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Introduction

In this paper we provide complete proofs of the results announced in 1989 by Mikhail Gromov [Gr1] on the *h-principle*, or *Oka's principle*, for sections of certain holomorphic submersions onto Stein manifolds. The validity of the h-principle means that *any continuous section can be homotopically deformed to a holomorphic section, and any continuous homotopy between two fixed holomorphic sections can be deformed to a homotopy consisting of holomorphic sections.* The main result is that the h-principle holds when the base of the submersion is a Stein manifold and the fibers are 'uniformly parabolic', in the sense that they admit nondegenerate entire images of Euclidean spaces depending holomorphically on the base point (Gromov calls such spaces 'elliptic'). It suffices for instance that, over small subsets of the base manifold, the total space of the submersion admits finitely many **C**-complete holomorphic vector fields which are vertical (tangent to the fibers of the submersion) and which span the vertical tangent space at each point. Precise statements are given by theorems 1.2, 1.4 and 1.6 in section 1 below. These results include Grauert's theorem [Gra] on sections of principal bundles whose fiber is any complex Lie group.

To put these result in context we briefly describe the main developments in this direction. By its very definition a Stein manifold admits plenty of global holomorphic functions; it is not surprising, then, that related analytic objects such as holomorphic vector bundles and, more generally, coherent analytic sheaves on Stein manifolds also admit plenty of global sections. This is the classical theory of Henri Cartan whose crown jewels are his Theorems A and B [GRo, Hör, GRe]. A characteristic feature of Cartan's theory is the presence of linear structure on the fibers which simplies the analysis significantly (even though it remains far from simple!). However we are often faced with the problem of finding holomorphic sections in a more general situation when the fibers have no linear structure. For instance the problem of constructing a holomorphic isomorphism between two holomorphic vector bundles of the same rank q over X reduces to that of constructing a holomorphic section of an associated fiber bundle over X whose fiber is the Lie group of invertible $q \times q$ complex matrices. This led Hans Grauert to prove in 1957 that the h-principle holds for sections of locally trivial holomorphic G-bundles over Stein manifolds with fiber any complex Lie group or a homogeneous space [Gra, Car, Ram, HL, HK]. Consequently the topological and holomorphic classifications of holomorphic vector bundles over Stein manifolds coincide (for line bundles this was first proved by Oka). Using the Lie group structure on the fiber G one can often reduce the analytic problems on such bundles to a neighborhood of the identity section where one can use the local linear structure induced from the Lie algebra of G by the exponential map.

More recently the question of embedding Stein manifolds into Euclidean spaces $\mathbf{C}^N$ of minimal dimension (Eliashberg and Gromov [EGr], Schürmann [Sch]) led to the problem of constructing holomorphic sections of holomorphic vector bundles $\pi: V \to X$ with Stein base whose graphs avoid a certain analytic subset $\Sigma \subset V$. In this case the fibers $V_x \backslash \Sigma_x$ ($x \in X$) have no Lie group structure in general and their holomorphic type changes from point to point. This minimal embedding dimension was established to be $N = [\frac{3n}{2}] + 1$ for a general Stein manifold of dimension $n \geq 2$, but the proof essentially depended on the validity of the h-principle in this situation (see theorem 1.6 in the present paper).

With all due respect to Gromov it must be said that, while the paper [Gr1] contains deep ideas and useful suggestions on how to prove the stated results, it does not contain proofs in the sense in which this word is commonly understood in contemporary mathematics. We offer a proof of Gromov's Main Theorem in this paper. In the special case of locally trivial fiber bundles the analogous results have been proved in [FPr].

The spirit of the proof of is reminiscent of Cartan's work leading to Theorems A and B. The main idea is to patch pieces of holomorphic sections together using a homotopy version of the Runge's approximation theorem and an analogue of the classical Cartan lemma (lemma 1.5.C in [Gr1] and proposition 5.2 in [FPr]). For this one needs holomorphic homotopies between the individual sections on intersections of their domains, as well as homotopies between the initial homotopies to insure that the 'triangles of homotopies' which appear in the construction are contractible. This idea is presented in [Gr1] but is not much elaborated. While Gromov talks in section 4 of [Gr1] about *C-fibrations* and *continuous sheaves*, we develop here a more concrete (albeit related) notion of *holomorphic complexes* and *prisms*. Roughly speaking, a holomorphic complex is a parametrized family of local holomorphic sections and homotopies between them, defined over sets (or their intersections) in a specially chosen covering of the base X, so that the family is continuous with respect to a parameter which belongs to the nerve of the given covering (an infinite simplicial complex). The strategy is to initially deform the given continuous section into a holomorphic complex and subsequently modify this complex by homotopies so as to combine the local sections together into semi-global sections over increasingly larger compact subsets of X. In the limit we obtain a holomorphic section over all of X which is homotopic to the initial continuous section.

Classical examples of the h-principle include the Smale–Hirsch theory of immersions and the theory of totally real and lagrangian immersions of Lees and Gromov. For maps between Riemann surfaces the h-principle has been studied recently by Winkelmann [Win]. A good encyclopaedic reference on the h-principle is Gromov's monograph [Gr2].

&1. The results.

Let $h: Z \to X$ be a holomorphic submersion onto a Stein manifold X. For each $x \in X$ we denote by $Z_x = h^{-1}(x)$ the fiber over x. At each point $z \in Z$ the tangent space $T_z Z$ contains the **vertical tangent space**

$$VT_z(Z) = \{e \in T_z Z : Dh(z)e = 0\} = T_z Z_{h(z)}. \quad (1.1)$$

If $p: E \to Z$ is a holomorphic vector bundle over Z, we denote by $E_z = p^{-1}(z) \subset E$ its fiber over $z \in Z$ and by $0_z \in E_z$ the zero element of E_z.

We begin by recalling from [Gr1] the definition of a spray associated to a submersion as above. More precisely this is what Gromov calls *fiber dominating sprays*; compare with definitions 1.2 and 3.1 in [FPr].

1.1 Definition. (Gromov [Gr1]) *Let $h: Z \to X$ be a holomorphic submersion and $U \subset X$ an open subset. A* **spray** *on $Z|U = h^{-1}(U)$ associated to h (or a fiber-spray) is a triple (E, p, s), where $p: E \to Z|U$ is a holomorphic vector bundle and $s: E \to Z|U$ is a holomorphic map such that for each $z \in Z|U$ we have*

(i) $s(E_z) \subset Z_{h(z)}$ (equivalently, $h \circ p = h \circ s$),

(ii) $s(0_z) = z$, and

(iii) the restriction of the derivative $Ds(0_z): T_{0_z}E \to VT_z(Z)$ to the subspace $E_z \subset T_{0_z}E$ maps E_z surjectively onto $VT_z(Z)$.

We denote the restriction in (iii) by

$$VDs(z) = Ds(0_z)|E_z: E_z \to VT_z(Z) \tag{1.2}$$

and call it the **vertical derivative** of s at the point $0_z \in E$. We can now state the main theorem. For locally trivial bundles with sprays this had been proved earlier in [FPr, theorem 1.3] (thus clarifying the results stated in [Gr1, sect. 2.9]).

1.2 Theorem. (Gromov [Gr1], 4.5 Main Theorem.) *Let $h: Z \to X$ be a holomorphic submersion onto a Stein manifold X. Assume that each point $x \in X$ has an open neighborhood $U \subset X$ such that $Z|U = h^{-1}(U)$ admits a spray. Then the inclusion between the spaces of holomorphic and continuous sections of $Z \to X$*

$$\mathrm{Holo}(X, Z) \hookrightarrow \mathrm{Cont}(X, Z) \tag{1.3}$$

is a weak homotopy equivalence, i.e., it induces an isomorphism of all homotopy groups of the two spaces (endowed with the compact–open topology). In particular, the path connected components of these spaces are in one-to-one correspondence, which means that

(i) each continuous section of $Z \to X$ can be homotopically deformed to a holomorphic section, and

(ii) any homotopy of sections $f_t: X \to Z$ ($0 \le t \le 1$) such that f_0 and f_1 are holomorphic can be deformed (keeping the end sections f_0 and f_1 fixed) into another homotopy consisting of holomorphic sections.

When the conclusion of theorem 1.2 holds we shall say, following Gromov [Gr1, Gr2], that *the submersion $h: Z \to X$ satisfies the h-principle.* Likewise we say that *the h-principle holds for maps of a Stein manifold X into a complex manifold F* if it holds for sections of the trivial fibration $X \times F \to X$.

Clearly the existence of a spray implies that the infinitesimal Kobayashi metric on each fiber Z_x is completely degenerate. Several natural examples of spaces with sprays have been pointed out by Gromov [Gr1, section 4.6.B]. The most common way to obtain a spray is the following.

Example 1. Suppose that Z admits finitely many $\mathbf{C}$-complete holomorphic vector fields $V_1, V_2, \cdots, V_N$ which are vertical (tangent to $VT(Z)$) and which span $VT_z(Z)$ at each point $z \in Z$. $\mathbf{C}$-completeness means that the flow θ_j^t of V_j is defined for all complex values of the time parameter t. The map $s\colon Z \times \mathbf{C}^N \to Z$ defined by

$$s(z; t_1, \ldots, t_N) = \theta_1^{t_1} \circ \theta_2^{t_2} \circ \cdots \circ \theta_N^{t_N}(z)$$

satisfies $s(z; 0, \ldots, 0) = z$ and $\frac{\partial}{\partial t_j} s(z; 0, \ldots, 0) = V_j(z)$ for each $z \in Z$ and each $j = 1, \ldots, N$. Since these vectors span $VT_z(Z)$, s is a spray on Z. Of course we only need such vector fields on $Z|U$ for small open sets $U \subset X$. ♠

Example 1 gives the following corollary to theorem 1.2:

Corollary. *Let $h\colon Z \to X$ be a holomorphic submersion onto a Stein manifold X. Assume that each point $x \in X$ has an open neighborhood $U \subset X$ such that there exist finitely many $\mathbf{C}$-complete holomorphic vector fields on $Z|U$ which are vertical with respect to h and which span the vertical tangent space $VT_z(Z)$ (1.1) at each point $z \in Z|U$. Then the h-principle holds for sections of h, i.e., the conclusion of theorem 1.2 holds.*

Since any complex Lie group is parallelized by finitely many $\mathbf{C}$-complete (left or right invariant) holomorphic vector fields obtained from a basis of its Lie algebra, the corollary applies in such situation. Thus theorem 1.2 includes the theorems of Grauert [Gra] and Ramspott [Ram]. (For an equivariant version of Grauert's theorem see Heinzner and Kutzschebauch [HK].)

We now describe a more general parametric version of the h-principle. This should be compared with theorem 1.4 and corollary 1.5 in [FPr] where the same result was proved for locally trivial bundles whose fiber admits a spray. Let $h\colon Z \to X$ be a holomorphic submersion and let P be a compact Hausdorff space (the parameter space). Our basic objects now are continuous maps $f\colon X \times P \to Z$ such that $f(\cdot\,, p)\colon X \to Z$ is a section of h for each fixed $p \in P$. A *homotopy* of such maps is a continuous map $H\colon X \times P \times [0,1] \to Z$ such that $H_t(\cdot\,, p) = H(\cdot\,, p, t)\colon X \to Z$ is a section of h for all $p \in P$ and $t \in [0,1]$.

