.

The word 'complete' in (a) must be understood in the sense of $\mathbf{C}$-complete, i.e., the flow ϕ_j^t associated to V_j extends holomorphically to all complex values of the time parameter t and hence defines a holomorphic map $\phi_j \colon \mathbf{C} \times F \to F$ such that $\frac{d}{dt}|_{t=0}\phi_j^t(x) = V_j(x)$ for all $x \in F$. The map $s \colon F \times \mathbf{C}^J \to F$ defined by

$$s(x; t_1, \ldots, t_J) = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \cdots \circ \phi_J^{t_J}(x)$$

satisfies $s(x; 0, \ldots, 0) = x$ and $\frac{\partial}{\partial t_j} s(x; 0, \ldots, 0) = V_j(x)$ for each $x \in F$ and each j. Since these vectors span T_xF, s is a spray on F. (For a discussion of $\mathbf{R}$-complete versus $\mathbf{C}$-complete holomorphic vector fields see [Fo1].)

In case (b) we think of Σ as a set which should be avoided by the image of a map $X \to \mathbf{C}^n$. The h-principle asserts that we can avoid Σ by a holomorphic map $f \colon X \to \mathbf{C}^n$ if we can do so by a continuous map. This result, with Σ being a finite union of affine complex subspaces, was used in an essential way by Eliashberg and Gromov [EGr] and Schürmann [Sch] in the construction of proper holomorphic embeddings of Stein manifolds (and Stein spaces) into Euclidean spaces of minimal dimension. In fact (b) is a special case of (a) which can be seen as follows. A generic linear projection $\pi \colon \mathbf{C}^n \to \mathbf{C}^{n-1}$ maps Σ properly to $\mathbf{C}^{n-1}$ and hence $\pi(\Sigma) = \Sigma'$ is a proper closed subvariety of $\mathbf{C}^{n-1}$. Choose a vector $v \in \mathbf{C}^n \backslash \{0\}$ such that $\pi(v) = 0$. For each holomorphic polynomial (or entire function) $f \colon \mathbf{C}^{n-1} \to \mathbf{C}$ the vector field $V(z) = f(\pi(z))v$ is complete on $\mathbf{C}^n$, with the flow $\phi^t(z) = z + tf(\pi(z))v$. If we choose f so that it vanishes on Σ' then ϕ^t maps $\mathbf{C}^n \backslash \Sigma$ to itself for each $t \in \mathbf{C}$, so V is complete when restricted to $\mathbf{C}^n \backslash \Sigma$. It is easily seen that there are finitely many vector fields of this type which span $T_z\mathbf{C}^n$ for each $z \in \mathbf{C}^n \backslash \Sigma$, so (a) holds. This argument works as long as there exist sufficiently many linear projection $\pi \colon \mathbf{C}^n \to \mathbf{C}^{n-1}$ which are proper when restricted to Σ. However one cannot completely dispose with this last condition even if Σ is a very simple complex manifold as the following example shows.

Example 2. For any integer $N > 0$ there exist discrete sets $\Sigma \subset \mathbf{C}^N$ for which there are no non-degenerate maps $\mathbf{C}^N \to \mathbf{C}^N \backslash \Sigma$ [RRu]. Also for any $1 \le n < N$ there exist proper holomorphic embeddings $\sigma \colon \mathbf{C}^n \to \mathbf{C}^N$ such that $\mathbf{C}^N \backslash \Sigma$ (where $\Sigma = \sigma(\mathbf{C}^n)$) admits no nondegenerate holomorphic images of $\mathbf{C}^{N-n}$ [BFo], [Fo2]. In both cases $\mathbf{C}^n \backslash \Sigma$ admits no spray. Also no Kobayashi-hyperbolic complex manifold admits a spray (since it even admits no nondegenerate holomorphic images of $\mathbf{C}$). ♠

In sections 3–5 of [Gro] Gromov considers generalizations of theorem 1.3 in several directions: inclusion of additional parameters, approximation on a holomorphically convex subset, matching a given holomorphic section on a subvariety $X_0 \subset X$, replacing fiber bundles by holomorphic submersions onto Stein manifolds in which small pieces admit a spray (this will be discussed later), and others. His discussion, which is quite intuitive and plausible, is most of the time not supported by detailed proofs. We do not attempt to prove all possible results alluded to by Gromov; instead we concentrate on those which we can prove by relatively simple additional arguments. Since the addition of a parameter is very natural and even unavoidable in these constructions, we prove all results for families of sections which depend continuously on a parameter in a compact Hausdorff space Y. The basic objects now are continuous maps $f: X \times Y \to Z$ such that $f(\cdot\,, y): X \to Z$ is a section of $h: Z \to X$ for each fixed $y \in Y$. A *homotopy* of such maps is a continuous map $H: X \times Y \times [0,1] \to Z$ such that $H_t(\cdot\,, y) = H(\cdot\,, y, t): X \to Z$ is a section of $h: Z \to X$ for all $y \in Y$ and $t \in [0,1]$.

Recall that a compact set $K \subset X$ is *holomorphically convex* in X if for each $x \in X \backslash K$ there is a holomorphic function f on X such that $|f(x)| > \sup_K |f|$. If X is Stein then by the Oka-Weil theorem each function holomorphic in a neighborhood of a holomorphically convex set $K \subset X$ can be approximated on K by functions holomorphic on X [Hö2].

Here is the parametric h-principle with approximation which includes theorem 1.3.

1.4 Theorem. *Let X be a Stein manifold and $h: Z \to X$ a locally trivial holomorphic fiber bundle whose fiber admits a spray. Let Y be a compact Hausdorff space (the parameter space), $Y_0 \subset Y$ a compact subset and $Y' \subset Y$ an open set containing Y_0. Assume that $f: X \times Y \to Z$ is a continuous map such that $f(\cdot\,, y): X \to Z$ is a section of $h: Z \to X$ for each $y \in Y$, and $f(\cdot\,, y)$ is holomorphic on X for each $y \in Y'$. Then there is a homotopy $H_t: X \times Y \to Z$ $(0 \le t \le 1)$ such that $H_0 = f$, $H_1(\cdot\,, y): X \to Z$ is holomorphic on X for each $y \in Y$, and the homotopy is fixed on Y_0 (i.e., $H_t(x, y)$ is independent of t for $y \in Y_0$). Moreover, if K is a compact holomorphically convex subset in X and we assume that there is a neighborhood $V \subset X$ of K such that each section $f(\cdot\,, y)$ $(y \in Y)$ is holomorphic in V, then for each metric d on Z and each $\epsilon > 0$ there is a homotopy H such that, in addition to the above,*

$$d\big(H_t(x,y), f(x,y)\big) < \epsilon \quad (x \in K,\ y \in Y,\ 0 \le t \le 1).$$

1.5 Corollary. *If $h: Z \to X$ is as in theorem 1.3, then the inclusion map between the spaces of holomorphic and continuous sections*

$$\mathrm{Holo}(X, Z) \subset \mathrm{Cont}(X, Z)$$

is a weak homotopy equivalence, i.e., it induces an isomorphism of the homotopy groups of the two spaces. (These spaces are endowed with the usual compact–open topology.)

Remarks. 1. In most applications of theorem 1.4 the subset $Y_0 \subset Y$ is a strong deformation retraction of some neighborhood $Y' \subset Y$. In such case the conclusion of theorem 1.4 holds if we assume only that the sections $f(\cdot\,, y)$ for $y \in Y_0$ are holomorphic on X since we can use the deformation retraction of Y' onto Y_0 to reparametrize the family so that the sections $f(\cdot\,, y)$ for y sufficiently near Y_0 become holomorphic (and approximate the original sections on K). In our applications Y will be a polyhedron and Y_0 a subpolyhedron.

2. The basic h-principle given by definition 1.1 is equivalent to saying that *each path connected component of the space of continuous sections* $\mathrm{Cont}(X, Z)$ *contains precisely*

one path connected component of the space of holomorphic sections Holo(X,Z). This is the most common application of the h-principle.

Proof of corollary 1.5. If we take Y to be the n-sphere S^n and $Y' = \emptyset$, theorem 1.4 implies that each (continuous) map $S^n \to \mathrm{Cont}(X,Z)$ can be homotopically deformed to a map $S^n \to \mathrm{Holo}(X,Z)$. Similarly, if we take Y to be the closed $(n+1)$-ball $B^{n+1} \subset \mathbf{R}^{n+1}$ and $Y_0 = \partial B^{n+1} = S^n$, we conclude that each map $S^n \to \mathrm{Holo}(X,Z)$ which extends to a map $B^{n+1} \to \mathrm{Cont}(X,Z)$ also extends to a map $B^{n+1} \to \mathrm{Holo}(X,Z)$. This is precisely the content of corollary 1.5. ♠

Our main reason for writing the present paper was that Gromov's result has been of great interest to complex analysis, but unfortunately his paper does not contain complete proofs of certain steps. Moreover, because of certain technical difficulties we had to modify the approach outlined in sections 1-2 of [Gro] to prove theorems 1.3 and 1.4. The main difference between [Gro] and the present paper is in the method for extending holomorphic sections across critical points of a given strongly plurisubharmonic exhaustion function on X. In spite of our criticism it must be said that we completed the proof by sharper versions of the tools developed in [Gro]. We believe that it would not be comprehensible to the reader if we only tried to modify the relevant part of Gromov's paper, and hence we decided to give a complete and self contained exposition.

To explain the main difficulty with the exposition in sect. 1-2 in [Gro] we recall the outline of the proof. We follow Grauert's 'bump method' in the form explained by Henkin and Leiterer [HL2] (see sect. 2 below for the details). We construct a holomorphic section $f\colon X \to Z$ which is homotopic to the given continuous section f_0 by successively modifying f_0 so as to make it holomorphic on increasingly larger compact subset $A_k \subset X$ that exhaust X (i.e., $X = \bigcup_{k=1}^{\infty} A_k$). The construction is such that, if we denote by $f_k\colon X \to Z$ the section which is holomorphic in a neighborhood of A_k for some k, then the sequence f_k converges locally uniformly on X to a holomorphic section $f = \lim_{k\to\infty} f_k\colon X \to Z$. The sets A_k are closed strongly pseudoconvex domains in X which are obtained from a strongly plurisubharmonic exhaustion function $\rho\colon X \to \mathbf{R}$ with 'nice' critical points. For each $k = 0, 1, 2, \ldots$ the set A_{k+1} is obtained by attaching to A_k in a special way a small strongly pseudoconvex domain $B_k \subset X$ (a *special pseudoconvex bump* in the terminology of [HL2]; see def. 2.6 there or def. 2.2 in the present paper) such that the bundle Z is trivial over a neighborhood of B_k. The next section $f_{k+1}\colon X \to Z$ must be holomorphic in a neighborhood of $A_{k+1} = A_k \cup B_k$, it must approximate f_k on A_k, and it must be homotopic to f_k by a homotopy $H^k_t\colon X \to Z$ $(0 \le t \le 1)$ that stays close to f_k on A_k. Such a section f_{k+1} is obtained by 'gluing' $f_k|A_k$ with some holomorphic section $b\colon B_k \to Z$ that approximates f_k very well on $C_k = A_k \cap B_k$. The gluing of f_k and b (theorem 5.1 below) is achieved by first reducing the problem to a certain model situation by using sprays. In the model case the solution is obtained by solving a certain $\overline{\partial}$-problem a neighborhood of $A_{k+1} = A_k \cup B_k$ and applying the implicit function theorem in Banach space (proposition 5.2).

In order to find a holomorphic section $b\colon B_k \to Z$ which approximates f_k on C_k we choose any holomorphic section $b_0\colon B_k \to Z$ (such sections exist since Z is trivial over B_k) and construct a homotopy $b_t\colon C_k \to Z$ $(0 \le t \le 1)$ of holomorphic sections over C_k

connecting $b_0|C_k$ with $b_1 = f_k|C_k$ (proposition 6.1). Once we have such a homotopy, we can apply the *h-Runge theorem* (theorem 4.1) to approximate the homotopy b_t uniformly on C_k by a homotopy $\tilde{b}_t\colon B_k \to Z$ consisting of sections that are holomorphic on B_k, with $\tilde{b}_0 = b_0$. The section $b = \tilde{b}_1$ is then holomorphic on B_k and approximates f_k on C_k, so we can glue them.

In the special *non-critical case* both B_k and $C_k = A_k \cap B_k$ are biholomorphic to strongly convex domains in $\mathbf{C}^n$ (with $n = \dim X$). Such B_k is called a *convex bump* on A_k (def. 2.2 below). In this case we immediatelly get a holomorphic homotopy b_t as above by using the triviality of Z over C_k and the holomorphic contractions of C_k to a point. Note that the homotopy type of the set does not change when attaching to it a convex bump. This non-critical case suffices to extend a holomorphic section in a finite number of steps described above from a certain sublevel set $\{\rho \le c_0\}$ to a higher sublevel set $\{\rho \le c_1\}$ when ρ has no critical values on the interval $[c_0, c_1] \subset \mathbf{R}$. A similar method (with $C_k = \emptyset$) allows us to cross the critical points of ρ which are local minima.

However, to jump across a critical point x_0 of ρ which is not a local minimum we must attach a cell B_k to A_k in a more general way as described by def. 2.2 below (such B_k is called a *pseudoconvex bump*). Now the set $C_k = A_k \cap B_k$ is no longer contractible and it is a nontrivial problem to find the required holomorphic homotopy b_t as above. In this *critical case* Gromov suggests (sec. 2.7 in [Gro]) to jump across x_0 by attaching to A_k a suitable real-analytic totally real disc B_k containing x_0 (whose dimension depends on the index of ρ at x_0) such that $A_k \cup B_k$ has arbitrarily small Stein neighborhoods V which can be chosen such that we can reach a suitable higher sublevel set $\{\rho \le c\}$ for $c > \rho(x_0)$ from V by attaching convex bumps (i.e., as a non-critical extension). Once we have such neighborhoods, the idea is that $f_k|B_k$ can be approximated by a real-analytic section which can then be complexified (since B_k is totally real), thus giving a section b that approximates f_k on $A_k \cap B_k$ and hence also on some small neighborhood of this set. Choosing a sufficiently small neighborhood $V \supset A_k \cup B_k$ as above one should then glue f_k with b to get a holomorphic section on V. No further details are provided in [Gro].

This suggestion seems problematic for two reasons. The first problem is to find small strongly pseudoconvex neighborhoods V of $A_k \cup B_k$ (when B_k is a totally real disc as above) such that a certain higher level set of ρ is a non-critical strongly pseudoconvex extension of V (i.e., it can be reached by adding convex bumps to V). To our knowledge there exists no explicit reference to such a result in the literature (for partial results see see Eliashberg [Eli] and Rosay [Ros]). Another independent problem is that the required rate of approximation of two sections near $A_k \cap B_k$ which is needed to glue them into a single section over $A_k \cup B_k$ depends on the set $A_k \cup B_k$ since we must solve a certain $\overline{\partial}$-equation *with sup norm estimates.* This can be done on any strongly pseudoconvex domain, but the constant in such estimate depends (unlike for the L_2 estimate!) on the geometry of the set. It is well known that, in general, there are no bounded solutions to the $\overline{\partial}$-equation on smooth pseudoconvex domains which fail to be strongly pseudoconvex even at a single point (Sibony [Si1, Si2]). However, when B_k is a totally real disc attached to A_k, we cannot control the rate of approximation of f_k by b in any *fixed* neighborhood of $A_k \cap B_k$ when b is obtained by complexification, no matter how good control we have on $A_k \cap B_k$. See also remark 1 following the proof of lemma 2.4 below.