Recall that a compact set $K \subset X$ is **holomorphically convex** in X if for each $x \in X \backslash K$ there is a holomorphic function f on X such that $|f(x)| > \sup_K |f|$. If X is Stein then by the Oka-Weil theorem each function holomorphic in a neighborhood of a holomorphically convex set $K \subset X$ can be approximated on K by functions holomorphic on X [Hör].

1.3 Definition. *A subset P_0 in a topological space P is called* **nice** *if there exists an open set $U \subset P$ containing P_0 and a strong deformation retraction of U onto P_0. The empty set is nice.*

1.4 Theorem. *Let X be a Stein manifold, $K \subset X$ a compact holomorphically convex subset and $h\colon Z \to X$ a holomorphic submersion onto X. Assume that for each point*

$x \in X \backslash K$ there is a neighborhood $U_x \subset X$ such that $Z|U_x = h^{-1}(U_x)$ admits a spray. Let P be a compact Hausdorff space and $P_0 \subset P$ a nice compact subset (def. 1.3). Assume that $a: X \times P \to Z$ is a continuous map such that for each $p \in P$, $a(\cdot\,, p): X \to Z$ is a section of $h: Z \to X$ which is holomorphic in an open set $U_0 \supset K$, and $a(\cdot\,, p)$ is holomorphic on X for each $p \in P_0$. Let d be a metric on Z compatible with the manifold topology of Z and let $\epsilon > 0$. Then there is a homotopy $H_t: X \times P \to Z$ $(0 \le t \le 1)$ such that

(i) $H_0 = a$,

(ii) the section $H_1(\cdot\,, p): X \to Z$ is holomorphic for each $p \in P$,

(iii) the homotopy is fixed on P_0, i.e., $H_t(x, p)$ is independent of t for $p \in P_0$, and

(iv) $d\big(H_t(x, p), a(x, p)\big) < \epsilon$ *for all $x \in K$, $p \in P$ and $0 \le t \le 1$.*

Theorem 1.4 implies theorem 1.2 which can be seen as follows. If we take P to be the n-sphere S^n and $P_0 = \emptyset$, theorem 1.4 shows that each continuous map $S^n \to \mathrm{Cont}(X, Z)$ can be homotopically deformed to a map $S^n \to \mathrm{Holo}(X, Z)$. Similarly, if P is the closed $(n+1)$-ball $B^{n+1} \subset \mathbf{R}^{n+1}$ and $P_0 = \partial B^{n+1} = S^n$, theorem 1.4 shows that each map $S^n \to \mathrm{Holo}(X, Z)$ which extends to a map $B^{n+1} \to \mathrm{Cont}(X, Z)$ also extends to a map $B^{n+1} \to \mathrm{Holo}(X, Z)$. This means that the inclusion (1.3) induces the isomorphism of all homotopy groups of the two spaces as claimed by theorem 1.2.

As an application, as well as an important special case, we now consider the problem of avoiding analytic subsets in vector bundles by graphs of sections. We begin with the simplest case of avoiding a closed analytic subset $\Sigma \subset \mathbf{C}^q$ by maps $f: X \to \mathbf{C}^q \backslash \Sigma$. Simple examples show that we cannot expect the h-principle to hold when Σ is a complex hypersurface except if it is very special (such as a complex hyperplane). The situation remains complicated even for subsets of higher codimension. The validity of the h-principle in this context really depends on the complex analytic properties of $\mathbf{C}^q \backslash \Sigma$ and not on the structure of Σ or its complex codimension, no matter how large it may be. The good situation is when Σ admits many proper linear projections onto complex hyperplanes since this guaranties the existence of many complete holomorphic vector fields on $\mathbf{C}^q \backslash \Sigma$. Here is a summary of the results.

1.5 Theorem. *(a) If $q \ge 2$ and $\Sigma \subset \mathbf{C}^q$ is a closed analytic subset of complex codimension ≥ 2 in $\mathbf{C}^q$ such that, with respect to some holomorphic coordinates $z = (z', z_q) \in \mathbf{C}^q$ and some constant $C > 0$ we have*

$$\Sigma \subset \{z \in \mathbf{C}^q : |z_q| \le C(1 + |z'|)\},$$

then the h-principle holds for maps from any Stein manifold into $\mathbf{C}^q \backslash \Sigma$. This is the case in particular if Σ is an algebraic subset of codimension at least two in $\mathbf{C}^q$.

(b) For each $q \ge 1$ there exist countable discrete sets $\Sigma \subset \mathbf{C}^q$ such that the h-principle fails for maps of certain Stein manifolds into $\mathbf{C}^q \backslash \Sigma$.

(c) For each $1 \le k < q$ there exist proper holomorphic embeddings $g: \mathbf{C}^k \hookrightarrow \mathbf{C}^q$ such that the h-principle fails for maps of certain Stein manifolds into $\mathbf{C}^q \backslash g(\mathbf{C}^k)$.

In fact we show in sect. 7 that there exist subvarieties $\Sigma \subset \mathbf{C}^q$ as in (b) or (c) such that, *if X is a Stein manifold whose universal covering space is a Euclidean space (such as*

$X = (\mathbf{C}^*)^n$), then any holomorphic map $f: X \to \mathbf{C}^q \backslash \Sigma$ is holomorphically contractible to a point inside $\mathbf{C}^q \backslash \Sigma$, but for some such manifolds X there exist homotopically nontrivial real-analytic maps $X \to \mathbf{C}^q \backslash \Sigma$.

We now consider the same problem in vector bundles. Let $\pi: V \to X$ be a holomorphic vector bundle of rank q over a Stein base X. By adding to each fiber $V_x = \pi^{-1}(x) \cong \mathbf{C}^q$ the hyperplane at infinity we obtain a projective space $\overline{V}_x \cong \mathbf{CP}^q$. Since every complex linear automorphism of $\mathbf{C}^q$ extends to a unique projective automorphism of $\mathbf{CP}^q$, the fibers $\overline{V}_x$ patch together into a holomorphic projective bundle $\overline{\pi}: \overline{V} \to X$.

1.6 Theorem. (Avoiding analytic sets by holomorphic sections.) *Let $\pi: V \to X$ be a holomorphic vector bundle of rank $q \geq 2$ over a Stein manifold X and let $\overline{\pi}: \overline{V} \to X$ be the associated projective bundle as above. If $\Sigma \subset \overline{V}$ is a closed analytic subset such that for each $x \in X$ the fiber $\Sigma_x = \Sigma \cap \overline{V}_x$ is of complex codimension at least two in $\overline{V}_x$, then the h-principle holds for sections $f: X \to V \backslash \Sigma$ whose graphs avoid Σ. Moreover, if $X_0 \subset X$ is a closed analytic subset and $K \subset X$ is a compact holomorphically convex subset of X, then each continuous section $f_0: X \to V$ which is holomorphic in a neighborhood of $X_0 \cup K$ and whose graph avoids Σ can be homotopically deformed into a holomorphic section $f: X \to V \backslash \Sigma$ such that f approximates f_0 uniformly on K and it matches f_0 to a prescribed order $k \in \mathbf{Z}_+$ on X_0.*

The hypothesis implies that each fiber Σ_x is an algebraic subset of $\overline{V}_x$ (by Chow's theorem [Chi]) which does not contain the hyperplane at infinity (since its complex codimension is assumed to be at least two). Of course Σ_x may be empty for some $x \in X$.

The condition on Σ in theorem 1.6 can be equivalently described as follows. Choose a small open set $U \subset X$ such that $\pi: V|U \to U$ is a trivial bundle which we identify with $U \times \mathbf{C}^q \to U$. Let $z = (x, w)$ be complex coordinates on $U \times \mathbf{C}^q$. If U is sufficiently small then $\Sigma \cap \pi^{-1}(U)$ is given by finitely many equations

$$g_j(x, w) = \sum_{|I| \leq d_j} g_{j,I}(x) w^I = 0 \quad (1 \leq j \leq m)$$

which are polynomial in $w \in \mathbf{C}^q$ with coefficients holomorphic in $x \in U$. The closure of Σ in $U \times \mathbf{CP}^q$ is defined by the corresponding homogenized equations obtained by adding a suitable power of an additional variable w_0 to each term (so that $w_0 = 0$ is the hyperplane at infinity). Let $P_j(x, w)$ be the top order homogeneous part of g_j; this is precisely the part of g_j which does not get any w_0 term after homogenization. We now see that the fiber $\Sigma_x \subset \mathbf{CP}^q$ contains the hyperplane at infinity $w_0 = 0$ (which is the bad case) if and only if $P_j(x, w) = 0$ for all $w \in \mathbf{C}^q$ and $j = 1, \ldots, m$. Therefore we must avoid this situation in order to insure the validity of the h-principle.

Remarks. 1. The second part of theorem 1.6 is precisely what was needed (in a special situation when the fibers of Σ are finite unions of affine complex subspaces) in the proofs of the embedding theorem for Stein manifolds into Euclidean spaces of minimal dimension given by Eliashberg and Gromov [EGr] and Schürmann [Sch]. The existence of continuous sections of $V \to X$ avoiding Σ has been clarified in [Sch] using results of Hamm.

2. Theorem 1.6 remains valid, with the same proof as given in sect. 7 below, if Σ is a closed analytic subset of $\overline{V}|(X\backslash X_0) = \overline{\pi}^{-1}(X\backslash X_0)$ satisfying the stated properties over $X\backslash X_0$ such that the closure of Σ in $\overline{V}$ (which need not be analytic) does not intersect $f_0(X_0)$. ♠

The paper is organized as follows. In section 2 we explain the idea of the proof of theorem 1.4 (and hence of theorem 1.2). A sketch of this can be found in section 4 of Gromov's paper [Gr1]. In section 3 we introduce the concept of a holomorphic (resp. continuous) complex and prism, and in sect. 4 we introduce the notion of a Cartan string. The heart of the proof is proposition 5.1 and the subsequent results of section 5. We conclude the proof of theorem 1.4 by an inductive construction in section 6. In section 7 we prove theorems 1.5 and 1.6.

&2. Outline of proof of the main theorem.

In this section we explain the main idea of the proof of theorem 1.4. We concentrate on the case without parameters (i.e., when P consists of one point). Thus we are given a continuous section $a: X \to Z$ of a holomorphic submersion $h: Z \to X$ such that a is holomorphic in an open set $U_0 \subset X$ containing a given holomorphically convex set $K \subset X$. Our goal is to construct a homotopy of sections $H^s: X \to Z$ $(0 \le s \le 1)$ such that $H^0 = a$, the section $f = H^1$ is holomorphic in X, and every section H^s is holomorphic near K and approximates a on K.

The idea of the construction is to replace the continuous section a by a collection of holomorphic sections $a_{(j)}: U_j \to Z$ $(j = 0, 1, 2, \ldots)$, where $\mathcal{U} = \{U_0, U_1, U_2, \ldots\}$ is a suitably chosen cover of X by small open sets, such that for each j there is a homotopy of continuous sections $a_{(j),s}: U_j \to Z$ $(0 \le s \le 1)$ satisfying $a_{(j),0} = a|U_j$ and $a_{(j),1} = a_{(j)}$. For $j = 0$ we take $a_{(0),s} = a|U_0$ for all s. The collection $\{a_{(j)}: j \in \mathbf{Z}_+\}$ is a holomorphic puzzle whose pieces should be rearranged into a holomorphic section $f: X \to Z$ which is homotopic to the initial section a.

Let us first consider what is involved in patching two such pieces into a single section over the union of the two sets. To simplify the notation let's assume that $A, B \subset X$ are compact subsets, $U \supset A$ and $V \supset B$ are open sets and $a: U \to Z$, $b: V \to Z$ are holomorphic sections of $Z \to X$ over U resp. V. We would like to replace the pair (a, b) by a single holomorphic section $\tilde{a}$ which is holomorphic in a neighborhood of $A \cup B$ and which approximates a on A. This will be done in two steps. In the first step we replace b by another holomorphic section $b_1: V \to Z$ (perhaps shrinking V around B if necessary) such that b_1 approximates a very closely in some neighborhood $W \supset A \cap B$. In the second step we 'glue' a with b_1 into a new holomorphic section $\tilde{a}$ over $A \cup B$.

Of course the pair (A, B) must satisfy certain analytic conditions in order to be able to accomplish these two steps. The first step (approximation) can be achieved if the following three conditions hold:

- the set $C = A \cap B$ is Runge in B (we can approximate holomorphic functions in small neighborhoods of C by functions holomorphic in a neighborhood of B),
- the submersion $Z \to X$ admits a spray over a neighborhood of B, and

– there is a neighborhood $W \supset A \cap B$ and a homotopy of holomorphic sections $b_t: W \to Z$ connecting $b_0 = b|W$ and $b_1 = a|W$.

Granted these conditions the h-Runge theorem (theorems 4.1 and 4.2 in [FPr]) shows that we can approximate the homotopy b_t uniformly in a neighborhood of C by a homotopy of sections $\tilde{b}_t$ $(0 \le t \le 1)$ which are holomorphic in a neighborhood of B. In particular the section $\tilde{b}_1$ approximates a as well as desired in some fixed neighborhood of C. Replacing b_1 by $\tilde{b}_1$ we may thus assume that b_1 approximates a near C.

For the second step (gluing a and b_1) we linearize the problem and solve a certain $\overline{\partial}$-equation in a neighborhood of $A \cup B$. This is accomplished by theorem 5.1 (or theorem 5.5) in [FPr] and requires that (A, B) is a *Cartan pair* in X (definition 4.1 below). This means that each of the sets A, B and $A \cup B$ has a basis of Stein neighborhoods (so we can solve the $\overline{\partial}$-equation), C is Runge in B, and the sets $\overline{A \backslash B}$, $\overline{B \backslash A}$ are disjoint (so we can separate them by a smooth function which equals 1 on one set and 0 on the other).

In order to glue the local holomorphic sections $a_{(j)}: U_j \to Z$ into a single holomorphic section $f: X \to Z$ we perform the steps described above inductively. At each step the sets U_j on which our sections are defined may shrink, and we must control the shrinking so that at the end we still have a cover of X. For this reason we initially choose a special locally finite cover $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ of X by compact sets such that

(i) $K \subset A_0 \subset U_0$;

(ii) for each $n \ge 1$ the ordered collection of sets $(A_0, A_1, \ldots, A_n)$ is a *Cartan string* (definition 4.2). This property allows to carry on the gluing procedure by induction on n. In particular this means that for each $n \ge 1$ the pair $A^{n-1} = A_0 \cup A_1 \cup \cdots \cup A_{n-1}$ and A_n is a Cartan pair;

(iii) for each $j \in \mathbf{Z}_+$ there is a holomorphic section $a_{(j)}$ in an open neighborhood $U_j \supset A_j$ which is homotopic to $a|U_j$, and for $j \ge 1$ the restriction $Z|U_j$ admits a spray;

(iv) for each pair $i \ne j$ such that $A_i \cap A_j \ne \emptyset$ there is a holomorphic homotopy between $a_{(i)}$ and $a_{(j)}$ in $U_{(i,j)} = U_i \cap U_j$, etc.

More generally, for each multiindex $J = (j_0, j_1, \ldots, j_n)$ such that $A_J = A_{j_0} \cap \cdots \cap A_{j_n} \ne \emptyset$ there is an n-dimensional holomorphic homotopy $a_J(t)$ in $U_J = U_{j_0} \cap \cdots \cap U_{j_n}$, with the parameter t belonging to the standard n-simplex $\triangle^n \subset \mathbf{R}^n$, such that for t belonging to a boundary face of $\triangle^n$ determined by a shorter multiindex $I \subset J$ we have $a_J(t) = a_I(t)|U_J$. Intuitively speaking, the parameter set of our entire colection of holomorphic sections and homotopies between them is a special simplicial complex, namely the nerve of the covering $\mathcal{A}$ (see section 3 for the details). The sets U_j will shrink but the A_j's will stay the same during the entire construction.

Suppose inductively that we have already joined the sections $a_{(0)}, \ldots, a_{(n-1)}$ into a holomorphic section f^{n-1} in a neighborhood of $A^{n-1} = A_0 \cup A_1 \cup \cdots \cup A_{n-1}$ using the homotopies between them and the gluing procedure. We emphasize that all modifications are done by holomorphic homotopies. In the next step we try to glue f^{n-1} with the section $a_{(n)}$. For this we need a holomorphic homotopy between the two sections in a neighborhood of $A^{n-1} \cap A_n$. In the special case when $h: Z \to X$ is a *locally trivial fiber bundle* whose fiber admits a spray such a homotopy can be constructed from a continuous homotopy between the two sections, provided that our sets are chosen sufficiently carefully (A_n must be a pseudoconvex bump on A^{n-1}). This was explained in [FPr]; see especially proposition 6.1 there. The argument given in [FPr] does not carry over to the present situation and we need an alternative systematic way of insuring the existence of such a homotopy.

Our inductive construction is such that for each $j = 0, 1, \ldots, n-1$ we have a holomorphic homotopy between f^{n-1} and $a_{(n)}$ in a neighborhood of $A_j \cap A_n$ (which is inherited from the initial homotopy between $a_{(j)}$ and $a_{(n)}$). We now patch these n partial homotopies into a single homotopy defined in a neighborhood of $A^{n-1} \cap A_n$. This is done in the same way as above by induction on n, and for this we need that the ordered collection of sets $(A_0 \cap A_n, A_1 \cap A_n, \ldots, A_{n-1} \cap A_n)$ is also a Cartan string. Once we have such a homotopy, we replace the pair f^{n-1} and $a_{(n)}$ by a section f^n in a neighborhood of A^n which approximates f^{n-1} on A^{n-1}. The sequence of sections f^n obtained in this way converges uniformly on compacts in X to a holomorphic section $f \colon X \to Z$ which solves the problem.

&3. Holomorphic complexes and prisms.

3.1 Definition. *Let $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ be any finite or countable family of nonempty subsets of X which is locally finite. The* **nerve** *of $\mathcal{A}$ is the (combinatorial) simplicial complex $\mathcal{K}(\mathcal{A})$ consisting of precisely those multiindices $J = (j_0, j_1, \ldots, j_k) \in \mathbf{Z}_+^{k+1}$ ($k \in \mathbf{Z}_+$) with increasing entries $0 \le j_0 < j_1 < \cdots < j_k$ for which*

$$A_J = A_{j_0} \cap A_{j_1} \cap \cdots \cap A_{j_k} \neq \emptyset.$$

We denote the geometric realization of $\mathcal{K}(\mathcal{A})$ by $K(\mathcal{A})$.

Recall that the geometric realization of a simplicial complex is a topological space which is a union of topological simplexes of various dimensions, and these intersect along simplices of lower dimension. Each multiindex $J = (j_0, j_1, \ldots, j_k) \in \mathcal{K}(\mathcal{A})$ determines a (closed) k-dimensional face $|J| \subset K(\mathcal{A})$ which is homeomorphic to the standard k-simplex $\triangle^k \subset \mathbf{R}^k$ (the closed convex hull of the set $\{0, e_1, e_2, \ldots, e_k\} \subset \mathbf{R}^k$ containing the origin and the standard basis vectors). Conversely, each k-dimensional face in $K(\mathcal{A})$ is of this form. We shall call the simplex $|J| \subset K(\mathcal{A})$ the **body** (or **carrier**) of $J \in \mathcal{K}(\mathcal{A})$. Conversely, J is the **vertex scheme** of $|J|$. Thus $K(\mathcal{A})$ (which is a topological space) is the body of $\mathcal{K}(\mathcal{A})$ (which is a combinatorial object). The vertices of the space $K(\mathcal{A})$ correspond to the individual sets $A_j \in \mathcal{A}$, i.e., to singletons $\{j\} \in \mathcal{K}(\mathcal{A})$. Given $I, J \in \mathcal{K}(\mathcal{A})$ we have $|I| \cap |J| = |I \cap J|$. Thus for any two (bodies of) simplexes in $K(\mathcal{A})$ either one is a subset of the other or else their intersection is a simplex of lower dimension (possibly empty). We refer the reader to [HW] for further details on simplicial complexes and their geometric realizations.

If the family $\mathcal{A}$ is infinite (countable), we denote for each $n \in \mathbf{Z}_+$ by

$$\mathcal{K}^n = \mathcal{K}(A_0, A_1, \ldots, A_n) \subset \mathcal{K}(\mathcal{A})$$

the nerve of the finite collection $\mathcal{A}_n = \{A_0, \ldots, A_n\}$ and by K^n its body. Clearly $\mathcal{K}^n \subset \mathcal{K}^{n+1}$ for each n and $\mathcal{K}(\mathcal{A}) = \bigcup_{n=0}^{\infty} \mathcal{K}^n$. Also for each multiindex $J = (j_0, j_1, \ldots, j_k) \in \mathbf{Z}_+^{k+1}$ (not necessarily belonging to $\mathcal{K}(\mathcal{A})$) we let

$$\mathcal{K}_J(\mathcal{A}) = \mathcal{K}(A_{j_0}, A_{j_1}, \ldots, A_{j_k})$$

be the nerve of the indicated collection of sets and $K_J(\mathcal{A})$ its body. Note that $\mathcal{K}_J(\mathcal{A})$ is a finite subcomplex of $\mathcal{K}(\mathcal{A})$ whose body is a k-dimensional simplex if and only if $J \in \mathcal{K}(\mathcal{A})$; otherwise it is a union of simplexes of lower dimension. Occasionally we shall write simply $\mathcal{K}$ instead of $\mathcal{K}(\mathcal{A})$ when it is absolutely clear from the context which collection $\mathcal{A}$ is meant.

From now on we assume that $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ is a (finite or countable) locally finite family of compact subsets of X. (Later $\mathcal{A}$ will be a covering of X with certain additional properties, but this is not necessary for the moment.) An **open neighborhood** of $\mathcal{A}$ in X is a collection $\mathcal{U} = \{U_0, U_1, \ldots\}$ of open sets in X (with the same index set) such that $A_i \subset U_i$ for each i. Such an open neighborhood $\mathcal{U}$ is called **faithful** if $\mathcal{K}(\mathcal{U}) = \mathcal{K}(\mathcal{A})$, that is, the families $\mathcal{A}$ and $\mathcal{U}$ have the same nerve. Clearly any such $\mathcal{A}$ has an open faithful neighborhood $\mathcal{U}$. As before we write for each $J = (j_0, j_1, \ldots, j_k)$

$$U^J = U_{j_0} \cup U_{j_1} \cup \cdots \cup U_{j_k}, \quad U_J = U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_k}.$$

If $h\colon Z \to X$ is a holomorphic submersion onto X and $U \subset X$ is an open subset, we denote by $\mathcal{O}_h(U, Z)$ (resp. $\mathcal{C}_h(U, Z)$) the set of all holomorphic (resp. continuous) sections $f\colon U \to Z$ of $h\colon Z \to X$ over U.