In view of these difficulties we modified the above approach to extending the a holomorphic section across a critical point as follows. It was proved in [HLe] that we can jump over a critical point of ρ by attaching to A_k a pseudoconvex bump B_k for which $C_k = A_k \cap B_k$ is a sublevel set $\{\tau \le 1\}$ of some strongly plurisubharmonic function $\tau \ge 0$ defined in a neighborhood of B_k such that, in some holomorphic coordinate system in a neighborhood of B_k, the set $S = \tau^{-1}(0)$ is a totally real sphere contained in some affine totally real subspace, and τ has no critical points on $C_k \backslash S$. In particular, C_k is obtained from a suitable tubular neighborhood of S by attaching a convex bump finitely many times. We initially deform the given continuous homotopy b_t so as to make it holomorphic in a neighborhood of S, and subsequently we extend it to C_k in a finite number of steps. In each step we extend the homotopy (by approximation) across a convex bump, using parametric versions of the h-principle and of the gluing result (corollary 5.6). This allows us to complete the proof of theorems 1.3 and 1.4.

The paper is organized as follows. In section 2 we explain the underlying geometric approach, the so called Grauert's *bump method*, following Henkin and Leiterer [HLe], and we reduce the proof of the main result to theorem 2.6 concerning the extension of holomorphic sections across pseudoconvex bumps. Theorem 2.6 is proved in sect. 6 after we develop the necessary tools in sect. 3–5. In sect. 3 we recall the concept of sprays and show how the homotopies of sections of $Z \to X$ can be lifted to homotopies of sections of certain iterated spray bundles over Z. In sect. 4 we prove Runge type approximation theorems for sections of holomorphic submersions which admit a spray. In sect. 5 we prove results on gluing holomorphic sections over Cartan pairs.

We wish to thank G. Henkin and J. Leiterer for their interest in this subject and for several useful discussions, and to the participants in the Seminar for complex analysis at the University of Ljubljana who were very patient during our lectures on this topic. The first author acknowledges partial support by the National Science Foundation, by the Vilas Foundation at the University of Wisconsin, and by the Ministry of Science and Technology of the Republic of Slovenia. The second author was supported by a grant from the Ministry of Education of the Republic of Slovenia.

&2. Pseudoconvex bumps and Cartan pairs.

Let X be a complex manifold. We say that a compact set $C \subset X$ is Runge in another compact set $B \supset C$ if C has a basis of neighborhoods C_j which are Runge in some open neighborhood $\tilde{B}$ of B, i.e., each function holomorphic in C_j can be approximated uniformly on compacts in C_j by functions holomorphic in $\tilde{B}$. This is the case for instance if C is holomorphically convex in some Stein neighborhood of B.

2.1 Definition. *Let X be a complex manifold.*

(i) A **compact strongly pseudoconvex domain** *in X is a compact set of the form $A = \{\rho \le 0\} \subset X$, where $\rho: X \to \mathbf{R}$ is a $\mathcal{C}^2$ function which is strongly plurisubharmonic and has no critical points in a neighborhood of $bA = \{\rho = 0\}$. (A need not be connected, but the definition implies that it has at most finitely many connected components each of which is again a compact strongly pseudoconvex domain.)*

(ii) *We say that a compact set $A' \subset X$ is a* **strongly pseudoconvex extension** *of a compact set $A \subset A'$ if there is a $\mathcal{C}^2$ function $\rho: X \to \mathbf{R}$ which is strongly plurisubharmonic in a neighborhood of $\overline{A' \backslash A}$ such that 0 and 1 are regular values of ρ and*

$$A = \{x \in X: \rho(x) \le 0\}, \quad A' = \{x \in X: \rho(x) \le 1\}.$$

If ρ can be chosen such that it has no critical points on $\overline{A' \backslash A}$ then A' is called a **non-critical strongly pseudoconvex extension** *of A.*

2.2 Definition. *Let X be a complex manifold. A pair (A, B) of compact subsets of X is a* **pseudoconvex bump** *(or B is a pseudoconvex bump on A) if the following hold:*

(i) *The sets A, B, $A \cup B$ and $C = A \cap B$ are compact strongly pseudoconvex domains (C may be empty);*

(ii) $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$;

(iii) *there are holomorphic coordinates in a neighborhood of B in which B is starshaped (and hence holomorphically contractible);*

(iv) *C is Runge in B;*

(v) *if $C \ne \emptyset$ there is a $\mathcal{C}^2$ strongly plurisubharmonic function $\tau \ge 0$ defined in a neighborhood $U \subset X$ of C such that $C = \{z \in U: \tau(z) \le 1\}$, the set $S = \tau^{-1}(0)$ is a compact totally real submanifold which is contained in an affine totally real subspace with respect to some holomorphic coordinates in a neighborhood of C, and τ has no critical points in $C \backslash S = \{z \in U: 0 < \tau(z) \le 1\}$.*

We say that B is a **convex bump** *on A (or the pair (A, B) is a convex bump) if, in addition to the above, there are holomorphic coordinates in a neighborhood of B in which both B and C are strongly convex domains. (The set C may be empty.)*

For certain purposes we can relax the above conditions somewhat and consider *Cartan pairs*. Gromov's definition of a Cartan pair is not very precise (sec. 1.5.A in [Gro]), but the way it is used suggests the following definition.

2.3 Definition. *A* **Cartan pair** *in X is a pair of compact sets $A, B \subset X$ such that*

(i) *A, B, and $A \cup B$ have bases of Stein neighborhoods, and*

(ii) $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$.

Clearly each pseudoconvex bump is a Cartan pair. We denote by $H^\infty(\Omega)$ the algebra of bounded holomorphic functions in a domain Ω. The following lemma will be used in sec. 5 for gluing holomorphic sections over Cartan pairs. (This is similar to what Gromov takes as his definition of a Cartan pair.)

2.4 Lemma. *If (A, B) is a Cartan pair in a Stein manifold such that $C = A \cap B \ne \emptyset$, there are bases of Stein open neighborhoods $A_j \supset A$, $B_j \supset B$, $C_j = A_j \cap B_j \supset A \cap B$, and bounded linear operators $\mathcal{A}_j: H^\infty(C_j) \to H^\infty(A_j)$, $\mathcal{B}_j: H^\infty(C_j) \to H^\infty(B_j)$, satisfying*

$$c = \mathcal{A}_j(c) - \mathcal{B}_j(c), \qquad c \in H^\infty(C_j),\ j = 1, 2, 3, \dots \tag{2.1}$$

Proof. Let $U \supset A$ and $V \supset B$ be open neighborhoods of A resp. B. Choose Stein open sets $\tilde{A}$ and $\tilde{B}$ in X so that $A \subset \tilde{A} \subset U$ and $B \subset \tilde{B} \subset V$. Set $\tilde{C} = \tilde{A} \cap \tilde{B}$. By the separation condition (ii) in def. 2.3 there is a smooth function $\chi: X \to [0,1]$ such that $\chi = 0$ in a neighborhood of $\overline{A \backslash B}$ and $\chi = 1$ in a neighborhood of $\overline{B \backslash A}$. Hence there are open sets $A_0, B_0 \subset X$, with $A \subset A_0 \subset \tilde{A}$ and $B \subset B_0 \subset \tilde{B}$, so that $\chi = 0$ on $A_0 \backslash \tilde{B}$ and $\chi = 1$ on $B_0 \backslash \tilde{A}$. Choose a smooth strongly pseudoconvex set Ω with $A \cup B \subset \Omega \subset A_0 \cup B_0$ and set

$$A' = \tilde{A} \cap \Omega, \quad B' = \tilde{B} \cap \Omega, \quad C' = A' \cap B' = \tilde{C} \cap \Omega.$$

Then A', B' and C' are Stein domains containing A, B, C respectively, with $A' \subset \tilde{A} \subset U$, $B' \subset \tilde{B} \subset V$, and $A' \cup B' = \Omega$. Moreover by the choice of Ω we have

$$A' \backslash C' = A' \backslash B' = \Omega \backslash \tilde{B} \subset A_0 \backslash \tilde{B} \subset \{\chi = 0\}$$

and similarly $B' \backslash C' \subset \{\chi = 1\}$. If c is any bounded holomorphic function in C' then the above implies that χc extends to a bounded smooth function in A' which vanishes in $A' \backslash C'$, $(1-\chi)c$ extends to a bounded smooth function in B' which vanishes in $B' \backslash C'$, and $\overline{\partial}(\chi c) = c \overline{\partial} \chi$ extends to a bounded smooth $\overline{\partial}$-closed $(0,1)$-form in Ω. Let T_Ω be a linear solution operator for the $\overline{\partial}$-equation in Ω (i.e., $\overline{\partial}(T_\Omega \alpha) = \alpha$ for each $\overline{\partial}$-closed smooth $(0,1)$-form α on Ω) which is bounded in the sup norm (and also in any $\mathcal{C}^k$ norm); see [HL1], p. 82 or [Hö1]. The linear operators

$$\mathcal{A}\, c = \chi c - T_\Omega(c \overline{\partial} \chi), \qquad \mathcal{B}\, c = (\chi - 1)c - T_\Omega(c \overline{\partial} \chi)$$

then satisfy the required properties with respect to the neighborhoods A', B', C'. In fact, since T_Ω is bounded in all $\mathcal{C}^k$ norms, so are the operators $\mathcal{A}$ and $\mathcal{B}$. ♠

Remark 1. Actually Gromov requires that there exist operators as above whose norms only depend on A and B and not on the chosen neighborhoods (see sect. 1.5.A in [Gro]). The proof of lemma 2.4 shows that such operators exist if we can choose a system of neighborhoods Ω of $A \cup B$ on which the T_Ω operator satisfies the sup norm estimates with a constant independent of Ω. However, this can be achieved only rarely, for instance when $A \cup B$ is itself a smooth strongly pseudoconvex domain. This problem causes serious difficulties in sect. 2.7 of [Gro] where Gromov proposes to glue two holomorphic sections in small neighborhoods of the set $A \cup B$ where A is a compact strongly pseudoconvex domain and B is a smooth totally real disc. The remark 1.5.A' in [Gro] is incorrect, with immediate counterexamples. (The union of two holomorphically convex sets in X need not have a basis of Stein neighborhoods, and there may be no decomposition (2.1).)

Remark 2. If A, B, $C = A \cap B$ are smoothly bounded domains in X such that $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$ and $\Omega = A \cup B$ is a smooth strongly pseudoconvex domain (this is the case for instance if (A, B) is a pseudoconvex bump), we can use a solution operator T_Ω for the $\overline{\partial}$-equation on Ω which is bounded in any $\mathcal{C}^k$-norm to prove exactly as above that there exist bounded linear operators $\mathcal{A}: A^k(C) \to A^k(A)$, $\mathcal{B}: A^k(C) \to A^k(B)$ such that $c = \mathcal{A}(c) - \mathcal{B}(c)$ for each $c \in A^k(C)$. Here $A^k(A)$ denotes the space of $\mathcal{C}^k$ functions on A which are holomorphic in the interior of A. This approach has been used in [HLe]. ♠

The inductive construction of sections of $Z \to X$ is based on the following geometric tool whose proof is given in [HLe].

2.5 Theorem. *(*Henkin and Leiterer [HLe].) *Let X be a complex manifold and $A' \subset X$ a strongly pseudoconvex extension of $A \subset A'$. Then there exist finitely many pseudoconvex bumps (A_j, B_j) in X $(0 \le j \le k)$ such that*

$$A = A_0, \quad A_{j+1} = A_j \cup B_j \ \text{ for } \ 0 \le j \le k-1, \quad \text{and} \quad A' = A_k.$$

Moreover for any open covering $\{U_i\}$ of X we can choose the bumps such that each B_j is contained in some U_i. If A' is a non-critical strongly pseudoconvex extension of A then we may choose each B_j to be a convex bump on A_j.

The following is the main technical result of the paper. It allows us to extend a holomorphic section of $Z \to X$ from a neighborhood of A to a neighborhood of $A \cup B$ for each pseudoconvex bump (A, B) in X, with approximation on A.

2.6 Theorem. *Let $h: Z \to X$ be a holomorphic submersion onto a Stein manifold X and let (A, B) be a pseudoconvex bump in X. Assume that there is an open neighborhood $\tilde{B} \subset X$ of B such that $Z|\tilde{B} = h^{-1}(\tilde{B})$ is isomorphic to a trivial bundle $\tilde{B} \times F$, where F admits a spray. Let d be a metric on Z. Let Y be a compact Hausdorff space (the parameter space), $Y_0 \subset Y$ a compact subset, and $Y' \subset Y_0$ an open set containing Y_0. Let $U \subset X$ be a neighborhood of A. Suppose that $a: X \times Y \to Z$ is a continuous map such that for each $y \in Y$, $a(\cdot, y): X \to Z$ is a section of $h: Z \to X$ which is holomorphic in U, and the sections a_y for $y \in Y'$ are holomorphic on X. Then for each $\epsilon > 0$ there exists a homotopy $a_t: X \times Y \to Z$ $(0 \le t \le 1)$ satisfying*

(i) $a_t(\cdot, y): X \to Z$ is a section of $h: Z \to X$ for each $y \in Y$ and $0 \le t \le 1$,

(ii) $a_0 = a$,

(iii) each section $a_1(\cdot, y)$ is holomorphic in a neighborhood of $A \cup B$ (independent of y),

(iv) the homotopy is fixed for y in a neighborhood of Y_0, i.e., $a_t(\cdot, y) = a(\cdot, y)$ for y near Y_0 and $0 \le t \le 1$, and

(v) $d\big(a_t(x, y), a(x, y)\big) < \epsilon$ for $x \in A$, $y \in Y$ and $0 \le t \le 1$.

Theorem 2.6 is proved in sect. 6 below. In the rest of this section we assume that theorem 2.6 holds and prove theorem 1.4. For simplicity of notation we write the proof in the case without the parameter y; in the general case the proof is exactly the same.

Proof of theorem 1.4. We may assume that d is complete metric on Z. Fix an open cover $\mathcal{U} = \{U_i\}$ of X such that $Z|U_i$ is a trivial bundle with fiber F for each i. Since X is Stein and K is holomorphically convex in X, there is a smooth strongly plurisubharmonic exhaustion function $\rho: X \to \mathbf{R}$ such that $\rho < 0$ on K, 0 is a regular value of ρ, and the given section f_0 is holomorphic in a neighborhood of $A_0 = \{x \in X \colon \rho(x) \le 0\}$. By theorem 2.5 there is a sequence of compact strongly pseudoconvex domains

$$A_0 \subset A_1 \subset A_2 \subset \ldots \subset \bigcup_{k=0}^{\infty} A_k = X$$

such that $A_{k+1} = A_k \cup B_k$ for each k, where (A_k, B_k) is a pseudoconvex bump and $B_k \subset U_i$ for some i (so that Z is trivial over a neighborhood of B_k).