3.2 Definition. *Let $h\colon Z \to X$ be a holomorphic submersion of a complex manifold Z onto a complex manifold X and let $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ be a locally finite family (finite or countable) of compact sets in X.*

(i) A **holomorphic** $\mathcal{K}(\mathcal{A})$**-complex** *with values in Z is a collection of continuous maps*

$$f_* = \{f_J\colon |J| \to \mathcal{O}_h(U_J, Z), \quad J \in \mathcal{K}(\mathcal{A})\},$$

where $\mathcal{U} = \{U_0, U_1, U_2, \ldots\}$ is a faithful neighborhood of $\mathcal{A}$ (see above), such that the following compatibility condition is satisfied:

$$I, J \in \mathcal{K}(\mathcal{A}),\ I \subset J \Longrightarrow f_J(t) = f_I(t)|_{U_J},\ t \in |I|. \tag{3.1}$$

A **continuous** $\mathcal{K}(\mathcal{A})$**-complex** *with values in Z is a collection of continuous maps*

$$f_* = \{f_J\colon |J| \to \mathcal{C}_h(U_J, Z), \quad J \in \mathcal{K}(\mathcal{A})\}$$

satisfying the compatibility conditions (3.1).

(ii) If f_* is a (holomorphic or continuous) $\mathcal{K}(\mathcal{A})$-complex and $\mathcal{K}' \subset \mathcal{K}(\mathcal{A})$ is a subcomplex of $\mathcal{K}(\mathcal{A})$, we denote by $f_*|\mathcal{K}'$ the restriction of f_* to $\mathcal{K}'$.

(iii) A holomorphic (resp. continuous) $\mathcal{K}(\mathcal{A})$-complex f_* is **constant** *if there is an open neighborhood $\mathcal{U} = \{U_i\colon i \in \mathbf{Z}_+\}$ of $\mathcal{A}$ and a holomorphic (resp. continuous) section $g\colon \bigcup\{U\colon U \in \mathcal{U}\} \to Z$ of $Z \to X$ such that $f_J(t)|U_J = g|U_J$ for each $J \in \mathcal{K}(\mathcal{A})$ and each $t \in |J|$.*

Remark. We shall consider the $\mathcal{K}(\mathcal{A})$-complexes defined above in the sense of their *germs* on the sets in $\mathcal{A}$. More precisely, $\mathcal{K}(\mathcal{A})$-complexes f_* and g_* are considered **equivalent** if there is an open neighborhood $\mathcal{U} = \{U_i\}$ of $\mathcal{A} = \{A_i\}$ such that for each $J \in \mathcal{K}(\mathcal{A})$ and each $t \in |J|$ the sections $f_J(t)$ and $g_J(t)$ are defined and equal in the set U_J. We shall not distinguish between equivalent complexes (so a complex must be understood in the sense of germs). ♠

A holomorphic complex is precisely the right tool to keep track of all sections and homotopies between them (at all levels) alluded to in section 2. For each $J = (j_0, \ldots, j_k) \in \mathcal{K}(\mathcal{A})$ we have a family of holomorphic sections

$$f_J(t): U_J \to Z, \quad t \in |J|,$$

depending continuously on the parameter $t \in |J|$, which we may think of as a homotopy of holomorphic sections over the set $U_J = U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_k}$, with the parameter t belonging to the simplex $|J| \subset K(\mathcal{A})$. For each face $I \subset J$ of J and for $t \in |I| \subset |J|$ the section $f_J(t)$ agrees with the section $f_I(t)|U_J$ restricted from its natural domain $U_I \subset X$ to the smaller set U_J. It is worthwile to consider separately the one dimensional case.

Example 2. Let $J = (j_0, j_1) \in \mathcal{K}(\mathcal{A})$ be a simplex of length two. Its body $|J| \subset K(\mathcal{A})$ is a segment which we can represent by $[0,1] \subset \mathbf{R}$. J contains two zero dimensional faces, namely the vertices (j_0) and (j_1) (corresponding to the sets A_{j_0} resp. A_{j_1}), and the bodies of these faces can be identified with the endpoints $\{0\}$ resp. $\{1\}$ of $[0,1]$. The restriction of a (holomorphic or continuous) $\mathcal{K}(\mathcal{A})$-complex f_* to the subcomplex $\mathcal{K}_J(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$ determined by $J = (j_0, j_1)$ consists of a one parameter family (homotopy) $f_J(t)$ ($t \in |J| = [0,1]$) of sections over $U_J = U_{j_0} \cap U_{j_1}$ such that $f_J(0)$ is the restriction to U_J of a section $f_{j_0}: U_{j_0} \to Z$ and likewise $f_J(1)$ is the restriction to U_J of a section $f_{j_1}: U_{j_1} \to Z$. ♠

We shall also need the notion of a multiparameter homotopy of $\mathcal{K}(\mathcal{A})$-complexes. A suitable concept for this is that of a holomorphic prism defined as follows.

3.3 Definition. *(i) Let $h: Z \to X$ and $\mathcal{A}$ be as in def. 3.2 and let $k \in \mathbf{Z}_+$. A* **holomorphic** $(\mathcal{K}(\mathcal{A}), k)$**-prism** *(or a k-prism over $\mathcal{K}(\mathcal{A})$) with values in Z is a collection of continuous maps*

$$f_* = \{f_J: |J| \times [0,1]^k \to \mathcal{O}_h(U_J, Z), \quad J \in \mathcal{K}(\mathcal{A})\},$$

where $\mathcal{U}$ is a faithful open neighborhood of $\mathcal{A}$, such that for each fixed $y \in [0,1]^k$ the associated family

$$f_{*,y} = \{f_{J,y} = f_J(\cdot\,, y): |J| \times \{y\} \to \mathcal{O}_h(U_J, Z), \quad J \in \mathcal{K}(\mathcal{A})\},$$

is a holomorphic $\mathcal{K}(\mathcal{A})$-complex (i.e., it satisfies the compatibility conditions (3.1)). Similarly a **continuous** $(\mathcal{K}(\mathcal{A}), k)$**-prism** *with values in Z is a collection of continuous maps $f_J: |J| \times [0,1]^k \to \mathcal{C}_h(U_J, Z)$ ($J \in \mathcal{K}(\mathcal{A})$) such that $f_{*,y}$ is a continuous $\mathcal{K}(\mathcal{A})$-complex for each $y \in [0,1]^k$.*

(ii) A prism f_* is **sectionally constant** if there is an open set $U \subset X$ containing $\bigcup_{i \geq 0} A_i$ such that the complex $f_{*,y}$ is represented by a section $f_y: U \to Z$ for each fixed

$y \in [0,1]^k$. If this holds only for y in a certain subset $Y \subset [0,1]^k$ we say that f_* is sectionally constant on Y.

Thus a $(\mathcal{K}(\mathcal{A}), 0)$-prism is a $\mathcal{K}(\mathcal{A})$-complex, a $(\mathcal{K}(\mathcal{A}), 1)$-prism is the same thing as a homotopy of $\mathcal{K}(\mathcal{A})$-complexes, a 2-prism is a homotopy of 1-prisms, etc. A sectionally constant holomorphic (resp. continuous) $(\mathcal{K}(\mathcal{A}), k)$-prism is the same thing as a homotopy with k parameters consisting of holomorphic (resp. continuous) sections of $Z \to X$ over an open neighborhood of the union of sets in the family $\mathcal{A}$.

&4. Cartan strings and the initial holomorphic complex.

Let X be a complex manifold. A compact set $C \subset X$ contained in another compact set $B \subset X$ is said to be **Runge** in B if C has a basis of open neighborhoods each of which is Runge in a fixed neighborhood of B. A compact set $K \subset X$ is said to be a **Stein compactum** if it has a basis of open neighborhoods which are Stein.

We now recall from [FPr] the definition of a Cartan pair.

4.1 Definition. *An ordered pair of compact sets (A, B) in a complex manifold X is said to be a* **Cartan pair** *or a* **Cartan string of length 2** *if*

(i) each of the set A, B, and $A \cup B$ has a basis of Stein neighborhoods,

(ii) $\overline{A\backslash B} \cap \overline{B\backslash A} = \emptyset$ (separation condition), and

(iii) the set $C = A \cap B$ is Runge in B. (C may be empty.)

Note that our definition of a Cartan pair in [FPr] did not include property (iii). We now recall Gromov's definition of a *Cartan string*. The definition is by induction on $n \in N$; a Cartan string of length 2 ($n = 1$) is just a Cartan pair.

4.2 Definition. ([Gr1], 4.2.D'.) *Let X be a complex manifold and $A_0, A_1, \ldots, A_n \subset X$ compact subsets ($n \geq 1$). The sequence $(A_0, A_1, \ldots, A_n)$ is a* **Cartan string** *of length $n+1$ if*

(i) $(A_0 \cup \ldots \cup A_{n-1}, A_n)$ is a Cartan pair, and

(ii) if $n \geq 2$, the sequences $(A_0, \ldots, A_{n-1})$ and $(A_0 \cap A_n, \ldots, A_{n-1} \cap A_n)$ are Cartan strings of length n.

We say that a (locally finite) cover $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ of X by compact sets is a **Cartan cover** *if $(A_0, A_1, \ldots, A_n)$ is a Cartan string for each $n \in \mathbf{N}$.*

In particular each set A_k in a Cartan string and each union $A^k = \bigcup_{i=0}^k A_i$ is a Stein compactum. Note that the order of sets in a Cartan string is important because of property (iii) in definition 4.1; hence a Cartan string or cover is really a *sequences* of sets. Cartan strings satisfy the following *hereditary property:*

4.3 Proposition. *If $(A_0, A_1, \ldots, A_n)$ is a Cartan string in a complex manifold X and if $B \subset X$ is a Stein compactum then $(A_0 \cap B, \ldots, A_n \cap B)$ is also a Cartan string.*

Proof. Induction on $n \in \mathbf{N}$. Note first that if A and B are two compact sets in X which have a basis of Stein neighborhoods then $A \cap B$ also has a basis of Stein neighborhoods (since the intersection of Stein domains is again Stein). Consider first the case $n = 1$, i.e., we have a Cartan pair (A_0, A_1) and we wish to prove that $(A_0 \cap B, A_1 \cap B)$ is also a Cartan pair. Clearly it satisfies (i) and (ii) in def. 4.1. Property (iii) follows from

4.4 Lemma. *If $D_0 \subset D_1 \subset X$ and $\Omega \subset X$ are open Stein domains in a complex manifold X and if D_0 is Runge in D_1, then $D_0 \cap \Omega$ is Runge in $D_1 \cap \Omega$.*

Proof of Lemma 4.4. Choose any compact set $K \subset\subset D_0 \cap \Omega$. We denote by $\hat{K}_D$ the holomorphically convex hull of K with respect to a domain $D \subset X$ containing K. Clearly $\hat{K}_{D_1 \cap \Omega} \subset \hat{K}_{D_1} \cap \hat{K}_\Omega$. Since D_0 is Runge in D_1 and both domains are Stein, we have $\hat{K}_{D_0} = \hat{K}_{D_1}$ ([Hör], Theorem 2.7.3) and therefore

$$\hat{K}_{D_1 \cap \Omega} \subset \hat{K}_{D_1} \cap \hat{K}_\Omega = \hat{K}_{D_0} \cap \hat{K}_\Omega \subset\subset D_0 \cap \Omega.$$

It follows (Theorem 2.7.3 in [Hör]) that $D_0 \cap \Omega$ is Runge in $D_1 \cap \Omega$. ♠

This completes the proof of proposition 4.3 when $n = 1$. Suppose the result holds for some n and let $(A_0, A_1, \ldots, A_{n+1})$ be a Cartan string of length $n + 2$. To see that $(A_0 \cap B, \ldots, A_{n+1} \cap B)$ is a Cartan string we must verify that

(i) the pair of sets $(A_0 \cap B) \cup \cdots \cup (A_n \cap B) = (A_0 \cup \cdots \cup A_n) \cap B$ and $A_{n+1} \cap B$ is a Cartan pair. Since $(A_0 \cup \cdots \cup A_n, A_{n+1})$ is a Cartan pair, this follows from the case $n = 1$ proved above;

(ii) the strings $(A_0 \cap B, \cdots, A_n \cap B)$ and $(A_0 \cap A_{n+1} \cap B, \ldots, A_n \cap A_{n+1} \cap B)$ are Cartan strings of length $n + 1$. This follows immediatelly from definition 4.3 and from the inductive hypothesis. ♠

4.5 Corollary. *If $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ is a sequence of compact sets in a complex manifold X such that for each $n \in \mathbf{N}$ the pair $(A_0 \cup \cdots \cup A_{n-1}, A_n)$ is a Cartan pair, then for each $n \in \mathbf{N}$ the string $(A_0, A_1, \cdots, A_n)$ is a Cartan string.*

Proof. This follows from proposition 4.3 by induction on n. ♠

4.6 Theorem. *For each open cover $\mathcal{U} = \{U_j\}$ of a Stein manifold X there exists a Cartan cover $\mathcal{A} = \{A_i : i = 0, 1, \ldots\}$ of X which is subordinate to $\mathcal{U}$, i.e., such that each set A_i is contained in U_j for some $j = j(i)$. Moreover, if $K \subset X$ is a compact holomorphically convex subset and $U_0 \subset X$ is an open set containing K, we can choose $\mathcal{A}$ such that $K \subset A_0 \subset U_0$ and $A_i \cap K = \emptyset$ for $i \geq 1$.*

Proof. This was proved by Henkin and Leiterer in sect. 2 of [HL]. We recall briefly the main idea. The conditions imply that there exists a smooth strongly plurisubharmonic exhaustion function $\rho : X \to \mathbf{R}$ with nondegenerate critical points (a Morse function) such that $K \subset \{\rho < 0\} \subset\subset U_0$ and 0 is a regular value of ρ. Set $A_0 = \{\rho \leq 0\}$. One can reach any higher sublevel set $\{\rho \leq c\}$ for $c > 0$ a regular value of ρ by successively adding (finitely many times) small strongly pseudoconvex domains A_k to the union of previous sets $A^{k-1} = \bigcup_{i=0}^{k-1} A_i$ in such a way that for each $k = 1, 2, 3, \ldots$ the pair (A^{k-1}, A_k) is a *special pseudoconvex bump* in the terminology of [HL]. It is clear from their definition that such a pair is also a Cartan pair. Corollary 4.5 implies that the cover $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ of X that one builds in this way is a Cartan cover. If ρ has no critical values on an interval $[c_0, c_1] \subset \mathbf{R}$ then we can actually reach $\{\rho \le c_1\}$ from $\{\rho \le c_0\}$ by only adding finitely many *convex bumps* (this is called a noncritical pseudoconvex extension); in order to cross critical points of ρ one must attach more general pseudoconvex bumps. ♠

The following proposition provides a homotopy of complexes (a 1-prism) from the initial continuous section to a holomorphic complex over a Cartan cover of X.

4.7 Proposition. (Construction of the initial holomorphic complex.) *Let X be a Stein manifold, $K \subset X$ a compact holomorphically convex subset and $h\colon Z \to X$ a holomorphic submersion onto X with the property that each point $x \in X \backslash K$ has an open neighborhood $U_x \subset X$ such that $Z|U_x$ admits a spray. Let $a\colon X \to Z$ be a continuous section which is holomorphic in an open set $U_0 \supset K$. Then there exists a Cartan cover $\mathcal{A} = \{A_i\colon i = 0, 1, \ldots\}$ of X and a continuous $(\mathcal{K}(\mathcal{A}), 1)$-prism $a_* = \{a_{*,s}\colon s \in [0,1]\}$ with values in Z such that*

(i) $K \subset A_0 \subset U_0$, $K \cap A_i = \emptyset$ for $i \ge 1$, and $a_{(0),s} = a|U_0$ for each $s \in [0,1]$,

(ii) $a_{*,0}$ is a constant $\mathcal{K}(\mathcal{A})$-complex given by the section $a\colon X \to Z$,

(iii) $a_{*,1}$ is a holomorphic $\mathcal{K}(\mathcal{A})$-complex,

(iv) for each $j \ge 1$ the submersion $Z \to X$ admits a spray over an open set $U_j \supset A_j$.

Proof. Let $N = \dim Z = n + m$ where $n = \dim X$ and m is the dimension of the fibers $h^{-1}(x)$ $(x \in X)$. Denote by P^N the unit open polydisc in $\mathbf{C}^N$ with complex coordinates $\zeta = (\zeta', \zeta'')$, where $\zeta' \in \mathbf{C}^n$ and $\zeta'' \in \mathbf{C}^m$. Let $\pi\colon P^N \to P^n$ be the projection $\pi(\zeta', \zeta'') = \zeta'$. Since $h\colon Z \to X$ is a submersion, there exist for each point $z_0 \in Z$ open neighborhoods $V \subset Z$ of z_0, $U \subset X$ of $x_0 = h(z_0)$, and biholomorphic maps $\Phi\colon V \to P^N$, $\phi\colon U \to P^n$, such that $\pi \circ \Phi = \phi \circ h$ on V and $\Phi(z_0) = 0$. Such map Φ induces a linear structure on the fibers of $h|V$ which lets us add sections of $h\colon V \to U$ and take their convex linear combinations.

If z_0 belongs to the graph of a, we can choose these neighborhoods and maps such that $a(U) \subset V$. In this case Φ maps $a(U)$ onto the graph of a section $\tilde{a}(\zeta') = (\zeta', a''(\zeta'))$ $(\zeta' \in P^n)$ of the projection $\pi\colon P^N \to P^n$. The family $a_s\colon U \to V$ given by

$$a_s(x) = \Phi^{-1}\big(\phi(x), (1-s)a''(\phi(x))\big) \quad (x \in U,\ 0 \le s \le 1)$$

is a homotopy of continuous sections of h over U such that $a_s(U) \subset V$ for each $s \in [0,1]$, $a_0 = a|U$, and the section a_1 is holomorphic. By shrinking U around x_0 and replacing V by $V \cap h^{-1}(U)$ we may insure in addition that the graph of the entire homotopy a_s stays in a prescribed open neighborhood of $\overline{a(U)}$.

Let $U_0 \subset X$ be an open set containing K such that a is holomorphic in U_0. Set $V_0 = h^{-1}(U_0) \subset Z$ and $a_{(0),s} = a|U_0$ for $s \in [0,1]$. Using the above argument we can cover the graph of a outside V_0 by open neighborhoods $V_j \subset Z$ $(j = 1, 2, 3, \ldots)$ biholomorphic to P^N, with $U_j = h(V_j) \subset X$ biholomorphic to P^n, such that for each $j \in \mathbf{N}$ we have a homotopy of continuous sections $a_{(j),s}\colon U_j \to V_j$ $(0 \le s \le 1)$ of h satisfying

(a) $a_{(j),0} = a|U_j$,

(b) the section $a_{(j),1}$ is holomorphic in U_j,

(c) $U_j \cap K = \emptyset$ and $Z|U_j$ admits a spray for each $j \geq 1$,

(d) if $U_{(i,j)} = U_i \cap U_j \neq \emptyset$ for some $i, j \geq 0$ then $a_{(i),s}(U_{(i,j)} \subset V_j$ for each $s \in [0,1]$.

The property (d) insures that for each pair of indices $i, j \in \mathbf{Z}_+$ such that $U_{(i,j)} \neq \emptyset$ there is a homotopy of sections $a_{(i,j),s}(t): U_{(i,j)} \to Z$, depending continuously on $t, s \in [0,1]$, such that $a_{(i,j),s}(0) = a_{(i),s}|U_{(i,j)}$, $a_{(i,j),s}(1) = a_{(j),s}|U_{(i,j)}$, $a_{(i,j),0}(t) = a|U_{(i,j)}$, and the section $a_{(i,j),1}(t)$ is holomorphic on $U_{(i,j)}$ for each $t \in [0,1]$. We get $a_{(i,j),s}(t)$ by taking the convex linear combinations in $t \in [0,1]$ of the sections $a_{(i),s}$ and $a_{(j),s}$, restricted to $U_{(i,j)}$, where the combinations are taken with respect to the linear structure on the fibers of $h|V_i$ induced by Φ_i (or with respect to the linear structure on the fibers of $h|V_j$ induced by Φ_j). If one of the indices is zero, say $i = 0$ and $j > 0$, we must use the linear structure on V_j induced by Φ_j since there is no such structure on V_0.

Likewise if $U_{(i,j,k)} \neq \emptyset$ for some multiindex $J = (i,j,k)$ we can use the linear structure on the fibers in any one of the sets V_i, V_j, V_k to get a homotopy of sections $a_{J,s}(t): U_J \to Z$, with t belonging to the standard 2-simplex $\triangle^2 \subset \mathbf{R}^2$, whose restriction to the sides of the simplex equals the respective homotopy obtained in the previous step. Continuing this way we build a 1-prism a_* on the cover $\mathcal{U} = \{U_0, U_1, U_2, \ldots\}$ of X. By theorem 4.6 there is a Cartan cover $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ of X subordinate to $\mathcal{U}$, with $K \subset A_0 \subset U_0$. Then a_* induces in a natural way a $(\mathcal{K}(\mathcal{A}), 1)$-prism with the required properties. This proves proposition 4.7. ♠

&5. Modifying holomorphic prisms over Cartan strings.

This section contains the heart of the proof of theorem 1.4 (and hence also of theorem 1.2). In proposition 5.1 we show by induction on n how to modify a holomorphic $\mathcal{K}(\mathcal{A})$-complex f_* (definition 3.2) over any finite subcomplex $\mathcal{K}^n = \mathcal{K}(A_0, \cdots, A_n)$ into an honest holomorphic section defined over a neighborhood of the union $A^n = A_0 \cup A_1 \cup \cdots \cup A_n$, provided that $(A_0, \ldots, A_n)$ is a Cartan string. Since the inductive procedure requires us to solve the problem for parametrized families of complexes, we consider holomorphic prisms from the outset. The initial case is $n = 1$ when the string (A_0, A_1) is a *Cartan pair*; here we need the analytic tools outlined in [Gr1] and proved in [FPr].