Applying theorem 2.6 to each pseudoconvex bump (A_k, B_k) we can inductively construct a sequence of continuous sections $f_k \colon X \to Z$ (with f_0 being the given initial section) and homotopies of continuous sections $H^k_t \colon X \to Z$ $(0 \le t \le 1)$ satisfying the following properties for each $k = 0, 1, 2, \ldots$ and $0 \le t \le 1$:

(i) f_k and H^k_t are holomorphic in a neighborhood of A_k;

(ii) $H^k_0 = f_k$, $H^k_1 = f_{k+1}$;

(iii) $d\bigl(H^k_t(x), f_k(x)\bigr) < \epsilon/2^{k+1}$ for all $x \in A_k$ and $0 \le t \le 1$.

We begin by applying theorem 2.6 to the data $a = f_0$, $A = A_0$, $B = B_0$, and ϵ replaced by $\epsilon/2$ to get a homotopy H^0_t such that the section $H^0_1 = f_1$ is holomorphic in a neighborhood of $A_0 \cup B_0 = A_1$ and the properties (i)–(iii) hold with $k = 0$.

Suppose inductively that $k > 0$ and we have already constructed sections f_l and H^l_t for $l < k$ and $0 \le t \le 1$. Then we take $f_k = H^{k-1}_1$ (this section is holomorphic in a neighborhood of A_k) and theorem 2.6 to the data $a = f_k$, $A = A_k$, $B = B_k$, and ϵ replaced by $\epsilon/2^{k+1}$ to get H^k_t and $f_{k+1} = H^k_1$ satisfying (i)–(iii). This completes the induction step.

Property (iii) with $t = 1$ implies for all k

$$d\bigl(f_{k+1}(x), f_k(x)\bigr) < \epsilon/2^{k+1} \quad (x \in A_k).$$

Since the metric d is complete, it follows that the limit $f = \lim_{k\to\infty} f_k \colon X \to Z$ exists uniformly on compacts in X, it is holomorphic on X since f_k is holomorphic on A_l for each $k \ge l$, and it satisfies the estimate $d\bigl(f(x), f_0(x)\bigr) < \epsilon$ for $x \in A_0$.

To construct a homotopy H_t between $H_0 = f_0$ and $H_1 = f$ we divide the t-interval $[0, 1)$ into subintervals $I_k = [1 - 2^{-k}, 1 - 2^{-k-1}]$ $(k = 0, 1, 2, \ldots)$ and take H_t for $t \in I_k$ to be the homotopy H^k_t, suitably rescaled. To be precise, let $\lambda_k \colon I_k \to [0, 1]$ be the linear bijection $\lambda_k(t) = 2^{k+1}\bigl(t - 1 + 2^{-k}\bigr)$. For each $k = 0, 1, 2, \ldots$ we set

$$H_t(x) = H^k_{\lambda_k(t)}(x) \quad (t \in I_k,\ x \in X).$$

Clearly this defines a homotopy $H_t \colon X \to Z$ for $0 \le t < 1$. The property (iii) above implies that $\lim_{t\to 1} H_t = f$ uniformly on compacts in X, so by setting $H_1 = f$ we obtain the required homotopy from f_0 to f. This completes the proof of theorem 1.4 provided that theorem 2.6 holds. ♠

&3. Holomorphic submersions with sprays.

In this section we recall from [Gro] the notion of a *fiber dominating spray* associated to a holomorphic submersion $h: Z \to X$ and we prove results on lifting homotopies of sections of Z to homotopies of sections of iterated spray bundles over Z. This roughly corresponds to sections 1.1–1.3 in [Gro].

Let $h: Z \to X$ be a holomorphic submersion between complex manifolds (not necessarily Stein). For $x \in X$ we denote by $Z_x = h^{-1}(x) \subset Z$ the fiber over x. At each point $z \in Z$ the tangent space $T_z Z$ contains a well defined **vertical tangent space**

$$VT_z(Z) = \{e \in T_z Z: Dh(z)e = 0\} = T_z Z_{h(z)}. \tag{3.1}$$

We denote by $VT(Z)$ the corresponding **vertical tangent bundle** to Z which is a holomorphic subbundle of the tangent bundle TZ. Since Z is not assumed to be Stein, there is in general no splitting of TZ into a direct sum $VT(Z) \oplus E'$ for some holomorphic vector bundle $E' \mapsto Z$. However such a splitting exists over any open Stein subset (or a Stein submanifold) $V \subset Z$ (see [GRo], p.256). Also if $f: X \to Z$ is a holomorphic section then along the graph $f(X) \subset Z$ the tangent bundle has a canonical splitting

$$TZ|f(X) = VT(Z)|f(X) \oplus Tf(X).$$

If $p: E \to Z$ is a holomorphic vector bundle over Z, we denote by $E_z = p^{-1}(z) \subset E$ its fiber over $z \in Z$ and by $0_z \in E_z$ the zero element of E_z.

3.1 Definition. (Gromov [Gro]) *A* **spray** *on Z associated to the submersion $h: Z \to X$ (or a* **fiber-spray***) is a tripple (E, p, s), where $p: E \to Z$ is a holomorphic vector bundle and $s: E \to Z$ is a holomorphic map such that for each $z \in Z$ we have*

(i) $s(E_z) \subset Z_{h(z)}$ *(equivalently, $h \circ p = h \circ s$),*

(ii) $s(0_z) = z$, *and*

(iii) the restriction of the derivative $Ds(0_z): T_{0_z} E \to VT_z(Z)$ to the subspace $E_z \subset T_{0_z} E$ maps E_z surjectively onto $VT_z(Z)$.

We denote the restriction in (iii) by

$$VDs(z) = Ds(0_z)|E_z: E_z \to VT_z(Z) \tag{3.2}$$

and call it the **vertical derivative** of s at the point $0_z \in E$. Gromov [Gro] calls such a map s a *fiberwise dominating spray*, the word dominating referring to the property (iii). We shall call it simply a spray when there is no danger of confusion with def. 1.2, or a fiber-spray if we wish to emphasize the difference between the two notions.

Example. Each spray $s: E \to F$ in the sense of definition 1.2 induces a fiber-spray $(\tilde{E}, \tilde{p}, \tilde{s})$ associated to the trivial fibration $h: Z = X \times F \to X$ by taking

$$\tilde{E} = X \times E, \quad \tilde{p}(x, e) = (x, p(e)) \in X \times F = Z, \quad \tilde{s}(x, e) = (x, s(e)) \in Z.$$

Hence if $Z \mapsto X$ is a locally trivial bundle whose fiber admits a spray, then X can be covered by open sets U_i such that each restriction $Z|U_i$ admits a (fiber-) spray (but in general there is no global spray over Z). ♠

The main use of sprays is to lift homotopies of sections of $h: Z \to X$ to homotopies of sections of a certain vector bundles, thereby linearizing the approximation and gluing problems for such sections. The first result in this direction is

3.2 Lemma. (Gromov [Gro], sec. 1.2.) *Let X be a Stein manifold and $h: Z \to X$ a holomorphic submersion which admits a spray (E, p, s). Then for each holomorphic section $f: X \to Z$ there exists a holomorphic vector subbundle E' of the restricted bundle $E|f(X)$ such that $s: E' \to Z$ maps a neighborhood of the zero section in E' biholomorphically onto a neighborhood of $f(X)$ in Z. In particular, if $f_t: X \to Z$ $(0 \le t \le 1)$ is a homotopy of holomorphic sections, then for each $t_0 \in [0,1]$ and each open relatively compact subset $V \subset\subset X$ there is neighborhood $I_0 \subset [0,1]$ of t_0 and a homotopy of holomorphic sections ξ_t $(t \in I_0)$ of $E' \subset E|f(X)$ over the set $f(V)$ such that ξ_{t_0} is the zero section and $s \circ \xi_t(z) = f_t(h(z))$ for $t \in I_0$ and $z \in f(V)$.*

Proof. By definition of the spray the map $s: E|f(X) \to Z$ is the identity on the zero section (which we identify with $f(X) \subset Z$) and it is a submersion near the zero section. Denote by $E_0 = \ker VDs \subset E$ the kernel of the vertical derivative (3.2) and let $\tilde{E} = E/E_0$ be the quotient bundle with the quotient projection $\pi: E \to \tilde{E}$. Since X is Stein, this projection splits over $f(X)$, i.e., there is a holomorphic vector bundle homomorphism $G: \tilde{E}|f(X) \to E|f(X)$ such that $\pi \circ G$ is the identity on $\tilde{E}|f(X)$. If we denote by E' the image of G, we have a direct sum decomposition

$$E|f(X) = \ker VD(s)|f(X) \oplus E'. \tag{3.3}$$

The restriction $s|E': E' \to Z$ maps the zero section of E' onto $f(X)$ and its derivative is an isomorphism at each point of the zero section. Hence $s|E'$ is biholomorphic near the zero section. The second statement follows immediatelly from this. ♠

Lemma 3.2 allows us to lift short pieces of a homotopy of sections of Z to a homotopy of sections of a vector bundle. In order to lift the entire homotopy we recall from [Gro] the concept of composed and iterated sprays.

3.3 Definition. (Gromov [Gro], sec. 1.3.) *(a) Let (E_1, p_1, s_1) and (E_2, p_2, s_2) be sprays on Z associated to a submersion $h: Z \to X$. The* **composed spray** *(E^*, p^*, s^*) over Z is defined by*

$$E^* = \{(e_1, e_2) \in E_1 \times E_2 : \ s_1(e_1) = p_2(e_2)\},$$
$$p^*(e_1, e_2) = p_1(e_1), \quad s^*(e_1, e_2) = s_2(e_2).$$

(b) Let (E, p, s) be a spray on Z associated to $h: Z \to X$. For each integer $k = 1, 2, 3, \ldots$ the k-th **iterated spray** *$(E^{(k)}, p^{(k)}, s^{(k)})$ is defined by*

$$\begin{aligned} E^{(k)} = \{e = (e_1, e_2, \ldots, e_k) : e_j \in E \ \text{ for } j = 1, 2, \ldots, k, \\ s(e_j) = p(e_{j+1}) \text{ for } j = 1, 2, \ldots, k-1\}, \\ p^{(k)}(e) = p(e_1), \quad s^{(k)}(e) = s(e_k). \end{aligned} \tag{3.4}$$

Note that the composed spray is *not a spray* over Z in the sense of def. 3.1 because E^* does not have a natural structure of a holomorphic vector bundle over Z with respect to the projection $p^*: E^* \to Z$ (the other requirements are satisfied). In fact E^* is the pullback of the vector bundle $p_2: E_2 \to Z$ by the spray map $s_1: E_1 \to Z$, so it is a holomorphic vector bundle over E_1 with the projection $(e_1, e_2) \mapsto e_1$. Similarly we can define the iterated sprays inductively as the composition of k copies of (E, p, s). We begin by taking $(E^{(1)}, p^{(1)}, s^{(1)}) = (E, p, s)$. Suppose that $(E^{(k-1)}, p^{(k-1)}, s^{(k-1)})$ has already been defined. Let $q^{(k)}: E^{(k)} \to E^{(k-1)}$ be the pullback of the bundle $p: E \to Z$ by the spray map $s^{(k-1)}: E^{(k-1)} \to Z$. Set $p^{(k)} = p^{(k-1)} \circ q^{(k)}: E^{(k)} \to Z$ and let $s^k: E^{(k)} \to Z$ be the map induced by $s: E \to Z$ under the pullback. This gives the next iteration $(E^{(k)}, p^{(k)}, s^{(k)})$.

The following lemma implies that the restriction of composed and iterated sprays to Stein subsets of Z admit a holomorphic vector bundle structure.

3.4 Lemma. *Let Y be a Stein manifold and let $p_1: E_1 \to Y$ resp. $p: E \to E_1$ be holomorphic vector bundles over Y resp. E_1. Then E has the structure of a holomorphic vector bundle over Y with respect to the projection $p_1 \circ p: E \to Y$. In fact, this bundle is isomorphic to the Whitney sum $E_1 \oplus E|Y$, where $Y \subset E_1$ is the zero section.*

3.5 Corollary. (Gromov [Gro], sec. 1.3.A') *The restriction of any composed or iterated spray bundle on Z to any Stein subset of Z admits a structure of a holomorphic vector bundle.*

Proof of lemma 3.4. Since E_1 is a holomorphic vector bundle over a Stein manifold, it is itself Stein. Denote the points of E_1 by (y, e), where $y \in Y$ and $p_1(y, e) = y$. Let $h_t: E_1 \to E_1$ $(t \in \mathbf{C})$ be the holomorphic homotopy defined by $h_t(y, e) = (y, te)$. Consider the pull-back by h_t of the bundle $p: E \to E_1$. By a standard result of Cartan theory these pull-backs $h_t^* E$ are isomorphic as holomorphic vector bundles over E_1 [Osb]. At $t = 1$ the map h_1 is the identity on E_1 so we get $h_1^* E = E$. At $t = 0$ the map h_0 is the projection of E_1 onto the zero section which we identify with Y (hence $h_0 = p_1$), so we get $h_0^*(E) = h_0^*(E|Y) = p_1^*(E|Y)$. The latter bundle is clearly isomorphic to the Whitney sum $E_1 \oplus (E|Y)$ which is a holomorphic vector bundle over Y. This proves lemma 3.4. ♠

The next result indicates the main use of iterated sprays.

3.6 Proposition. *Let $f_t: X \to Z$ be a homotopy of holomorphic sections of a holomorphic submersion $h: Z \to X$. Assume that the base X is Stein and that $h: Z \to X$ admits a spray (E, p, s). Then for each open relatively compact subset $V \subset\subset X$ there are an integer $k > 0$ and a homotopy of holomorphic sections ξ_t $(0 \le t \le 1)$ of the iterated spray bundle $E^{(k)}$ (3.4) over the set $f_0(V) \subset Z$ such that*

$$\xi_0(z) = z, \quad s^{(k)}(\xi_t(z)) = f_t(h(z)) \qquad (z \in f_0(V),\ 0 \le t \le 1).$$

Proof. For each fixed $t \in [0,1]$ we can apply lemma 3.2 to lift the sections f_τ for τ near t by the spray map s to a homotopy of holomorphic sections of $E|f_t(V)$. Hence by compactness of $[0,1]$ there are numbers $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ such that for each $j = 0, 1, \ldots, k-1$ there exists a homotopy of holomorphic sections ξ_t^j of $E|f_{t_j}(V)$ for $t_j \le t \le t_{j+1}$ satisfying

$$s \circ \xi_t^j(f_{t_j}(x)) = f_t(x) \quad (x \in V,\ t_j \le t \le t_{j+1}).$$

In particular we have $s \circ \xi_{t_{j+1}}^j(f_{t_j}(x)) = f_{t_{j+1}}(x)$ for $j = 0, 1, \ldots, k-1$. Comparing these compatibility conditions with those defining the iterated spray bundle $E^{(k)}$ (3.4) we see that these k families can be joined into a single family of sections ξ_t $(0 \le t \le 1)$ of $E^{(k)}|f_0(V)$. Explicitly we define for $x \in V$ and $t_j \le t \le t_{j+1}$:

$$\xi_t(f_0(x)) = \big(\xi_{t_1}^0(f_0(x)), \xi_{t_2}^1(f_{t_1}(x)), \ldots, \xi_{t_j}^{j-1}(f_{t_{j-1}}(x)), \xi_t^j(f_{t_j}(x)), 0, \ldots, 0\big) \in E^{(k)}$$

(the last $k-j-1$ components are the zero elements in the fiber of E over $s \circ \xi_t^j(f_{t_j}(x)) = f_t(x)$). One easily verifies that these sections satisfy the stated conditions. ♠

&4. The h-Runge theorems.

In this section we prove Runge type approximation theorems for holomorphic sections of submersions with a spray over a Stein base. We essentially follow section 1.4 of [Gro], except that some results, in particular theorems 4.2 and 4.5, are more general than what is proved there.

To motivate the discussion we recall that, if X is a Stein manifold and $K \subset X$ is a compact holomorphically convex subset then we can approximate each function holomorphic in a neighborhood of K uniformly on K by functions holomorphic on X. This is the Oka-Weil theorem [Hö2] which extends the classical Runge theorem for planar sets K with connected complement $\mathbf{C}\backslash K$. Of course we cannot expect such results for sections of an arbitrary holomorphic submersion over X; in particular this fails for maps of Stein manifolds into a hyperbolic complex manifold F. On the other hand, if $h: Z \to X$ is a submersion which admits a fiber-spray (def. 3.1) then it turns out that the *Runge approximation property is homotopy independent* in the following sense: If f_t is a homotopy of sections in a neighborhood of K such that f_0 has a holomorphic extension to X, then each section in the homotopy can be approximated on K by sections holomorphic on X. Following Gromov we call such results *h-Runge theorems.* We first state the special case (theorem 4.1) in which there are no additional parameters other than the homotopy parameter t. The general case is explained in theorem 4.2.

4.1 Theorem. (Gromov [Gro], sec. 1.4.) *Let X be a Stein manifold and $h: Z \to X$ a holomorphic submersion which admits a fiber-spray (def. 3.1). Let d be a metric on Z. Let $K \subset X$ be a compact holomorphically convex set. Assume that $U \subset X$ is an open set containing K and $f_t: U \to Z$ $(0 \le t \le 1)$ is a homotopy of holomorphic sections of $h: Z \to X$ over U such that f_0 extends to a holomorphic section over X. Then for each $\epsilon > 0$ there exists a continuous family of holomorphic sections $\tilde f_t: X \to Z$ $(0 \le t \le 1)$ such that $\tilde f_0 = f_0$ and*

$$d\big(\tilde f_t(x), f_t(x)\big) < \epsilon \quad (x \in K,\ 0 \le t \le 1).$$

Remark. In fact we will prove the following stronger result: *There is a continuous family of continuous sections $g_{t,u}: X \to Z$ $(0 \le t, u \le 1)$ which are holomorphic in a neighborhood of K and satisfy*

(a) $g_{t,0} = f_t$ *on* K *for all* $0 \le t \le 1$,

(b) the section $\tilde f_t = g_{t,1}$ *is holomorphic on* X *for each* t,

(c) $g_{0,u} = f_0$ *for all* $0 \le u \le 1$, *and*

(d) $d\big(g_{t,u}(x), f_t(x)\big) < \epsilon$ *for* $x \in K$ *and* $0 \le t, u \le 1$.

The existence of such a homotopy $g_{t,u}$ connecting f_t and $\tilde f_t$ comes for free from the proof of theorem 4.1 and will be useful to us in sect. 6.

Proof. Let (E, p, s) be the spray on Z associated to h. After shrinking U around K we obtain by proposition 3.6 an integer $k > 0$ and a homotopy of sections ξ_t over the set $f_0(U) \subset Z$ of the iterated spray bundle $p^{(k)}: E^{(k)} \to Z$ which are mapped back to f_t by $s^{(k)}: E^{(k)} \to Z$. In particular, ξ_0 is the zero section. Since X is Stein, the restriction $E^{(k)}|f_0(X)$ admits the structure of a holomorphic vector bundle by corollary 3.5. It now suffices to approximate ξ_t on the holomorphically convex subset $f_0(K)$ of $f_0(X)$ by a homotopy of holomorphic sections of $E^{(k)}|f_0(X)$ (keeping the zero section fixed) and to take $\tilde f_t$ to be their images in Z by the spray map $s^{(k)}$. This can be done by the usual Oka-Weil approximation theorem for sections of a holomorphic vector bundle, and it can be reduced to the aproximation of functions by inserting the given bundle as a subbundle of a trivial bundle. Even though this is standard, we outline the proof since we will later need a stronger result with parameters.

For convenience we let $p': E' = f_0^*(E^{(k)}) \to X$ to be the pull-back of $E^{(k)}$ by the section $f_0: X \to Z$, and we denote by $s': E' \to Z$ the holomorphic map induced by the spray $s^{(k)}: E^{(k)} \to Z$. We may then consider ξ_t as sections of the bundle $p': E' \to X$ over $U \subset X$ such that ξ_0 is the zero section and $s' \circ \xi_t(x) = f_t(x)$ for all $x \in U$ and $0 \le t \le 1$. Choose a smooth function $\chi: X \to [0, 1]$ which is identically one in a neighborhood of K and has compact support contained in U. Since K is holomorphically convex in X, there is a smooth plurisubharmonic exhaustion function $\rho: X \to \mathbf{R}_+$ which vanishes in a neighborhood $U_0 \subset\subset U$ of K and is strictly positive on the support of $d\chi$. For each fixed value of $\tau > 0$ there is a section v_t of E' which solves the equation

$$\overline\partial v_t = \overline\partial(\chi\xi_t) = \xi_t\, \overline\partial\chi \tag{4.1}$$

and whose L_2 norm with weight $e^{-\tau\rho}$ (measured in a fixed hermitean metric) is bounded on each compact set in X by a constant times the norm of the data $\xi_t\overline\partial\chi$ (with a constant independent of τ). By Hörmander [Hö1] such a solution is given by a linear operator $v_t = T_\tau(\xi_t\overline\partial\chi)$. We have $v_0 = 0$, each v_t is smooth, and the family is continuous in t. Set

$$\tilde g_{t,u} = \chi\xi_t - uv_t, \quad g_{t,u} = s' \circ \tilde g_{t,u} \quad (0 \le t, u \le 1).$$

Clearly $g_{t,u}: X \to Z$ is a continuous family of sections of $Z \to X$ which satisfies (a)–(c) in the remark following theorem 4.1. By choosing τ sufficiently large (depending on ϵ) the family $g_{t,u}$ will also satisfy (d) which can be seen as follows. When $\tau \to \infty$, the L_2 norm with weight $e^{-\tau\rho}$ of $\xi_t\overline{\partial}\chi$ tends to zero since $\rho > 0$ on $\mathrm{supp}\,\overline{\partial}\chi$. Since ρ vanishes in U_0, it follows that unweighted $L_2(U_0)$ norm $||v_t||_{L_2(U_0)}$ tends to zero as $\tau \to +\infty$. By the Cauchy estimates the sup norm of $v_t|K$ tends to zero and hence the sections $g_{t,u}$ converge to f_t as $\tau \to +\infty$, uniformly in t and u. In particular, $\tilde{f}_t = g_{t,1}$ $(0 \le t \le 1)$ is a homotopy of holomorphic sections of $Z \to X$ satisfying theorem 4.1. ♠

We will also need the following parametric version of the h-Runge theorem.

4.2 Theorem. *Let X be a Stein manifold and $h: Z \to X$ a holomorphic submersion which admits a spray (def. 3.1). Let $K \subset X$ be a compact holomorphically convex set and let $U, V \subset X$ be open, relatively compact subsets in X such that $K \subset U \subset V$. Let Y be a compact Hausdorff space (the parameter space), $Y_0 \subset Y$ a compact subset, and $Y' \subset Y$ an open set containing Y_0. Assume that $f_{y,t}: U \to Z$ is a family of holomorphic sections of $h: Z \to X$, depending continuously on $y \in Y$ and $0 \le t \le 1$, such that the sections $f_{y,0}$ $(y \in Y)$ and $f_{y,t}$ $(y \in Y'$, $0 \le t \le 1)$ extend to holomorphic sections over X. Then for each $\epsilon > 0$ there exists a continuous family of holomorphic sections $\tilde{f}_{y,t}: V \to Z$ over V $(y \in Y,\ 0 \le t \le 1)$ satisfying*

(a) $\tilde{f}_{y,0} = f_{y,0}$ for all $y \in Y$,

(b) $\tilde{f}_{y,t} = f_{y,t}$ for all $y \in Y_0$ and $0 \le t \le 1$, and

(c) $d\big(\tilde{f}_{y,t}(x), f_{y,t}(x)\big) < \epsilon$ for all $x \in K$, $y \in Y$ and $0 \le t \le 1$.

Remark. As in theorem 4.1 the proof will show that the approximating family $\tilde{f}_{y,t}$ can be chosen so that it can be connected to the initial family $f_{y,t}$ by a homotopy of sections $g_{y,t,u}$ $(u \in [0,1])$ such that the homotopy is fixed for each $y \in Y_0$ (where $\tilde{f}_{y,t} = f_{y,t}$).

Proof. We begin by reducing to the approximation problem for families of sections of an iterated spray bundle. This is essentially proposition 3.6 with the addition of the parameter $y \in Y$. Let (E, p, s) be a spray on Z and $(E^{(k)}, p^{(k)}, s^{(k)})$ its k-th iterated spray (3.4).

4.3 Proposition. (Assumptions as in theorem 4.2.) *Let U' be an open set in X such that $K \subset U' \subset\subset U$. Then there is an integer $k > 0$ and a continuous family of holomorphic sections $\xi_{y,t}$ of $E^{(k)}|f_{y,0}(U')$ $(y \in Y,\ 0 \le t \le 1)$ such that*

(i) $\xi_{y,0}$ is the zero section for each $y \in Y$,

(ii) $\xi_{y,t}$ extends to a holomorphic section of $E^{(k)}|f_{y,0}(V)$ for each y in a neighborhood of Y_0 and $t \in [0,1]$, and

(iii) $s^{(k)} \circ \xi_{y,t}(f_{y,0}(x)) = f_{y,t}(x)$ for all $x \in U'$, $y \in Y$ and $0 \le t \le 1$. Moreover, for each y in a neighborhood of Y_0 this holds for all $x \in V$.

Proof. It suffices to prove that each fixed $t_0 \in [0,1]$ has a neighborhood $I_0 \subset [0,1]$ such that there exists a family of holomorphic sections $\xi_{y,t}$ of the vector bundle $E|f_{y,t_0}(U')$, depending continuously on $y \in Y$ and $t \in I_0$, such that ξ_{y,t_0} is the zero section and

$$s \circ \xi_{y,t}(z) = f_{y,t}(h(z)) \quad (z \in f_{y,t_0}(U'),\ y \in Y,\ t \in I_0). \tag{4.2}$$

Moreover, for y in a neighborhood of Y_0 the property (4.2) must hold over the larger set $z \in f_{y,t_0}(V)$. Proposition 4.3 then follows from this as in the proof of proposition 3.6 by using the compactness of $[0,1]$ and combining the finitely many families of sections $\xi^j_{y,t}$ obtained over k subintervals $t \in [t_j, t_{j+1}] \subset [0,1]$ into a single family of sections of the iterated spray bundle $E^{(k)}$ over the sets $f_{y,0}(U')$ resp. $f_{y,0}(V)$.

We will consider the case $t_0 = 0$ when all initial sections $f_{y,0}$ exist over V. The only difference for $t_0 > 0$ is that some sections only exist over the smaller set U', but the proof goes through in the same way. Denote by $E_0 = \ker VDs \subset E$ the kernel of the vertical derivative of s (3.2) and let $\pi: E \mapsto \tilde{E} = E/E_0$ be the quotient projection. Recall that a *holomorphic splitting* of π is a holomorphic vector bundle homomorphism $G: \tilde{E} \to E$ such that $\pi \circ G$ is the identity on $\tilde{E}$; in such case we have $E = E_0 \oplus G(\tilde{E})$. A splitting exists over any Stein subset of Z [GRo, p. 256].

4.4 Lemma. (Hypotheses as in theorem 4.2.) *There is a family of holomorphic splittings $G_y: \tilde{E}|f_{y,0}(V) \to E|f_{y,0}$ which depends continuously on $y \in Y$, and hence there is a holomorphic direct sum splitting*

$$E|f_{y,0}(V) = E_0|f_{y,0}(V) \oplus E'_y \tag{4.3}$$

depending continuously on $y \in Y$.

Remark. We may consider the restricted bundles in lemma 4.4 as subsets of the bundle E resp. $\tilde{E} = E/E_0$, and the continuity of the family G_y with respect to y should be understood in this sense. Lemma 4.4 also holds if we replace $t = 0$ by an arbitrary $t_0 \in [0,1]$, except that in this case one must replace for each $y \in Y \backslash Y'$ the set V in (4.3) by the smaller set U' (since the section f_{y,t_0} is only defined on U').