5.1 Proposition. *Let $h: Z \to X$ be a holomorphic submersion onto a complex manifold X. Let $(A_0, \ldots, A_n)$ be a Cartan string in X and $\mathcal{K}^n = \mathcal{K}(A_0, \ldots, A_n)$ its nerve. Assume that for each $i = 1, \ldots n$ there is an open set $U_i \supset A_i$ in X such that $Z|U_i$ admits a spray. If f_* is a holomorphic $(\mathcal{K}^n, k)$-prism with values in Z which is sectionally constant on a nice compact set $Y \subset [0,1]^k$, there exists a homotopy f_*^u $(0 \leq u \leq 1)$ of holomorphic $(\mathcal{K}^n, k)$-prisms such that*

(i) $f_*^0 = f_*$ is the given prism,

(ii) the prism f_*^1 is sectionally constant,

(iii) the section $f_{(0),y}^u$ approximates $f_{(0),y}$ on A_0, uniformly with respect to $y \in [0,1]^k$ and $u \in [0,1]$, and

(iv) $f^u_{*,y} = f_{*,y}$ *for all* $y \in Y$ *and* $0 \le u \le 1$ *(i.e., the homotopy is fixed on* Y*).*
Moreover, setting $\mathcal{K}^{n-1} = \mathcal{K}(A_0, \ldots, A_{n-1})$*, if the restriction* $f_*|\mathcal{K}^{n-1}$ *is sectionally constant on* $[0,1]^k$*, the homotopy* f^u_* *can be chosen such that, in addition to the above,* $f^u_*|\mathcal{K}^{n-1}$ *is sectionally constant on* $[0,1]^k$ *and the corresponding section* $f^u_{*,y}|\mathcal{K}^{n-1}$ *(which is holomorphic in a neighborhood of* $A^{n-1} = A_0 \cup \cdots \cup A_{n-1}$*) approximates* $f_{*,y}|\mathcal{K}^{n-1}$ *on* A^{n-1}*, uniformly with respect to* $u \in [0,1]$ *and* $y \in [0,1]^k$*.*

Proof. In the proof we may assume that X is a Stein manifold since everything is happening in a neighborhood of the set A^n which is a Stein compactum in X. The proof goes by induction on $n \ge 0$; for $n = 0$ there is nothing to prove.
Case $n = 1$*:* Our data consists of a Cartan pair (A_0, A_1) in X and a holomorphic $(\mathcal{K}(A_0, A_1), k)$-prism f_* which is sectionally constant on a nice compact set $Y \subset [0,1]^k$. Such an object f_* is determined by the following data:

(a) a pair of open sets $U_0 \supset A_0$ and $U_1 \supset A_1$,
(b) families of holomorphic sections $a_y = f_{(0),y} \colon U_0 \to Z$, $b_y = f_{(1),y} \colon U_1 \to Z$ depending continuously on $y \in [0,1]^k$,
(c) a family of holomorphic sections

$$c_{y,t} = f_{(0,1),y}(t) \colon U_{(0,1)} = U_0 \cap U_1 \to Z$$

depending continuously on $t \in [0,1]$ and $y \in [0,1]^k$,

such that $a_y|U_{(0,1)} = c_{y,0}$, $b_y|U_{(0,1)} = c_{y,1}$, and for each $y \in Y$ the section $c_{y,t}$ is independent of $t \in [0,1]$. Hence for $y \in Y$ the family $\{c_{y,t} \colon t \in [0,1]\}$ determines a holomorphic section $c_y \colon U_0 \cup U_1 \to Z$ such that $c_y|U_0 = a_y$ and $c_y|U_1 = b_y$. We shall write $f_* = (a_*, b_*, c_*)$, where $*$ indicates the missing parameters.

Our goal is to construct a homotopy $f^u_* = (a^u_*, b^u_*, c^u_*)$ $(0 \le u \le 1)$ of holomorphic $(\mathcal{K}(A_0, A_1), k)$-prisms (over smaller sets $U_0 \supset A_0$ and $U_1 \supset A_1$) such that $f^0_* = f_*$ and f^1_* is a constant prism, which really means that f^1_* is a collection of holomorphic sections $f^1_y \colon U_0 \cup U_1 \to Z$ (we have eliminated the t parameter!). Moreover the homotopy should be fixed for $y \in Y$ and it should approximate the sections a_y over A_0, uniformly with respect to y. We shall denote the data in the homotopy f^u_* by the same letters as above by simply adding the upper index u.

Such a homotopy f^u_* will be constructed in two steps. For convenience we use the parameter interval $u \in [0,2]$ and later rescale it to $[0,1]$. In the first step we apply the h-Runge theorem (theorem 4.2 in [FPr]) to get a homotopy f^u_*, $0 \le u \le 1$, from $f^0_* = f_*$ to another prism f^1_* so that we do not move the section a_y (i.e., $a^u_y = a_y \colon U_0 \to Z$ for all u and y) and so that the section $b^1_y \colon U_1 \to Z$ approximates a_y in a neighborhood of $A_0 \cap A_1$, uniformly with respect to $y \in [0,1]^k$. In the second step we apply the gluing theorem (theorem 5.5 in [FPr]) to get homotopies

$$a^u_y \colon U_0 \to Z, \quad b^u_y \colon U_1 \to Z, \quad c^u_{y,t} \colon U_0 \cap U_1 \to Z \qquad (1 \le u \le 2)$$

such that at $u = 2$, $a^2_y = b^2_y$ over $U_0 \cap U_1$ for each $y \in [0,1]^k$, so these two sections define a single holomorphic section $f^2_y \colon U_0 \cup U_1 \to Z$. (Of course we must shrink the sets U_0 and U_1 again.) Moreover, both homotopies will be fixed on Y.

Consider the first step. Since $A_0 \cap A_1$ is Runge in A_1 and the submersion $h: Z \to X$ admits a spray over a neighborhood of A_1, the h-Runge theorem with parameters (theorem 4.2 in [FPr] and the remark following it) shows that, after shrinking the sets U_0 and U_1, there exists a homotopy of holomorphic sections $g^s_{y,t}: U_0 \cap U_1 \to Z$ $(0 \le s \le 1)$, depending continuously on t, s, y, so that

(i) $g^0_{y,t} = c_{y,t}$ for each y and t,

(ii) $g^s_{y,1} = c_{y,1} = b_y | U_0 \cap U_1$ for each s and y,

(iii) $g^1_{y,t}$ extends to a holomorphic section over U_1 for each y and t,

(iv) the homotopy is fixed on Y, i.e., for $y \in Y$ we have $g^s_{y,t} = c_y: U_0 \cup U_1 \to Z$ for each s and t, and

(v) $g^s_{y,t}$ approximates $c_{y,t}$ in a neighborhood of $A_0 \cap A_1$ as close as desired, uniformly with respect to all parameters.

We define $f^u_* = (a^u_*, b^u_*, c^u_*)$ for $0 \le u \le 1$ by

$$a^u_y = a_y, \quad b^u_y = g^1_{y,1-u},$$

$$c^u_{y,t} = \begin{cases} c_{y,2t(1-u)} & \text{if } 0 \le t \le 1/2; \\ g^{2t-1}_{y,1-u} & \text{if } 1/2 \le t \le 1. \end{cases}$$

The reader will easily verify that this satisfies all required properties. In particular, at $u = 1$, $b^1_y = g^1_{y,0}$ approximates $c_{y,0} = a_y$ in a neighborhood of $A_0 \cap A_1$.

Next we apply theorem 5.5 in [FPr] (on gluing parametrized families of holomorphic sections over Cartan pairs) to get homotopies of sections $a^u_y: U_0 \to Z$ and $b^u_y: U_1 \to Z$ for $1 \le u \le 2$ such that a^u_y approximates $a^1_y = a_y$ on A_0 for each $u \in [1,2]$ and $a^2_y = b^2_y$ in $U_0 \cap U_1$. Moreover, over a neighborhood of the set A_0 the graphs of all sections a^u_y, b^u_y $(1 \le u \le 2)$ and $c^1_{y,t}$ $(0 \le t \le 1)$ lie in small neighborhood of the (graph of) the section a_y in Z, and such a neighborhood is equivalent to a neighborhood of the zero section in a holomorphic vector bundle over U_0 (see lemma 5.3 in [FPr]). Using such a vector bundle structure we see that the triangle of homotopies formed by these families is contractible, which means that it can be filled by a 2-parameter homotopy $c^u_{y,t}$ $(0 \le t \le 1,\ 1 \le u \le 2)$ over a neighborhood of $A_0 \cap A_1$. This completes the proof of proposition 5.1 for $n = 1$.

The induction step $n \Rightarrow n+1$: Suppose that the proposition holds for all Cartan strings of length $n+1$ for some $n \ge 1$ and for all $k \ge 0$. Let $\mathcal{A} = (A_0, \ldots, A_{n+1})$ be a Cartan string of length $n+2$ with the nerve $\mathcal{K}^{n+1} = \mathcal{K}(\mathcal{A})$, and let f_* be a holomorphic $(\mathcal{K}^{n+1}, k)$-prism with values in Z which is sectionally constant on a nice compact subset $Y \subset [0,1]^k$. Let $\mathcal{K}^n = \mathcal{K}(A_0, \ldots, A_n) \subset \mathcal{K}^{n+1}$. The proof consists of the following three steps, each of which is accomplished by constructiong a suitable homotopy of prisms.

Step 1: Reduction to the case when $f_* | \mathcal{K}^n$ is a sectionally constant prism.

Step 2: Reduction to the case when f_* represents a $(k+1)$-prism over the Cartan pair (A^n, A_{n+1}), where $A^n = A_0 \cup A_1 \cup \cdots \cup A_n$.

Step 3: Applying the case $n = 1$ proved above to the prism obtained in step 2 to get a sectionally constant $(\mathcal{K}^{n+1}, k)$-prism.

We begin by some general considerations. We denote the coordinates on $\mathbf{R}^{n+1}$ by $t = (t', t_{n+1})$ where $t' = (t_1, \ldots, t_n)$. We identify $\mathbf{R}^n$ with the coordinate hyperplane $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$. The body K^{n+1} of the nerve $\mathcal{K}^{n+1}$ can be represented as the union of certain faces of the standard simplex $\triangle^{n+1} \subset \mathbf{R}^{n+1}$. (In fact $K^{n+1} = \triangle^{n+1}$ if and only if $A_0 \cap A_1 \cap \cdots \cap A_{n+1} \neq \emptyset$.) The body $K^n \subset \mathbf{R}^n$ of the subcomplex $\mathcal{K}^n = \mathcal{K}(A_0, \cdots, A_n) \subset \mathcal{K}^{n+1}$ is precisely the base $K^{n+1} \cap \{t_{n+1} = 0\}$ of K^{n+1}. We shall also need the complex

$$\mathcal{K}_1^n = \mathcal{K}(A_0 \cap A_{n+1}, \ldots, A_n \cap A_{n+1}) \subset \mathcal{K}^n. \tag{5.1}$$

Note that $\mathcal{K}_1^n = \{J \in \mathcal{K}^n : (J, n+1) \in \mathcal{K}^{n+1}\}$. Its body K_1^n is a subset of K^n which equals $\overline{(K^{n+1} \backslash K^n)} \cap K^n$. Moreover for each $0 < s < 1$ the section $K^{n+1} \cap \{t_{n+1} = s\}$ is homeomorphic to K_1^n. The map

$$r : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}, \quad r(t, s) = (t(1-s), s) \quad (t \in \mathbf{R}^n,\ s \in \mathbf{R}) \tag{5.2}$$

maps the prism $\triangle^n \times [0,1] \subset \mathbf{R}^{n+1}$ onto the standard simplex $\triangle^{n+1}$ (it is homeomorphic outside the level $s = 1$), and it maps $K_1^n \times \{s\}$ homeomorphically onto the section $K^{n+1} \cap \{t_{n+1} = s\}$ for each $s \in (0,1)$.