Proof of lemma 4.4. For each fixed $y \in Y$ the section $f_{y,0}(X) \subset Z$ (which is a Stein submanifold of Z) is contained in an open Stein set $D_y \subset Z$ according to a theorem of Siu [Siu]. Hence by compactness of Y there is an open cover $\{Y_j : 1 \le j \le J\}$ of Y and a family of open Stein subsets $\{D_j : 1 \le j \le J\}$ of Z such that $f_{y,0}(V) \subset D_j$ when $y \in Y_j$. Let χ_j be a continuous partition of unity on Y subordinate to the cover $\{Y_j\}$. Let $H_j: \tilde{E}|D_j \to E|D_j$ be a splitting of $\pi: E \to \tilde{E}$ over D_j (such H_j exists since D_j is Stein). Then the family

$$G_y = \sum_{j=1}^{J} \chi_j(y) H_j: \quad \tilde{E}|f_{y,0}(V) \to E|f_{y,0}(V)$$

satisfies lemma 4.4. Note that the map G_y is well defined since the coefficient $\chi_j(y)$ vanishes when $f_{y,0}(V)$ is not contained in D_j, and G_y is a splitting of π since it is a convex linear combination of finitely many splittings. ♠

We continue with the proof of proposition 4.3. By the inverse function theorem the map $s: E'_y \to Z$ (the restriction of the spray map $s: E \to Z$ to E'_y) is a biholomorphic map from a neighborhood of the zero section of E'_y onto a neighborhood of the set $f_{y,0}(V)$ in Z (since the vertical derivative VDs is an isomorphism when restricted to E'_y). Denote the local inverse of this map by u_y. Because of continuous dependence on the data on $y \in Y$ and the compactness of Y the neighborhoods on which the maps u_y are defined can be chosen uniformly with respect to $y \in Y$. Hence there is a $t_1 > 0$ such that for $0 \le t \le t_1$ and $y \in Y$ the set $f_{y,t}(U')$ belongs to the domain of u_y. The sections $\xi_{y,t} = u_y \circ f_{y,t}: f_{y,0}(U') \to E'_y \subset E$ for $0 \le t \le t_1$ then satisfy (4.2). This completes the proof of proposition 4.3. ♠

To complete the proof of theorem 4.2 it remains to approximate the sections $\xi_{y,t}$ uniformly on $f_{y,0}(K)$ by holomorphic sections of $E^{(k)}|f_{y,0}(V)$. We shall first do this locally with respect to y and finally patch the approximations together. Let $Y' \supset Y_0$ be a neighborhood of Y_0 such that the sections $\xi_{y,t}$ for $y \in Y'$ are defined and holomorphic over the set $f_{y,0}(V) \subset Z$. As in the proof of lemma 4.4 there exist a finite open cover $\{Y_j: 1 \le j \le J\}$ of Y and open Stein subsets $D_j \subset Z$ such that $f_{y,0}(V) \subset D_j$ for all y in a neighborhood $Y'_j \supset Y_j$ of $\overline{Y}_j$. By corollary 3.5 the restriction $E^{(k)}|D_j$ has a vector bundle structure and we can embed it as a vector subbundle of a trivial bundle $D_j \times \mathbf{C}^N$. With this identification we may write

$$\xi_{y,t}(f_{y,0}(x)) = \big(f_{y,0}(x), g_{y,t}(x)\big)$$

where $g_{y,t}: U' \to \mathbf{C}^N$ is a holomorphic map. Note that $g_{y,0} = 0$ for all $y \in \overline{Y}_j$, and $g_{y,t}$ extends holomorphically to V for $y \in Y' \cap Y'_j$. Choose a continuous function $u: Y \to [0,1]$ which vanishes in smaller neighborhood of Y_0 and is identically one outside Y'. In now suffices to approximate the functions $g_{y,t}$ by functions holomorphic in V without changing their values for $y \in Y'$ (when they are already holomorphic in V). This is done by taking

$$\tilde g_{y,t}(x) = u(y)\chi(x) g_{y,t}(x) - v_{y,t}(x),$$

where $v_{y,t}$ is the solution of the $\overline{\partial}$-equation

$$\overline{\partial}_x v_{y,t}(x) = u(y) g_{y,t}(x) \overline{\partial}_x \chi(x)$$

with minimal L_2-norm with respect to a suitable weight and $\chi: X \to \mathbf{R}_+$ is a plurisubharmonic function as in the proof of theorem 4.1. Since $u(y) = 0$ for y near Y_0, we have $v_{y,t} = 0$ and hence $\tilde g_{y,t} = g_{y,t}$ for such y.

This gives for each $j = 1, 2, \ldots, J$ an approximating family of holomorphic sections $\xi^j_{y,t}$ of $E^{(k)}|f_{y,0}(V)$ for y in a neighborhood Y'_j of $\overline{Y}_j$. To conclude the proof of theorem 4.2 it suffices to combine these families into a single family of sections $\tilde\xi_{y,t}$ $(y \in Y)$ and then take

$$\tilde f_{y,t} = s^{(k)} \circ \tilde\xi_{y,t} \qquad (y \in Y,\ 0 \le t \le 1). \tag{4.4}$$

However we cannot do the patching by the usual partition of unity in the y variable since the vector bundle structures on different restrictions $E^{(k)}|D_j$ may not agree on their intersection. Instead we can do a stepwise extension as follows. For $y \in Y'_1$ we take $\tilde\xi_{y,t} = \xi^1_{y,t}$. Choose a continuous function $\chi_1: Y \to [0,1]$ such that $\chi_1(y) = 1$ for $y \in \overline{Y}_1$ and $\operatorname{supp}\chi_1 \subset Y'_1$. For $y \in Y'_2$ we then set

$$\tilde\xi_{y,t} = \chi_1(y)\xi^1_{y,t} + (1 - \chi_1(y))\xi^2_{y,t} \tag{4.5}$$

where the linear combination is taken with respect to a vector bundle structure on $E^{(k)}|D_2$. This definition is good since $\chi_1(y) = 0$ for those values of y (i.e., for $y \in Y \backslash Y_1'$) for which the section $\xi^1_{y,t}$ is not defined, and so the patching only occurs over the set $y \in Y_2'$. Clearly the family (4.5) extends continuously to all parameter values $y \in Y_1' \cup Y_2'$ and it equals the previously chosen family of sections for $y \in Y_1$. Moreover, for $y \in Y_1' \cap Y_2'$ both sections $\xi^1_{y,t}$ and $\xi^2_{y,t}$ approximate the initial section $\xi_{y,t}$ over $f_{y,0}(K)$, and hence the same is true for their convex combination (4.5). We now continue in the same way by patching the family (4.5) with $\xi^3_{y,t}$ over the parameter set $y \in Y_3'$ with respect to the vector bundle structure on $E^{(k)}|D_3$. In finite number in steps we obtain a continuous family of sections $\tilde{\xi}_{y,t}$ such that the sections (4.4) satisfy theorem 4.2. ♠

Our next result, which is an immediate application of theorem 4.2, is essential in our approach to extending a holomorphic section across a pseudoconvex bump.

4.5 Theorem. *Let X be a Stein manifold and $Z = X \times F \to X$ a trivial holomorphic fiber bundle whose fiber F admits a spray (def. 1.2). Let d be a metric on Z. Suppose that $U \subset X$ is a Stein domain which is Runge in X and $\Theta_u \colon X \to X$ is a family of holomorphic mappings, depending continuously on $u \in [0,1]$, such that*

(i) Θ_0 is the identity on X,

(ii) $\Theta_u(U) \subset U$ for all $u \in [0,1]$, and

(iii) $\Theta_1(X) \subset U$.

Let $f_t \colon U \to Z$ $(0 \le t \le 1)$ be a homotopy of holomorphic sections such that f_0 and f_1 extend holomorphically to X. For each compact set $K \subset U$, relatively compact set $V \subset\subset X$, and $\epsilon > 0$ there exists a homotopy of holomorphic sections $\tilde{f}_t \colon V \to Z$ $(0 \le t \le 1)$ such that $\tilde{f}_0 = f_0$, $\tilde{f}_1 = f_1$, and

$$d\big(\tilde{f}_t(x), f_t(x)\big) < \epsilon \quad (x \in K,\ 0 \le t \le 1).$$

Remark. In a typical application of theorem 4.5 the sets $U \subset X$ are bounded convex domains in $\mathbf{C}^n$ and Θ_u is a family of linear contractions to a point in U. By a limiting argument it is possible to prove that such a holomorphic homotopy $\tilde{f}_t$ exists on all of X, but we shall not need this.

Proof. By reparametrizing the family f_t we may assume that for some small $\delta > 0$ we have $f_t = f_0$ for $0 \le t \le \delta$ and $f_t = f_1$ for $1 - \delta \le t \le 1$. Choose a continuous function $u \colon [0,1] \to [0,1]$ such that $u(t) = 0$ for t near 0 or 1, and $u(t) = 1$ for $\delta \le t \le 1 - \delta$. We shall identify sections of the trivial bundle $X \times F \to X$ with mappings $X \to F$. Set

$$f_{t,s} = f_t \circ \Theta_{(1-s)u(t)} \quad (0 \le t, s \le 1).$$

This family of sections satisfies the following properties:

(a) $f_{t,s}$ is defined and holomorphic in U for each $0 \le t, s \le 1$,

(b) $f_{t,0} = f_t \circ \Theta_{u(t)}$ is defined and holomorphic on all of X for each $t \in [0,1]$,

(c) $f_{t,1} = f_t \circ \Theta_0 = f_t$ for each $t \in [0,1]$, and

(d) $f_{0,s} = f_0 \circ \Theta_{u(0)} = f_0$ and $f_{1,s} = f_1 \circ \Theta_{u(1)} = f_1$ for all $s \in [0,1]$.

It remains to apply theorem 4.2 with the parameter space $t \in Y = [0,1]$ and the subspace $Y_0 = \{0,1\}$. (Note that our current variable s plays the role of the time parameter t in theorem 4.2.) If $\tilde{f}_{t,s}$ is the approximating family of holomorphic sections in V as in theorem 4.2 then the sections $\tilde{f}_t = \tilde{f}_{t,1}$ $(0 \le t \le 1)$ satisfy theorem 4.5. ♠

&5. Gluing holomorphic sections over Cartan pairs.

The main results of this section are theorems 5.1 and 5.5 on gluing holomorphic sections over Cartan pairs.

5.1 Theorem. (Gromov [Gro], sec. 1.6) *Let $h: Z \to X$ be a holomorphic submersion onto a Stein manifold X, let d a metric on Z, and let (A,B) a Cartan pair in X (def. 2.3) such that the set $C = A \cap B$ is Runge in B. Suppose that $\tilde{B}$ is an open neighborhood of B in X such that the restriction $Z|\tilde{B} = h^{-1}(\tilde{B})$ admits a fiber-spray (def. 3.1). Let $\tilde{A} \subset X$ be an open neighborhood of A and $a: \tilde{A} \to Z$ a holomorphic section of $Z \to X$ over $\tilde{A}$. Then for each $\epsilon > 0$ there is a $\delta > 0$ satisfying the following property. If $b: \tilde{B} \to Z$ is a holomorphic section satisfying $d\big(a(x), b(x)\big) < \delta$ for $x \in \tilde{C} = \tilde{A} \cap \tilde{B}$, there exist homotopies a_t (resp. b_t), $0 \le t \le 1$, of holomorphic sections over a neighborhood A' of A (resp. over a neighborhood B' of B) such that $a_0 = a$, $b_0 = b$, $a_1 = b_1$ on $C' = A' \cap B'$, and*

$$d\big(a_t(x), a(x)\big) < \epsilon \quad (x \in A',\ 0 \le t \le 1);$$
$$d\big(b_t(x), b(x)\big) < \epsilon \quad (x \in C',\ 0 \le t \le 1).$$

Using sprays we shall reduce the proof of theorem 5.1 to the model situation described by the following proposition which is analogous to the classical Cartan–Grauert attaching lemma. We denote by $H^\infty(\Omega, \mathbf{C}^n)$ the Banach space of bounded holomorphic maps $\Omega \to \mathbf{C}^n$ equipped with the sup norm over all components.

5.2 Proposition. *Let (A,B) be a Cartan pair in a Stein manifold X such that $C = A \cap B$ is Runge in B. Let $\tilde{C} \subset X$ be an open neighborhood of C, $U \subset \mathbf{C}^n$ an open neighborhood of the origin in $\mathbf{C}^n$, and $\psi_0: \tilde{C} \times U \to \mathbf{C}^n$ a bounded holomorphic map such that for each $x \in \tilde{C}$, $\psi_0(x,0) = 0$ and $\psi_0(x,\cdot): U \to \mathbf{C}^n$ is injective (i.e., biholomorphic onto its image). Then there are neighborhoods $A' \supset A$ and $B' \supset B$ with $C' = A' \cap B' \subset\subset \tilde{C}$, a neighborhood W of ψ_0 in the Banach space $H^\infty(\tilde{C} \times U, \mathbf{C}^n)$, and smooth Banach space operators $\mathcal{A}': W \to H^\infty(A', \mathbf{C}^n)$, $\mathcal{B}': W \to H^\infty(B', \mathbf{C}^n)$, with $\mathcal{A}'(\psi_0) = 0$ and $\mathcal{B}'(\psi_0) = 0$, such that for each $\psi \in W$ the bounded holomorphic maps $\alpha = \mathcal{A}'(\psi): A' \to \mathbf{C}^n$, $\beta = \mathcal{B}'(\psi): B' \to \mathbf{C}^n$ satisfy*

$$\psi\big(x, \alpha(x)\big) = \beta(x) \quad (x \in A' \cap B'). \tag{5.1}$$

Moreover, if $\psi \in W$ satisfies $\psi(x,0) = 0$ for $x \in \tilde{C}$ then $\mathcal{A}'(\psi) = 0$ and $\mathcal{B}'(\psi) = 0$.

Remark. We can view a pair of maps satisfying (5.1) as a section of a nonlinear bundle over $A' \cup B'$ obtained by patching the trivial bundles over A' resp. B' by the map ψ.

For later application to parametrized families it is important to have a canonically given solution (i.e., by operators).

Proof of proposition 5.2. By shrinking $\tilde{C}$ we may assume that it is Runge in a neighborhood B_0 of B. We choose neighborhoods $A' \supset A$ and $B' \supset B$ as in lemma 2.4 so that $B' \subset B_0$, $C' = A' \cap B' \subset\subset \tilde{C}$, and there are bounded linear operators $\mathcal{A}: H^\infty(C', \mathbf{C}^n) \to H^\infty(A', \mathbf{C}^n)$ and $\mathcal{B}: H^\infty(C', \mathbf{C}^n) \to H^\infty(B', \mathbf{C}^n)$ satisfying $c = \mathcal{A}c - \mathcal{B}c$ for all $c \in H^\infty(C', \mathbf{C}^n)$. Consider first the case when $\psi_0(x, u) = u$ is the identity map in the u-variable for each $x \in \tilde{C}$. Consider the operator

$$\Phi: H^\infty(C', \mathbf{C}^n) \times H^\infty(\tilde{C} \times U, \mathbf{C}^n) \to H^\infty(C', \mathbf{C}^n),$$
$$\Phi(c, \psi)(x) = \psi\big(x, \mathcal{A}c(x)\big) - \mathcal{B}c(x) \qquad (x \in C').$$

We claim that Φ is defined and smooth for $c \in H^\infty(C', \mathbf{C}^n)$ in a neighborhood of the origin and for $\psi \in H^\infty(\tilde{C} \times U, \mathbf{C}^n)$. Clearly Φ is linear and hence smooth in ψ. To see that Φ is smooth in c we choose a neighborhood $U' \subset \mathbf{C}^n$ of 0 such that $\overline{U}' \subset U$. By Cauchy estimates the restriction map $\psi \to \psi | C' \times U'$ is a bounded linear operator from the space $H^\infty(\tilde{C} \times U, \mathbf{C}^n)$ to $\mathcal{C}^\infty(C' \times U', \mathbf{C}^n)$. On the set of c's for which $\mathcal{A}c(x) \in U'$ for all $x \in C'$ (these form an open neighborhood of the origin in $H^\infty(C', \mathbf{C}^n)$) the first term in Φ is the composition operator of a linear operator $\mathcal{A}$ with a smooth map ψ. Hence Φ is a smooth operator. In fact we only need that Φ is of class $\mathcal{C}^1$ which is seen directly from the formula for its differential

$$D\Phi(c, \psi)(c', \psi') = \psi'(\cdot\,, \mathcal{A}c) + D_2\psi(\cdot\,, \mathcal{A}c)\mathcal{A}c' - \mathcal{B}c'.$$

Note that $\Phi(c, \psi_0) = \mathcal{A}c - \mathcal{B}c = c$. Hence $D_c\Phi(0, \psi_0)$ (the partial derivative of Φ with respect to the first variable) is the identity map on $H^\infty(C', \mathbf{C}^n)$. By the implicit function theorem in Banach spaces there is an open set $W \subset H^\infty(\tilde{C} \times U, \mathbf{C}^n)$ containing ψ_0 and a smooth map $\mathcal{C}: W \to H^\infty(C', \mathbf{C}^n)$ such that $\Phi(\mathcal{C}(\psi), \psi) = 0$ and $\mathcal{C}(\psi_0) = 0$. Moreover, if $\psi \in W$ satisfies $\psi(x, 0) = 0$ for all $x \in \tilde{C}$, then $c = 0$ solves the equation $\Phi(c, \psi) = 0$ and hence by local uniqueness of solutions we have $\mathcal{C}(\psi) = 0$ for any such ψ. The operators $\mathcal{A}' = \mathcal{A} \circ \mathcal{C}$ and $\mathcal{B}' = \mathcal{B} \circ \mathcal{C}$ then satisfy proposition 5.2.