Proof of step 1: Since

$$\tilde{f}_*^0 = f_* | \mathcal{K}^n = \{f_J : J \in \mathcal{K}^n\}$$

is a k-prism over a Cartan string of length $n+1$, the induction hypothesis implies that there exists a holomorphic homotopy

$$\tilde{f}_* = \{\tilde{f}_*^u : u \in [-1, 0]\}$$

such that each $\tilde{f}_*^u$ is a $(\mathcal{K}^n, k)$-prism, the homotopy is fixed for all $y \in Y$, and the prism $\tilde{f}_*^{-1}$ is sectionally constant. The parameter space of the prism f_* is $K^{n+1} \times [0,1]^k$ while the parameter space of $\tilde{f}_*$ is $K' \times [0,1]^k$ where

$$K' = \{(t', u) \in \mathbf{R}^n \times \mathbf{R} :\ t' \in K^n,\ -1 \le u \le 0\}.$$

Note that f_* and $\tilde{f}_*$ agree on their common domain which is $K^n \times [0,1]^k$ and hence they define a collection of sections parametrized by the set $L \times [0,1]^k$ where $L = K^{n+1} \cup K' \subset \mathbf{R}^{n+1}$. We denote this collection by $\{g_y(t) : t \in L,\ y \in [0,1]^k\}$.

For each $s \in [0,1]$ we let $L_s \subset \mathbf{R}^{n+1}$ denote the set

$$L_s = (K^{n+1} \backslash K^n) \cup \{(t', t_{n+1}) : t' \in K_1^n,\ -s \le t_{n+1} \le 0\} \cup \{(t', -s) : t' \in K^n\}.$$

Intuitively speaking we get L_s by pushing the base K^n of K^{n+1} down (in the negative t_{n+1} direction) for s units and adding the vertical sides $K_1^n \times [-s, 0]$. Clearly $L_0 = K^{n+1}$ and each L_s is homeomorphic to K^{n+1}. In fact there is a continuous family of homeomorphisms $\Theta_s: K^{n+1} \to L_s$ $(0 \le s \le 1)$ such that Θ_0 is the identity, each Θ_s preserves the top vertex $(0,\ldots,0,1) \in K^{n+1}$ and the cellular structure of the two sets, and Θ_s maps K^n (the base of K^{n+1}) onto $K^n \times \{-s\}$ (the base of L_s) by a downward shift for s units. By 'respecting the cellular structure' we mean the following. Each face $J \in \mathcal{K}_1^n$ determines a face $\tilde J = (J, n+1) \in \mathcal{K}^{n+1}$, and Θ_s maps its body $|\tilde J| \subset K^{n+1}$ onto the set $|\tilde J| \cup \{(t', t_{n+1}): t' \in |J|,\ -s \le t_{n+1} \le 0\} \subset L_s$. Using this family we define a homotopy H_*^u $(0 \le u \le 1)$ of $(\mathcal{K}^{n+1}, k)$-prisms by

$$H_J^u(t) = g_y(\Theta_u(t)) \quad (J \in \mathcal{K}^{n+1},\ t \in |J| \subset K^{n+1}).$$

Clearly $H_*^0 = f_*$ and $H_*^1|\mathcal{K}^n = \tilde f_*^{-1}|\mathcal{K}^n$ is sectionally constant. This proves step 1.

Proof of step 2: By step 1 we may assume that the prism f_* is such that $f_*|\mathcal{K}^n$ is sectionally constant. The next step is to modify f_* by a homotopy of holomorphic prisms into another prism which is sectionally constant also in the direction of the last variable t_{n+1}. Let $\mathcal{K}_1^n$ be the complex (5.1). We associate to f_* a holomorphic $(\mathcal{K}_1^n, k+1)$-prism

$$F_* = \{F_{J,(y,s)}: |J| \to \mathcal{O}_h(U_{(J,n+1)}, Z), \quad J \in \mathcal{K}_1^n,\ y \in [0,1]^k,\ s \in [0,1]\}$$

where

$$F_{J,(y,s)}(t) = f_{(J,n+1),y}(r(t,s)), \quad t \in |J|,\ y \in [0,1]^k,\ s \in [0,1].$$

The set

$$Y_1 = (Y \times [0,1]) \cup ([0,1]^k \times \{0,1\}) \subset [0,1]^{k+1}$$

is a nice compact subset of $[0,1]^{k+1}$ (def. 1.3). Since $f_*|\mathcal{K}^n$ is sectionally constant, the prism F_* (which is associated to the complex $\mathcal{K}_1^n$ of length $n+1$) satisfies the induction hypothesis with respect to the set Y_1. Hence there is a homotopy F_*^u $(u \in [0,1])$ of holomorphic $(\mathcal{K}_1^n, k+1)$-prisms, beginning with $F_*^0 = F_*$, such that the homotopy is fixed for $(y,s) \in Y_1$ and such that F_*^1 is sectionally constant. This means that for each fixed $(y,s) \in [0,1]^{k+1}$ the $\mathcal{K}_1^n$-complex $F_{*,(y,s)}^1$ is constant, i.e., it represents a holomorphic section $F_y^1(s): V \to Z$ of $Z \to X$ over a neighborhood $V \subset X$ of the set $A^n \cap A_{n+1}$. In particular, since the homotopy is fixed for $s=0$ and $s=1$, the section $F_y^1(0)$ coincides with the section represented by the constant complex $f_{*,y}|\mathcal{K}^n$, and the section $F_y^1(1)$ coincides with the section $f_{(0,\ldots,0,1),y}$ associated to the set A_{n+1}.

Proof of step 3: We may consider the family of sections

$$F_*^1 = \{F_y^1(s): V \to Z|\ s \in [0,1],\ y \in [0,1]^k\}$$

constructed above as a holomorphic k-prism over the complex $\mathcal{K}' = \mathcal{K}(A^n, A_{n+1})$ determined by the Cartan pair (A^n, A_{n+1}). Here the parameter $s \in [0,1]$ represents the variable in the body $|\mathcal{K}'| = [0,1]$, and y is the space parameter. Note that the sections $F_y^1(0)$ extend holomorphically to a neighborhood of A^n and the sections $F_y^1(1)$ extend to a neighborhood of A_{n+1}.

The case $n=1$ of this proposition now applies and shows that there is a homotopy G_*^u $(0 \le u \le 1)$ of holomorphic $(\mathcal{K}', k)$-prisms such that $G_*^0 = F_*^1$, G_y^1 is a constant

$\mathcal{K}'$-complex for each $y \in [0,1]^k$ (i.e., a holomorphic section over an open neighborhood of $A^{n+1} = A_0 \cup \cdots \cup A_{n+1}$), and the homotopy is fixed for $y \in Y$ (where $G^0_y = F^1_y$ is already a section over A^{n+1}). Moreover, for each $u \in [0,1]$ the section $G^u_y(0)$ (which is holomorphic over a neighborhood of A^n) approximates the section $F^1_y(0) = f_{*,y}|\mathcal{K}^n$ on the set A^n, uniformly with respect to $u \in [0,1]$ and $y \in [0,1]^k$.

We can interpret the families $\{F^u_*: 0 \le u \le 1\}$ and $\{G^u_*: 0 \le u \le 1\}$ also as homotopies of holomorphic $(\mathcal{K}^{n+1}, k)$-prisms. By connecting these homotopies F^u_* and G^u_* (in this order) we obtain a homotopy f^u_* $(0 \le u \le 1)$ of holomorphic $(\mathcal{K}^{n+1}, k)$-prisms beginning at $u = 0$ with f_* and ending at $u = 1$ with the sectionally constant prism G^1_*.

If we assume in addition that the restriction $f_*|\mathcal{K}^n$ is sectionally constant on $[0,1]^k$ (so $f_{*,y}|\mathcal{K}^n$ is a holomorphic section in a neighborhood of A^n for each $y \in [0,1]^k$), we can skip the initial step in the proof of the inductive step. Note that, by construction, the restriction $F^u_*|\mathcal{K}^n$ is independent of $u \in [0,1]$ (since the homotopy F^u_* is fixed on the set Y_1), and the homotopy G^u_* is such that the complex $G^u_{*,y}|\mathcal{K}^n$ is represented by a holomorphic section in a neighborhood of A^n which approximates the section $F^1_{*,y}|\mathcal{K}^n = f_{*,y}|\mathcal{K}^n$ on A^n, uniformly with respect to $u \in [0,1]$ and $y \in [0,1]^k$. Hence the section $f^u_{*,y}|\mathcal{K}^n$ approximates $f_{*,y}|\mathcal{K}^n$ on A^n, the approximation being uniform with respect to $u \in [0,1]$ and $y \in [0,1]^k$.

This concludes the proof of the induction step and hence proposition 5.1 is proved. ♠

The next proposition shows that a holomorphic 1-prism can be extended from a finite subcomplex to the entire complex such that the 0-level of the prism matches a given complex. This does not require any analytic tools and hence the result applies to arbitrary locally finite covers.

5.2 Proposition. *Let $\mathcal{A} = \{A_0, A_1, A_2, \ldots\}$ be a locally finite family of compact sets in a complex manifold X with nerve $\mathcal{K}(\mathcal{A})$. Let $\mathcal{K}^n = \mathcal{K}(A_0, \ldots, A_n) \subset \mathcal{K}(\mathcal{A})$. Assume that $h: Z \to X$ is a holomorphic submersion onto X, f_* is a holomorphic $\mathcal{K}(\mathcal{A})$-complex with values in Z, and g_* is a holomorphic $(\mathcal{K}^n, 1)$-prism such that $g_{*,0} = f_*$. Then there exists a holomorphic $(\mathcal{K}(\mathcal{A}), 1)$-prism G_* such that $G_{*,0} = f_*$ and $G_*|\mathcal{K}^n = g_*$.*

Remark. A similar result holds if f_* is a holomorphic $(\mathcal{K}(\mathcal{A}), k)$-prism and g_* is a holomorphic $(\mathcal{K}^n, k+1)$-prism with base $f_*|\mathcal{K}^n$: such g_* extends to a holomorphic $(\mathcal{K}(\mathcal{A}), k+1)$-prism G_*.

Proof. Let $\mathcal{U}$ be an open neighborhood of $\mathcal{A}$ as in the definition of a holomorphic complex (so for each $J \in \mathcal{K}(\mathcal{A})$ and each $t \in |J|$, $f_J(t)$ is a holomorphic section in U_J, and likewise for g_*).