The general case (when ψ_0 is not the identity, or even close to the identity) can be reduced to the special case as follows. Since C is Runge in B, there are open sets $\tilde{C}_0 \subset X$, $B_0 \subset X$, $U_0, U_1 \subset \mathbf{C}^n$, satisfying $C \subset \tilde{C}_0 \subset\subset \tilde{C}$, $0 \in U_0 \subset\subset U_1 \subset\subset U$, $B \subset B_0$, such that we can approximate ψ_0 as well as desired on $\tilde{C}_0 \times U_1$ by a holomorphic map $\tilde{\psi}: B_0 \times \mathbf{C}^n \to \mathbf{C}^n$ such that $\tilde{\psi}(x, \cdot\,)$ is biholomorphic on U_1 for each $x \in \tilde{C}_0$ and $\tilde{\psi}(x, 0) = 0$ for all $x \in B_0$. If the approximation of ψ_0 by $\tilde{\psi}$ is sufficiently close on $\tilde{C}_0 \times U_1$, there is a unique holomorphic map $\psi_0': \tilde{C}_0 \times U_0 \to U_1$ which satisfies

$$\psi_0(x, u) = \tilde{\psi}(x, \psi_0'(x, u)) \quad (x \in \tilde{C}_0,\ u \in U_0)$$

and which is so close to the map $(x, u) \to u$ on the set $\tilde{C}_0 \times U_0$ that it belongs to the domain $W \subset H^\infty(\tilde{C}_0 \times U_0, \mathbf{C}^n)$ of the operators $\mathcal{C}$, $\mathcal{A}'$, and $\mathcal{B}'$ obtained in the special case (with respect to the smaller set $\tilde{C}_0 \times U_0$). We may assume that the sets $A' \supset A$, $B' \supset B$, related to the operators $\mathcal{A}'$ resp. $\mathcal{B}'$ as above, satisfy $B' \subset B_0$ and $A' \cap B' \subset\subset \tilde{C}_0$. Note that $\mathcal{C}(\psi'_0) = 0$ since $\psi'_0(x,0) = 0$. Moreover, for each ψ which is sufficiently uniformly close to ψ_0 on $\tilde{C} \times U$ we have

$$\psi(x,u) = \tilde{\psi}(x, \psi'(x,u)) \quad (x \in \tilde{C}_0,\ u \in U_0)$$

where $\psi' : \tilde{C}_0 \times U_0 \to U_1$ belongs to W. In fact, $\psi \to \psi'$ defines a smooth Banach space operator $\Psi: W_0 \subset H^\infty(\tilde{C} \times U, \mathbf{C}^n) \to H^\infty(\tilde{C}_0 \times U_0, \mathbf{C}^n)$ in an open neighborhood W_0 of ψ_0, with range in W. For each $\psi \in W_0$ and $\psi' = \Psi(\psi) \in W$ the holomorphic maps $\alpha = \mathcal{A}'(\psi'): A' \to \mathbf{C}^n$ and $\beta' = \mathcal{B}'(\psi'): B' \to \mathbf{C}^n$ satisfy $\psi'(x, \alpha(x)) = \beta'(x)$ for $x \in C'$. Hence the pair $\alpha(x)$ and $\beta(x) = \tilde{\psi}(x, \beta'(x))$ (the latter one is defined and holomorphic in B') satisfies

$$\psi(x, \alpha(x)) = \tilde{\psi}\big(x, \psi'(x, \alpha(x))\big) = \tilde{\psi}(x, \beta'(x)) = \beta(x)$$

which is precisely (5.1). By construction α and β are obtained from ψ by a composition of smooth Banach space operators. ♠

In order to reduce the proof of theorem 5.1 to proposition 5.2 we also need the following lemma. Denote by $B^n(\eta) \subset \mathbf{C}^n$ the open ball of radius η.

5.3 Lemma. *Let $h: Z \to X$ be a holomorphic submersion onto a Stein manifold X, let $A \subset X$ be a compact set with a Stein neighborhood basis, and let $a: \tilde{A} \to Z$ be a holomorphic section defined in an open set $\tilde{A} \supset A$. Then there are an integer $n > 0$, a number $\eta > 0$, a Stein open set $U \subset Z$ containing $a(A)$, and a holomorphic map $\tilde{s}: U \times B^n(\eta) \to Z$ such that for all $z \in U$ we have*

(i) $h \circ \tilde{s}(z,t) = h(z)$ for all $t = (t_1, t_2, \ldots, t_n) \in B^n(\eta)$,

(ii) $\tilde{s}(z,0) = z$, and

(iii) the vectors $\tilde{V}_j(z) = \frac{\partial}{\partial t_j} \tilde{s}(z,0)$ $(1 \le j \le n)$ span $VT_z(Z)$.

Remark. Note that $\tilde{s}$ satisfies all requirements for a fiber-spray except that it is not defined globally on $U \times \mathbf{C}^n$.

Proof of Lemma 5.3. The set $a(A) \subset Z$ has a basis of Stein neighborhoods in Z according to [Siu] and [Shd]. By Cartan's theory there exist finitely many holomorphic vector fields $\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_n$ in a neighborhood of $a(A)$ in Z which are tangent to $VT(Z)$ (i.e., they are holomorphic sections of the vertical tangent bundle $VT(Z)$) and which span $VT(Z)$ at each point in the given set. Let θ_j^t be the flow of $\tilde{V}_j$. There is a small Stein neighborhood $U \subset Z$ of $a(A)$ and an $\eta > 0$ such that the map $\tilde{s}: U \times B^n(\eta) \to \mathbf{C}^n$ given by

$$\tilde{s}(z,t) = \tilde{s}(z, t_1, \ldots, t_n) = \theta_1^{t_1} \circ \theta_2^{t_2} \circ \cdots \circ \theta_n^{t_n}(z)$$

satisfies all requirements. ♠

Proof of theorem 5.1. Let $\tilde{s}: U \times B^n(\eta) \to Z$ be the local spray given by lemma 5.3. Choose a Stein open set $\tilde{A} \supset A$ so that $a(\tilde{A}) \subset\subset U$. Set $s_1(x,t) = \tilde{s}(a(x), t)$ for $x \in \tilde{A}$ and

$t \in B^n(\eta)$. Then $s_1(x,0) = a(x)$ and s_1 is a submersion along $t = 0$ (i.e., a local spray onto a neighborhood of $a(\tilde{A})$ in Z). Suppose that $\tilde{B} \supset B$ is an open Stein set and $b: \tilde{B} \to Z$ is a holomorphic section such that b approximates a in an open neighborhood $\tilde{C}$ of $C = A \cap B$. Our goal is to construct a fiber preserving holomorphic map $s_2: \tilde{B} \times \mathbf{C}^n \to Z$ such that $s_2(x,0) = b(x)$ and such that s_2 is a submersion near $t = 0$ which approximates s_1 in a neighborhood of $C \times \{0\}^n$. We then solve the equation $s_2(x, \psi(x,t)) = s_1(x,t)$ to get a map ψ as in proposition 5.2. If α and β satisfy proposition 5.2 (i.e., $\psi(x, \alpha(x)) = \beta(x)$ for x near C) then

$$a_1(x) = s_1(x, \alpha(x)), \qquad b_1(x) = s_2(x, \beta(x)$$

are holomorphic sections of Z over neighborhoods of A resp. B which agree near $C = A \cap B$.

To construct s_2 we assume that $\tilde{B} \supset B$ is so small that there exists a fiber-spray (E, p, s) over $Z|\tilde{B} = h^{-1}(\tilde{B})$. Choose a Stein open set $U' \subset Z$ such that $a(C) \subset U' \subset\subset U \cap h^{-1}(\tilde{B})$ (so the spray s is defined over $Z|U'$). Let $\tilde{V}_j$ be the vector fields as in lemma 5.3. Since $VDs(0): E \to VT(Z)$ is a surjective vector bundle homomorphism and the set U' is Stein, we can split $E|U' = \ker VDs(0) \oplus E'$ and lift the sections $\tilde{V}_j$ to holomorphic sections V_j of $E' \subset E|U'$ such that $VDs(0_z)V_j(z) = \tilde{V}_j(z)$ for all $z \in U'$ and $1 \le j \le n$. For each $z \in U'$ and each collection of n vectors $\mathcal{W} = \{W_1, \ldots, W_n\} \subset E_z$ we define a map $s_{\mathcal{W}}(z, \cdot): \mathbf{C}^n \to Z_{h(z)}$ by

$$s_{\mathcal{W}}(z,t) = s\left(\sum_{j=1}^{n} t_j\, W_j\right).$$

We have $s_{\mathcal{W}}(z,0) = z$ and $\frac{\partial}{\partial t_j} s_{\mathcal{W}}(z,0) = VDs(0_z)W_j$. In particular, for the collection $\mathcal{V}(z) = \{V_1(z), \ldots, V_n(z)\}$ we get for $j = 1, 2, \ldots, n$ and $z \in U'$

$$\frac{\partial}{\partial t_j} s_{\mathcal{V}(z)}(z,0) = VDs(0_z)V_j(z) = \tilde{V}_j(z) = \frac{\partial}{\partial t_j} \tilde{s}(z,0). \tag{5.2}$$

The map $s_{\mathcal{W}}(z,t)$ is holomorphic in all arguments, including $\mathcal{W}$.

5.4 Lemma. *Let d be a metric on Z and d' a metric on E. There are numbers $\eta > 0$, $\delta > 0$ with the following property. For each pair of points $z, w \in U'$ such that $h(z) = h(w)$ and $d(z,w) < \delta$, and for each collection $\mathcal{W} = \{W_1, \ldots, W_n\} \subset E_w$ such that $d'(W_j, V_j(w)) < \delta$ for $j = 1, \ldots, n$, there is an injective holomorphic map $\phi_{\mathcal{W}}(z, w, \cdot): B^n(\eta) \to \mathbf{C}^n$ satisfying*

(i) $s_{\mathcal{W}}(w, \phi_{\mathcal{W}}(z,w,t)) = \tilde{s}(z,t)$,

(ii) *ϕ is holomorphic in all arguments $z, w, \mathcal{W}, t$, and*

(iii) $\phi_{\mathcal{W}}(z,z,0) = 0$.

Proof. Since $U \subset Z$ is Stein, we have a splitting $U \times \mathbf{C}^n = M \oplus N$ where M_z is the kernel of $D_t\tilde{s}(z,0)$ (the t-derivative of $\tilde{s}$ at the zero section) and N is some holomorphic complementary bundle. We split the fiber vectors $t = (t', t'') \in M_z \oplus N_z$ accordingly (so the splitting depends on the base point z). For each $z \in U$ the restriction of $\tilde{s}$ to the fiber N_z maps a neighborhood of $0_z \in N_z$ biholomorphically onto a neighborhood of z in the fiber $Z_{h(z)}$. The same is true for the restriction

$$t'' \in N_z \to \tilde{s}(z, (t', t'')) \in Z_{h(z)} \tag{5.3}$$

of $\tilde{s}$ to fibers $\{t'\} \oplus N_z$ for all sufficiently small vectors $t' \in M_z$.

Now (5.2) shows that for each pair of points $z, w \in U'$ in the same fiber $Z_{h(z)}$ which are sufficiently close together, for each sufficiently small vector $t' \in M_z$, and for each collection of n vectors $\mathcal{W} = \{W_1, \ldots, W_n\} \subset E_w$ which are sufficiently close to the corresponding vectors $\mathcal{V}(w) = \{V_1(w), \ldots, V_n(w)\}$, the map

$$t'' \in N_z \to s_{\mathcal{W}}(w, (t', t'')) \in Z_{h(z)} \tag{5.4}$$

takes a neighborhood of $0_z^{''} \in N_z$ in N_z biholomorphically onto a neighborhood of w in $Z_{h(z)} = Z_{h(w)}$ such that the image also contains the point z. For such choice of points and vectors we take $\phi'_{\mathcal{W}}(z, w, t', \cdot) \colon N_z \to N_z$ to be the map (5.3) followed by the (unique!) local inverse of (5.4) at $t'' = 0$, and then take

$$\phi_{\mathcal{W}}(z, w, (t', t'')) = \big(t', \phi'_{\mathcal{W}}(z, w, t', t'')\big).$$

This map is defined for $t = (t', t'') \in M_z \oplus N_z = \mathbf{C}^n$ in some neighborhood of the origin $0 \in \mathbf{C}^n$ which we may take to be independent of z, w, $\mathcal{W}$, provided that all conditions regarding closeness are satisfied. It is easily verified that this map satisfies all required properties. Since both maps (5.3) and (5.4) depend holomorphically on all arguments, so does ϕ. ♠

Suppose now that $\tilde{C} \subset \tilde{A} \cap \tilde{B}$ is an open Stein set containing C which is Runge in $\tilde{B}$. Assume that $b \colon \tilde{B} \to Z$ is a holomorphic section such that $b(\tilde{C}) \subset U'$, where $U' \subset Z$ is the neighborhood of $a(C)$ chosen above. We consider the restrictions $V_j | b(\tilde{C})$ as holomorphic sections of the bundle $E | b(\tilde{B})$ over the set $b(\tilde{C})$. By shrinking $\tilde{C}$ we get holomorphic sections W_j $(1 \le j \le n)$ of $E | b(\tilde{B})$ which approximate the sections V_j uniformly on $b(\tilde{C})$ as close as desired. Write $\mathcal{W}(z) = \{W_1(z), \ldots, W_n(z)\}$ for $z = b(x) \in b(\tilde{B})$. The maps $s_1 \colon \tilde{A} \times B^n(\eta) \to Z$ and $s_2 \colon \tilde{B} \times \mathbf{C}^n \to Z$ given by

$$\begin{aligned} s_1(x, t) &= \tilde{s}(a(x), t), \\ s_2(x, t) &= s_{\mathcal{W}(b(x))}(b(x), t) \end{aligned} \tag{5.5}$$

are holomorphic. If b is sufficiently uniformly close to a over $\tilde{C}$ and if $W_j(b(x))$ is sufficiently close to $V_j(b(x))$ for each $j = 1, \ldots, n$ and $x \in \tilde{C}$, then by lemma 5.4 we have for each $x \in \tilde{C}$ an injective holomorphic map

$$\psi(x, \cdot\,) = \phi_{\mathcal{W}(b(x))}(a(x), b(x), \cdot\,) \colon B^n(\eta) \to \mathbf{C}^n$$

which solves the equation

$$s_2(x, \psi(x, t)) = s_1(x, t) \quad (x \in C',\ t \in B^n(\eta)).$$

Moreover, if the approximations are sufficiently close, ψ is uniformly close to the map

$$\psi_0(x, t) = \phi_{\mathcal{V}(a(x))}(a(x), a(x), t)$$

which satisfies $\psi_0(x,0) = 0$ for $x \in \tilde{C}$. By proposition 5.3 we get open sets A', B', $C' = A' \cap B'$ in X, with $A \subset A' \subset \tilde{A}$ and $B \subset B' \subset \tilde{B}$, and holomorphic maps $\alpha\colon A' \to \mathbf{C}^n$, $\beta\colon B' \to \mathbf{C}^n$, such that $\psi(x,\alpha(x)) = \beta(x)$ for $x \in C'$. The homotopies of sections

$$a_t(x) = s_1(x, t\alpha(x)) \ \ (x \in A'), \qquad b_t(x) = s_2(x, t\beta(x)) \ \ (x \in B') \tag{5.6}$$

for $0 \le t \le 1$ then satisfy theorem 5.1.

Perhaps a word is in order regarding the proximity of the sections a_t to $a = a_0$ and of b_t to $b = b_0$. The rate of approximation of ψ_0 by ψ depends on the proximity of $b(x)$ to $a(x)$ and on the proximity of the vector fields $\mathcal{W}(b(x))$ to the fields $\mathcal{V}(a(x))$ for $x \in \tilde{C}$. This in turn determines the estimates on the norms $||\alpha||_{H^\infty(A')}$ and $||\beta||_{H^\infty(B')}$ (by proposition 5.2). Since the map s_1 (5.5) only depends on the section a, the definition (5.6) shows that the estimate of $d\big(a_t(x), a(x)\big)$ for $x \in A'$ and $t \in [0,1]$ depends only on $||\alpha||_{H^\infty(A')}$, and we get the stated approximation result over A'. However, the map s_2 (5.5) depends both on b and on $\mathcal{W}$. These quantities are under control only on $\tilde{C}$ and not on all of $\tilde{B}$ (since we apply Runge approximation). Therefore we can estimate $d\big(b_t(x), b(x)\big)$ in terms of $||\beta||_{H^\infty(B')}$ only for points $x \in C'$ and not on $B' \backslash C'$. ♠

The following is an extension of theorem 5.1 to parametrized families of sections.

5.5 Theorem. *Let $h\colon Z \to X$ be a holomorphic submersion onto a Stein manifold X, let d a metric on Z, and let (A,B) be a Cartan pair in X (def. 2.3) such that the set $C = A \cap B$ is Runge in B. Suppose that $\tilde{B}$ is an open neighborhood of B in X such that the restriction $Z|\tilde{B} = h^{-1}(\tilde{B})$ admits a fiber-spray (def. 3.1). Let Y be a compact Hausdorff space (the parameter space) and $Y_0 \subset Y$ a compact subset. Let $\tilde{A} \subset X$ be an open neighborhood of A and $a\colon \tilde{A} \times Y \to Z$ a continuous map such that for each $y \in Y$, $a(\cdot\,, y)$ is a holomorphic section of Z over $\tilde{A}$. Then for each $\epsilon > 0$ there is a $\delta > 0$ satisfying the following property. If $b\colon \tilde{B} \times Y \to Z$ is a continuous map such that for each $y \in Y$, $b(\cdot\,, y)$ is a holomorphic section of Z over $\tilde{B}$ satisfying*

$$\begin{aligned} &d\big(a(x,y), b(x,y)\big) < \delta && (x \in \tilde{C} = \tilde{A} \cap \tilde{B}, \ \ y \in Y), \\ &a(x,y) = b(x,y) && (x \in \tilde{C}, \ \ y \in Y_0), \end{aligned}$$

then there exist smaller neighborhoods $A' \supset A$, $B' \supset B$ and homotopies $a_t\colon A' \times Y \to Z$ resp. $b_t\colon B' \times Y \to Z$ $(0 \le t \le 1)$ of families of holomorphic sections such that $a_0 = a$, $b_0 = b$, $a_1 = b_1$ on $C' = A' \cap B'$, and for each $0 \le t \le 1$ we have

$$\begin{aligned} &d\big(a_t(x,y), a(x,y)\big) < \epsilon && (x \in A', \ y \in Y), \\ &d\big(b_t(x,y), b(x,y)\big) < \epsilon && (x \in C', \ y \in Y), \\ &a_t(x,y) = b_t(x,y) && (x \in C', \ y \in Y_0). \end{aligned}$$

Proof. This can done by essentially repeating the proof of theorem 5.1 with the addition of the parameter y. We shall only indicate a few critical places in the proof where it is not completely obvious what must be done. First one needs for each $y \in Y$ holomorphic maps

$$s_{1,y}\colon \tilde{A} \times B^n(\eta) \to Z, \qquad s_{2,y}\colon \tilde{B} \times \mathbf{C}^n \to Z \tag{5.7}$$

for some integer $n > 0$ and some $\eta > 0$ which are related to the sections $a_y = a(\cdot\,, y)$ resp. $b_y = b(\cdot\,, y)$ as in (5.5) and which depend continuously on the parameter $y \in Y$. To get $s_{1,y}$ we cover Y by finite number of open sets Y_j $(1 \le j \le J)$ such that for each j there is an open Stein set $U_j \subset Z$, with $a_y(\tilde{A}) \subset D_j$ for each $y \in Y_j$, and there are finitely many holomorphic vector fields $\tilde{V}^j_k$ $(1 \le k \le k_j)$ which generate $VT(Z)$ at each point of U_j. (This is similar to the proof of lemma 4.4 above.) Let $\chi_j\colon Y \to [0,1]$ $(1 \le j \le J)$ be a continuous partition of the unity on Y subordinate to the cover $\{Y_j\}$. Then the vector fields $\chi_j(y)\tilde{V}^j_k$ $(1 \le j \le J,\ 1 \le k \le k_j)$ are well defined and holomorphic in a neighborhood of the section $a_y(\tilde{A})$ for each $y \in Y$ (since $\chi_j(y) = 0$ for those $y \in Y$ for which the field $\tilde{V}^j_k$ is not defined near $a_y(\tilde{A})$, i.e., for y outside Y_j). Together these n fields generate the vertical tangent bundle $VT(Z)$ at each point $a_y(x)$ for $x \in \tilde{A}$ and $y \in Y$. Using these fields and their local flows we get as before a family of submersions $s_{1,y}$ (5.7) depending continuously on $y \in Y$.

With a similar argument (see lemma 4.4) we obtain a family of maps $s_{2,y}$ (5.7) which are submersions in a neighborhood of the zero section over the set $\tilde{C}$ and which approximate $s_{1,y}$ near $\tilde{C} \times \{0\}^n$. This gives a family of transition mappings ψ_y as in proposition 5.2 which are continuous in y and approximate a certain initial family $\psi_{0,y}$. By proposition 5.2 we obtain families of bounded holomorphic maps $\alpha_y\colon A' \to \mathbf{C}^n$, $\beta_y\colon B' \to \mathbf{C}^n$, depending continuously on $y \in Y$ and satisfying

$$\psi_y\big(x, \alpha_y(x)\big) = \beta_y(x) \quad (x \in C',\ y \in Y).$$

This gives for all $y \in Y$ homotopies of sections

$$a_t(x,y) = s_{1,y}\big(x, t\alpha_y(x)\big) \ \ (x \in A'), \qquad b_t(x,y) = s_{2,y}\big(x, t\beta_y(x)\big) \ \ (x \in B').$$

Moreover, for $y \in Y_0$ (when the sections a_y and b_y agree over $\tilde{C}$ and hence define a section over a neighborhood of $A \cup B$) we have by construction $\psi_y(x,0) = 0$ for all $x \in \tilde{C}$. Therefore $\alpha_y = 0$ and $\beta_y = 0$ for such y (prop. 5.2), and hence the above homotopies are fixed for $y \in Y_0$ as required. Everything else is clear from the earlier arguments. ♠

The last part of the proof of theorem 5.5 clearly shows why we need the solutions of the equation (5.1) to be given by operators.

5.6 Corollary. *Let X be a Stein manifold and $A' \subset X$ a non-critical strongly pseudoconvex extension of $A \subset A'$ (def. 2.1). Let F be a complex manifold with a spray and $Z = X \times F \to X$ the associated trivial bundle with fiber F. Let d be a metric on Z. Given a homotopy a_t $(0 \le t \le 1)$ of holomorphic sections of Z in a neighborhood of A such that a_0 and a_1 are holomorphic in a neighborhood of A', there is for each $\epsilon > 0$ a homotopy $\tilde{a}_t$ $(0 \le t \le 1)$ of holomorphic sections in a neighborhood of A' such that $\tilde{a}_0 = a_0$, $\tilde{a}_1 = a_1$, and*

$$d\big(\tilde{a}_t(x), a_t(x)\big) < \epsilon \quad (x \in A,\ 0 \le t \le 1).$$

Proof. By theorem 2.5 there is a finite sequence

$$A = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_k = A'$$

such that for each $j = 0, 1, \ldots, k-1$ we have $A_{j+1} = A_j \cup B_j$, where (A_j, B_j) is a convex bump. Suppose inductively that we have already approximated the initial homotopy $a_t = a_t^0$ on A by a homotopy a_t^j $(0 \le t \le 1)$ consisting of sections that are holomorphic in a neighborhood of A_j (with $a_0^j = a_0$ and $a_1^j = a_1$). Since B_j is a convex bump on A_j, we can apply theorem 4.5 to approximate the homotopy a_t^j in a neighborhood of $C_j = A_j \cap B_j$ by a homotopy b_t consisting of holomorphic sections in a neighborhood of B_j and such that $b_0 = a_0$, $b_1 = a_1$. If the approximation is sufficiently close, we can apply theorem 5.5 to glue the families a_t^j and b_t over C_j into a single family a_t^{j+1} $(0 \le t \le 1)$ consisting of sections which are holomorphic in a neighborhood of $A_{j+1} = A_j \cup B_j$, they equal a_0 resp. a_1 when $t = 0$ resp. $t = 1$, and they approximate the sections a_t^j on A_j. This completes the induction step. In finite number of steps we reach a desired homotopy $\tilde{a}_t = a_t^k$ satisfying corollary 5.4. ♠

Remark. The proof of corollary 5.6 remains valid if the bundle Z is trivial over a neighborhood of each bump B_j but not necessarily over A'. Thus the corollary provides an extension of the h-principle from A to any non-critical strongly pseudoconvex extension of A.

&6. Extending holomorphic sections across pseudoconvex bumps.

In this section we prove theorem 2.6, thereby concluding the proof of the h-principle (theorems 1.3 and 1.4). We write the proof in the case without parameters, but all arguments go through in the general case by using the parametric versions of the approximation and gluing theorems which were proved in sections 4 and 5. There are three main steps:

- approximate the given section a in a neighborhood of $C = A \cap B$ by a section b which is holomorphic in a neighborhood of B;
- glue the sections a and b by theorem 5.1 (or theorem 5.5) to obtain a holomorphic section $\tilde{a}$ in a neighborhood of $A \cup B$;
- show that the new section $\tilde{a}$ is homotopic to a and satisfies all required properties.

We present all arguments in the general case when (A, B) is a pseudoconvex bump and the set $C = A \cap B$ is not necessarily contractible. In the special case of convex bumps the arguments involving homotopies can be substantially simplied by using contractibility of C and the fact that A is a strong deformation retraction of $A \cup B$.

6.1 Proposition. *There exists a homotopy b_t $(0 \le t \le 1)$ of continuous sections in a neighborhood of B such that b_0 is holomorphic in a neighborhood of B, each b_t is holomorphic in a neighborhood of C, and $b_1 = a$.*

Proof. Since B is starshaped, it has an open contractible neighborhood $\tilde{B}$. By hypothesis we may choose $\tilde{B}$ so small that $Z|\tilde{B}$ is a trivial bundle. Hence there is a holomorphic section b_0 of Z over $\tilde{B}$ (we may simply take a constant section in a given trivialization) and a homotopy $b'_t: \tilde{B} \to Z$ $(0 \le t \le 1)$ of continuous sections connecting $b'_0 = b_0$ and $b'_1 = a$ (such a homotopy exists since $\tilde{B}$ is contractible and the bundle is trivial over $\tilde{B}$).

Recall (part (v) in def. 2.2) that we have a strongly plurisubharmonic function $\tau \ge 0$ in a neighborhood of $C = A \cap B$ such that $S = \{\tau = 0\}$ is a totally real sphere contained in an affine plane T (in some holomorphic coordinates in a neighborhood of C), $C = \{\tau \le 1\}$, and τ has no critical points on $C \backslash S$. Our first goal is to modify the homotopy b'_t so as to make it holomorphic in a neighborhood of S. Assume (as we may) that S is real-analytic. Let T' be the affine totally real subspace of maximal dimension containing T (and hence S). We can approximate $b'_t|S$ uniformly on S by a homotopy consisting of real-analytic sections over a neighborhood of B without changing the sections $b'_0 = b_0$ and $b'_1 = a$. We still denote this real-analytic homotopy by b'_t. For each t the section $b'_t|T'$ extends (by complexification) to a unique holomorphic section $\tilde{b}_t$ in a neighborhood of $T' \cap C$ (independent of t). Hence $\tilde{b}_t$ is a holomorphic homotopy in a neighborhood of $S_0 = \{\tau \le c_0\}$ for some sufficiently small $c_0 > 0$. Of course this process does not affect the sections that were already holomorphic, so we have $\tilde{b}_0 = b_0$ and $\tilde{b}_1 = a$ on S_0.

Since C is a non-critical strongly pseudoconvex extension of S_0, corollary 5.4 implies that $\tilde{b}_t$ can be approximated uniformly on S_0 by another homotopy b_t $(0 \le t \le 1)$ which is holomorphic in a neighborhood of C and connects b_0 and a.

To complete the proof we must show that the homotopy b_t (which has so far been defined and holomorphic in a neighborhood of C) extends to a continuous homotopy from b_0 to a in a neighborhood of B. To do this we first reparametrize both homotopies b_t and b'_t so that for some small $\delta > 0$ we have

$$b_t = b'_t = b_0 \quad \text{for } 0 \le t \le \delta, \qquad b_t = b'_t = a \quad \text{for } 1 - \delta \le t \le 1. \tag{6.1}$$

6.2 Lemma. *If $b_t|S$ is sufficiently uniformly close to $b'_t|S$ on $S = \{\tau = 0\} \subset C$ for each $t \in [0,1]$, and if (6.1) holds for some $\delta > 0$, then there exist a neighborhood $\tilde{C}$ of C and a two parameter homotopy $\tilde{b}_{t,s}$ $(0 \le t, s \le 1)$ of continuous sections of Z over $\tilde{C}$ satisfying*

$$\tilde{b}_{t,0} = b_t, \quad \tilde{b}_{t,1} = b'_t \ \text{ for } \ 0 \le t \le 1;$$
$$\tilde{b}_{t,s} = b_t = b'_t \ \text{ for } \ t \in \{0,1\} \ \text{ and } \ 0 \le s \le 1.$$

Proof. If $b_t|S$ is sufficiently uniformly close to $b'_t|S$ for each $t \in [0,1]$ (which we may assume to be the case), we can use the spray as in lemma 3.2 to obtain a two parameter homotopy of sections $b_{t,s}: S \to Z$ $(0 \le t, s \le 1)$ satisfying

$$b_{t,0} = b_t|S, \quad b_{t,1} = b'_t|S \ \text{ for } \ 0 \le t \le 1;$$
$$b_{t,s} = b_t = b'_t \ \text{ for } \ t \in [0,\delta] \cup [1-\delta, 1] \ \text{ and } \ 0 \le s \le 1.$$

Furthermore S is a strong deformation retraction of a neighborhood $\tilde{C}$ of C, i.e., there is a smooth family of maps $\Theta_t: \tilde{C} \to \tilde{C}$ $(0 \le t \le 1)$ such that Θ_0 is the identity on $\tilde{C}$, each Θ_t

is the identity on S, and $\Theta_1(\tilde{C}) = S$. Choose a smooth function $u: [0,1] \to [0,1]$ which is zero in a neighborhood of $t = 0$ and $t = 1$ and is identically one on $[\delta, 1-\delta]$. We identify sections of Z over $\tilde{C}$ (or over $\tilde{B}$) by mappings into the fiber F, using the triviality of $Z|\tilde{B}$. For each $x \in \tilde{C}$ and $0 \le t \le 1$ we define

$$\tilde{b}_{t,s}(x) = \begin{cases} b_t\big(\Theta_{3su(t)}(x)\big); & \text{if } 0 \le s \le 1/3, \\ b_{t,3s-1}\big(\Theta_{u(t)}(x)\big); & \text{if } 1/3 \le s \le 2/3, \\ b'_t\big(\Theta_{3(1-s)u(t)}(x)\big); & \text{if } 2/3 \le s \le 1. \end{cases}$$

Clearly these maps are defined for all $x \in \tilde{C}$. We have $\tilde{b}_{t,0} = b_t \circ \Theta_0 = b_t$ and $\tilde{b}_{t,1} = b'_t \circ \Theta_0 = b'_t$. Moreover, when t is near 0 or near 1, or when $x \in S$, we have $\tilde{b}_{t,s}(x) = b_{t,s}(x)$ for all $s \in [0,1]$. Thus $\tilde{b}_{t,s}$ satisfies lemma 6.2. ♠

Choose a smooth function $v: X \to [0,1]$ such that $v = 1$ in a neighborhood $C' \subset \tilde{C}$ of C and $\operatorname{supp} v \subset\subset \tilde{C}$. Let $\tilde{b}_{t,s}$ be as in lemma 6.2. Consider the homotopy

$$\tilde{b}_{t,1-v(x)}(x) \quad (x \in \tilde{B},\ 0 \le t \le 1).$$

By the choice of v this equals $\tilde{b}_{t,0}(x) = b_t(x)$ for $x \in C'$ (so it is holomorphic there), it is defined for all $x \in \tilde{B}$, and it equals $b'_t(x)$ for x outside $\tilde{C}$. This homotopy, which we again denote by b_t, satisfies proposition 6.1. ♠

In what follows we shall shrink the neighborhoods $\tilde{B}$ and $\tilde{C}$ several times without mentioning this again, and without changing the notation. We are now in position to apply the h-Runge approximation (theorem 4.1 or 4.2) to approximate the homotopy b_t from proposition 6.1 uniformly in $\tilde{C}$ by a holomorphic homotopy $\tilde{b}_t: \tilde{B} \to Z$ such that $\tilde{b}_0 = b_0$. The section $b = \tilde{b}_1$ then approximates a in $\tilde{C}$ as well as desired. Hence by theorem 5.1 (or 5.5) we can glue a and b into a single section $\tilde{a}$ which is holomorphic in a neighborhood A' of $A \cup B$ and which approximates a in a neighborhood of A.

It remains to show that there is a homotopy of sections a_t $(0 \le t \le 1)$ in a neighborhood of $A \cup B$ connecting $a_0 = a$ and $a_1 = \tilde{a}$ such that each a_t is holomorphic in a neighborhood of A and approximates a there (so a_t will satisfy theorem 2.6). In a neighborhood of A such a homotopy is provided by theorem 5.1. Our goal is to extend this homotopy to a neighborhood of $A \cup B$ by modifying it outside some neighborhood of A.

Recall that over $\tilde{B}$ we have the following homotopies:

(i) the homotopy b_t from b_0 to a given by proposition 6.1;

(ii) the homotopy $\tilde{b}_t$ from $\tilde{b}_0 = b_0$ to $\tilde{b}_1 = b$ obtained by approximating b_t in $\tilde{C}$;

(iii) the homotopy b'_t from $b'_0 = b(= \tilde{b}_1)$ to $b'_1 = \tilde{a}$ given by theorem 5.1.

Note that all these homotopies are holomorphic in $\tilde{C}$ and b'_t approximates b (and hence a) there. If we combine these three homotopies in the correct order (first follow b_{1-t} from a to b_0, then follow $\tilde{b}_t$ from b_0 to $\tilde{b}_1 = b$, and finally follow b'_t from b to $\tilde{a}$), we get a homotopy from a to $\tilde{a}$ over $\tilde{B}$ which is holomorphic over $\tilde{C}$. However, we must show that this homotopy over $\tilde{B}$ can be glued with the homotopy a_t into a single homotopy from a to $\tilde{a}$ in a neighborhood of $A \cup B$.

In order to do this we will first join the above homotopies (i)–(iii) over $\tilde{B}$ into a new homotopy h_t from a to $\tilde{a}$ which in addition will approximates a in $\tilde{C}$. For convenience we shall define h_t initially on the t-interval $[0,4]$ and subsequently rescale the parameter to $[0,1]$. According to the remark following theorem 4.1 we may assume that there is a two-parameter homotopy $g_{t,s}: \tilde{B} \to Z$ $(0 \le t, s \le 1)$ which is holomorphic in $\tilde{C}$ and satisfies

$$g_{t,0} = b_t, \quad g_{t,1} = \tilde{b}_t, \quad g_{0,s} = b_0,$$

and such that $g_{1,s}$ approximates $b_1 = a$ in $\tilde{C}$ for each $s \in [0,1]$. Choose a smooth function χ on X with values in $[0,1]$ such that $\chi = 1$ in a neighborhood of A and $\chi = 0$ in a neighborhood of $B \backslash \tilde{C}$. Such χ exists since the sets $A \backslash B$ and $B \backslash A$ are separated. We now define for each $x \in \tilde{B}$

$$h_t(x) = \begin{cases} b_{1-t+t\chi(x)}(x), & \text{if } 0 \le t \le 1; \\ g_{\chi(x),t-1}(x), & \text{if } 1 \le t \le 2; \\ \tilde{b}_{\chi(x)+(1-\chi(x))(t-2)}(x), & \text{if } 2 \le t \le 3; \\ b'_{t-3}(x), & \text{if } 3 \le t \le 4. \end{cases}$$

The reader may verify that this is indeed a homotopy from $h_0 = a$ to $h_4 = \tilde{a}$. For x in a neighborhood of C we have $\chi(x) = 1$ and hence

$$h_t(x) = \begin{cases} b_1(x) = a(x), & \text{if } 0 \le t \le 1; \\ g_{1,t-1}(x), & \text{if } 1 \le t \le 2; \\ \tilde{b}_1(x) = b(x), & \text{if } 2 \le t \le 3; \\ b'_{t-3}(x), & \text{if } 3 \le t \le 4. \end{cases}$$

Hence h_t is holomorphic near C and approximates a there (since all homotopies in question are close to a on $\tilde{C}$). Hence we assume that on $\tilde{C}$ both homotopies a_t and h_t (which we rescale to the t-interval $[0,1]$) approximate a so well that their images (over $\tilde{C}$) belong to a tubular neighborhood of $a(\tilde{C}) \subset Z$ in which we can apply lemma 3.2. This means that we can view these sections on $\tilde{C}$ as sections of a certain holomorphic vector bundle over $\tilde{C}$. This allows us to find a two parameter homotopy $k_{t,s}$ joining a_t and h_t over $\tilde{C}$ (we can simply use the convex combinations of the two sections in the given vector bundle.) Finally we patch a_t and h_t using $k_{t,s}$ into the homotopy

$$\tilde{a}_t(x) = k_{t,1-\chi(x)}(x)$$

where χ is a smooth function chosen as above. For x near A we have $\chi(x) = 1$ and hence $\tilde{a}_t(x) = k_{t,0}(x) = a_t(x)$, while for x near $B \backslash \tilde{A}$ we have $\chi(x) = 0$ ad hence $\tilde{a}_t(x) = k_{t,1}(x) = h_t(x)$. We denote this new homotopy again a_t.

Finally we choose a function smooth function η on X with values in $[0,1]$ such that $\eta = 1$ near $A \cup B$ and $\operatorname{supp} \eta \subset A'$, where A' is the neighborhood of $A \cup B$ on which the homotopy a_t has been defined. The homotopy $a_{t\eta(x)}(x)$ $(0 \le t \le 1)$ is now defined for all $x \in X$, it equals $a_0(x) = a(x)$ for $x \in X \backslash A'$ and $t \in [0,1]$, and it equals $a_t(x)$ for $x \in A \cup B$. This completes the proof of theorem 2.6. ♠

References.

[Car] H. Cartan: Espaces fibrés analytiques. Symposium Internat. de topologia algebraica, Mexico, 97–121 (1958). (Also in Oeuvres, vol. 2, Springer, New York, 1979.)

[BFo] G. Buzzard, J. E. Fornæss: An embedding of **C** in $\mathbf{C}^2$ with hyperbolic complement. Math. Ann. **306**, 539–546 (1996).

[Dem] J.-P. Demailly: Un exemple de fibré holomorphe non de Stein à fibre $\mathbf{C^2}$ ayant pour base le disque ou le plan. Invent. Math. **48**, 293–302 (1978).

[Eli] Y. Eliashberg: Topological characterization of Stein manifolds of dimension > 2. Internat. J. Math. **1**, 29–46 (1990).

[EGr] Y. Eliashberg, M. Gromov: Embeddings of Stein manifolds. Ann. Math. **136**, 123–135 (1992).

[Fr1] O. Forster: Plongements des variétés de Stein. Comment. Math. Helv. **45**, 170–184 (1970).

[Fr2] O. Forster: Topologische Methoden in der Theorie der Steinscher Räume. (Iternat. Congress in Math., Nice, 1970, pp. 613–618) Gauthier-Villars, Paris, 1971.

[FR1] O. Forster and K. J. Ramspott: Okasche Paare von Garben nicht-abelscher Gruppen. Invent. Math. **1**, 260–286 (1966).

[FR2] O. Forster and K. J. Ramspott: Analytische Modulgarben und Endromisbündel. Invent. Math. **2**, 145–170 (1966).

[Fo1] F. Forstnerič: Actions of $(\mathbf{R}, +)$ and $(\mathbf{C}, +)$ on complex manifolds. Math. Z. **223**, 123–153 (1996).

[Fo2] F. Forstnerič: Interpolation by holomorphic automorphisms and embeddings in $\mathbf{C}^n$. J. Geom. Anal., to appear.

[FGR] F. Forstnerič, J. Globevnik, J.-P. Rosay: Non straightenable complex lines in $\mathbf{C}^2$. Arkiv Mat. **34**, 97–101 (1996).

[Gr1] H. Grauert: Charakterisierung der holomorph vollständigen Räume. Math. Ann. **129**, 233–259 (1955).

[Gr2] H. Grauert: Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen. Math. Ann. **133**, 139–159 (1957).

[Gr3] H. Grauert: Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen. Math. Ann. **133**, 450–472 (1957).

[GRe] H. Grauert, R. Remmert: Theory of Stein Spaces. Grundl. Math. Wiss. **227**, Springer, New York, 1977

[Gro] M. Gromov: Oka's principle for holomorphic sections of elliptic bundles. J. Amer. Math. Soc. **2**, 851-897 (1989).

[GRo] C. Gunning, H. Rossi: Analytic functions of several complex variables. Prentice–Hall, Englewood Cliffs, 1965.

[HKu] P. Heinzner and F. Kutzschebauch: An equivariant version of Grauert's Oka principle. Invent. Math. **119**, 317–346 (1995).

[HL1] G. M. Henkin, J. Leiterer: Theory of functions on complex manifolds. Akademie-Verlag, Berlin, 1984.

[HL2] G. Henkin, J. Leiterer: The Oka-Grauert principle without induction over the basis dimension. Math. Ann. **311**, 71–93 (1998).

[Hö1] L. Hörmander: L^2 estimates and existence theorems for the $\overline{\partial}$ operator. Acta Math. **113**, 89–152 (1965).

[Hö2] L. Hörmander: An Introduction to Complex Analysis in Several Variables, 3rd ed. North Holland, Amsterdam, 1990.

[Osb] H. Osborn: Vector bundles. Vol. 1. Foundations and Stiefel–Whitney classes. Pure and Applied Mathematics, **101**, Academic Press, New York, 1982.

[Ram] K. J. Ramspott: Stetige und holomorphe Schnitte in Bundeln mit homogener Faser. Math. Z. **89**, 234–246 (1965).

[Ros] J.-P. Rosay: A counterexample related to Hartogs phenomenon (a question by E. Chirka). Michigan Math. J., to appear.

[RRu] J.-P. Rosay, W. Rudin: Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. Trans. Amer. Math. Soc. **310**, 47–86 (1988)

[Shd] M. Schneider: Tubenumgebungen Steinscher Räume. Manuscripta Math. **18**, 391–397 (1976).

[Sch] J. Schürmann: Embeddings of Stein spaces into affine spaces of minimal dimension. Math. Ann. **307**, 381–399 (1997).

[Si1] N. Sibony: Probleme de la couronne pour des domaines pseudoconvexes a bord lisse. Ann. of Math. **126**, 675–682 (1987).

[Si2] N. Sibony: Some aspects of weakly pseudoconvex domains. Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), 199–231, Proc. Sympos. Pure Math., **52**, Part 1, Amer. Math. Soc., Providence, RI, 1991.

[Siu] Y.T. Siu: Every Stein subvariety admits a Stein neighborhood. Invent. Math. **38**, 89–100 (1976).

[Ste] N. Steenrod: The Topology of Fibre Bundles. Princeton Univ. Press, Princeton, 1951.

Franc Forstnerič
Department of Mathematics
University of Wisconsin
Madison, WI 53706, USA

Current address:
IMFM, University of Ljubljana
Jadranska 19
1000–Ljubljana, Slovenia

Jasna Prezelj
Faculty of Mechanical Engineering
University of Ljubljana
Aškerčeva 6
1000–Ljubljana, Slovenia