Write $A^n = A_0 \cup \cdots \cup A_n$ as before. Let $m \ge n$ be the smallest integer such that $A_k \cap A^n = \emptyset$ for each $k \ge m$. We represent the set $K^m = K(A_0, \ldots, A_m)$ (the body of the subcomplex $\mathcal{K}^m$) as a subset of $\mathbf{R}^m$. We denote the coordinates on $\mathbf{R}^{m+1}$ by (t, s) with $t \in \mathbf{R}^m$ and $s \in \mathbf{R}$, and we identify $\mathbf{R}^m$ with the hyperplane $\mathbf{R}^m \times \{0\} = \{s = 0\} \subset \mathbf{R}^{m+1}$. Similarly we identify a set $K \subset \mathbf{R}^m$ with $K \times \{0\} \subset \mathbf{R}^{m+1}$ and we write $K \times [0,1] = \{(t,s): t \in K,\ s \in [0,1]\}$. For each face $J \in \mathcal{K}^m$ we denote by $b|J| \subset K^m$ the boundary of its body $|J|$.

5.3 Lemma. *There exists a retraction*

$$r: K^m \times [0,1] \to K^m \cup (K^n \times [0,1]) \subset \mathbf{R}^{m+1}$$

such that for each face $J \in \mathcal{K}^m$ *we have*

(i) $r(|J| \times [0,1]) \subset |J| \cup (b|J| \times [0,1])$,

(ii) if $|J| \cap K^n = \emptyset$ *then* $r(t,s) = t$ *for each* $t \in |J|$ *and* $s \in [0,1]$.

Proof. We first define r over those faces $J \in \mathcal{K}^m$ for which either $|J| \subset K^n$ (we let r be the identity on $|J| \times [0,1]$) or $|J| \cap K^n = \emptyset$ (we let $r(t,s) = t$ for $t \in |J|$). We also define r to be the identity map on the bottom side $K^m = K^m \times \{0\}$. On the remaining faces $J \in \mathcal{K}^m$ we define r inductively with respect to the dimension of J. Suppose that r has already been defined on all faces of dimension $< k$ and let $J = (j_0, \ldots, j_k) \in \mathcal{K}^m$. Then r is defined on $|J| \cup (b|J| \times [0,1])$ and satisfies (i), and it satisfies (ii) on those sides of $b|J|$ which are disjoint from K^n. Moreover r is the identity on $|J| = |J| \times \{0\}$. It is now clear that r extends from $|J| \cup (b|J| \times [0,1])$ to $|J| \times [0,1]$ so that (i) holds. ♠

Let r be as in lemma 5.3. Write $r(t,s) = (r_0(t,s), u(t,s))$ where $r_0(t,s) \in K^m$ and $u(t,s) \in [0,1]$. We define a holomorphic $(\mathcal{K}^m, 1)$-prism G_* by setting for each $J \in \mathcal{K}^m$, $t \in |J|$ and $s \in [0,1]$

$$G_{J,s}(t) = \begin{cases} f_J(r_0(t,s)) & \text{if } u(t,s) = 0; \\ g_{J,u(t,s)}(r_0(t,s)) & \text{if } u(t,s) > 0. \end{cases}$$

Property (i) in lemma 5.3 implies that the section $G_{J,s}(t)$ for $t \in |J|$ is defined (and holomorphic) in the set U_J (it may be holomorphic in a larger set if $r_0(t,s) \in b|J|$, but in such case we restrict it to U_J). Clearly the collection $G_* = \{G_{J,s}: J \in \mathcal{K}^m,\ s \in [0,1]\}$ is a holomorphic $(\mathcal{K}^m, 1)$-prism which extends g_* and satisfies $G_{*,0} = f_*$.

The property (ii) of the retraction r lets us extend G_* to a prism over the entire complex $\mathcal{K}(\mathcal{A})$ by observing that for those faces $J \in \mathcal{K}(\mathcal{A})$ which do not belong to $\mathcal{K}^m$ we have $|J| \cap K^n = \emptyset$ (by definition of m) and therefore $r(t,s) = t$ for $t \in |J| \cap K^m$. Thus we can simply take $G_{J,s}(t) = f_J(t)$ for $t \in |J|$ and $s \in [0,1]$. This completes the proof of proposition 5.2. ♠

&6. Proof of theorem 1.4.

In this section we complete the proof of theorem 1.4 using the tools developed in section 5. We shall concentrate on the case of a single section; the proof in the general parametric case is basically the same.

Thus we are given a continuous section $a: X \to Z$ which is holomorphic in an open set $U_0 \subset X$ containing the given holomorphically convex set $K \subset\subset X$. Our goal is to construct a homotopy $H_s: X \to Z$ $(0 \le s \le 1)$ of continuous sections such that $H_0 = a$, the section H_1 is holomorphic on X, and all sections H_s are holomorphic near K and approximate a on K.

Let $\mathcal{A} = \{A_0, A_1, \ldots\}$ be a Cartan cover of X given by theorem 4.6 such that $K \subset A_0 \subset U_0$ and $K \cap A_i = \emptyset$ for $i \geq 1$. Also let $a_* = \{a_{*,s}: 0 \leq s \leq 1\}$ be a continuous $(\mathcal{K}(\mathcal{A}), 1)$-prism provided by proposition 4.7. Thus the complex $a_{*,0}$ is constant and represents the section a, $a_{*,1}$ is a *holomorphic* $\mathcal{K}(\mathcal{A})$-complex, and $a_{(0),s} = a|U_0$ for each $s \in [0,1]$.

Let d be a complete metric on Z compatible with the manifold topology. Fix an $\epsilon > 0$. We shall inductively construct a sequence of holomorphic $\mathcal{K}(\mathcal{A})$-complexes f^n_* and holomorphic $(\mathcal{K}(\mathcal{A}), 1)$-prisms $G^n_* = \{G^n_{*,s}: 0 \leq s \leq 1\}$ for $n = 0, 1, 2, \ldots$ satisfying the following properties:

(a) $f^0_* = a_{*,1}$,

(b) $G^n_{*,0} = f^n_*$ and $G^n_{*,1} = f^{n+1}_*$ for each $n \in \mathbf{Z}_+$,

(c) for each $k \in \mathbf{Z}_+$, $n \geq k$ and $s \in [0,1]$ the complexes $f^n_*|\mathcal{K}^k$ and $G^n_{*,s}|\mathcal{K}^k$ are constant, i.e., they are given by holomorphic sections denoted by f^n resp. G^n_s in an open neighborhood of $A^k = A_0 \cup \cdots \cup A_k$,

(d) (approximation) for each $n \in \mathbf{Z}_+$ and $s \in [0,1]$ we have

$$d\big(G^n_s(x), f^n(x)\big) < \epsilon/2^{n+1} \quad (x \in A^n).$$

In particular we have $d\big(f^{n+1}(x), f^n(x)\big) < \epsilon/2^{n+1}$ for $x \in A^n$.

In (d) we are using the notation for sections established in (c). The property (d) implies that the the sequence of sections $f^n: A^n \to Z$ $(n = 0, 1, 2, \ldots)$ converges uniformly on compacts in X to a holomorphic section $f^\infty = \lim_{n\to\infty} f^n: X \to Z$ which satisfies

$$d\big(f^\infty(x), a(x)\big) = d\big(f^\infty(x), f^0(x)\big) < \epsilon \quad (x \in A_0).$$

To construct a homotopy $H_s: X \to Z$ $(0 \leq s \leq 1)$ between $H_0 = a$ and $H_1 = f^\infty$ we first construct a continuous $(\mathcal{K}(\mathcal{A}), 1)$-prism h_* such that $h_{*,0} = a$ and $h_{*,1} = f^\infty$. To do this we simply collect all individual 1-prisms a_* and G^n_* $(n \in \mathbf{Z}_+)$ into a single 1-prism as follows. For each $n \in \mathbf{Z}_+$ set $I_n = [1 - 2^{-n}, 1 - 2^{-n-1}]$ and let $\lambda_n: I_n \to [0,1]$ be the linear bijection $\lambda_n(s) = 2^{n+1}(s - 1 + 2^{-n})$. Then $\bigcup_{n=0}^\infty I_n = [0,1)$. For $s \in [0,1)$ we define

$$h_{*,s} = \begin{cases} a_{*,2s} & \text{if } s \in I_0 = [0, 1/2]; \\ G^{n-1}_{*,\lambda_n(s)} & \text{if } s \in I_n,\ n \geq 1. \end{cases}$$

The two definitions of $h_{*,s}$ at the values $s = 1 - 2^{-n}$ $(n \in \mathbf{Z}_+)$ are compatible by property (b). Properties (c) and (d) imply that $\lim_{s\to 1} h_{*,s} = f^\infty$ uniformly on compacts in X. In fact, each compact $L \subset\subset X$ is contained in some A^m, and for $n \geq m$ the complex $G^n_{*,s}$ is constant on A^n (a holomorphic section). Hence for $1 - 2^{-n-1} \leq s < 1$ the complex $h_{*,s}|\mathcal{K}^n$ is a holomorphic section in a neighborhood of A^n. As $s \to 1$, these sections converge uniformly on A^n (and hence on L) to the limit f^∞. This proves that, if we set $h_{*,1} = f^\infty$, the collection $h_* = \{h_{*,s}: 0 \leq s \leq 1\}$ is indeed a continuous $(\mathcal{K}(\mathcal{A}), 1)$-prism. Notice also that the restriction of h_* to A_0 (more precisely, to the complex $\mathcal{K}(A_0)$ represented by the first set A_0) is in fact a homotopy of holomorphic sections h_s $(0 \leq s \leq 1)$ in a neighborhood of A_0 which connects $h_0 = a$ and $h_1 = f^\infty$ and so that all sections in the family satisfy $d(h_s(x), a(x)) < \epsilon$ for $x \in A_0$.

To complete the proof of theorem 1.4 we apply the version of proposition 5.1 for continuous prisms to modify the 1-prism h_* by a homotopy of 1-prisms (keeping the ends $s = 0$ and $s = 1$ fixed) into a 1-prism H_* which is *sectionally constant*, i.e., such that H_* represents a homotopy of continuous sections $\{H_s: X \to Z: 0 \le s \le 1\}$. Moreover we can achieve that in a neighborhood of A_0 the two sections H_s and h_s agree.

This concludes the proof of theorem 1.4 in the case without parameters. The parametric case can be proved by the same tools by introducing the parameter space P into the definition of (holomorphic or continuous) complexes and prisms and then repeating the same arguments in this setting. Note that the analytic tools used in the proof (the h-Runge theorem and the gluing theorem) have been established in this generality in [FPr].

A final remark. We have described an infinite procedure in which the final holomorphic section of $Z \to X$ is obtained as a locally uniform limit of holomorphic sections defined over increasingly larger compacts in X. Often one need less: To modify a given continuous section which is holomorphic over a Stein compactum $K \subset X$ into another section that is holomorphic over a larger Stein compactum $L \supset K$ in X. Our proof shows that this can be done in a *finite number of steps* provided that L *is a finite strongly pseudoconvex extension of* K, in the sense that K and L are two regular sublevel sets of a smooth strongly plurisubharmonic function defined in a neighborhood of $\overline{L\backslash K}$. In such case the construction in [HL] gives a finite Cartan string $(A_0, A_1, \ldots, A_l)$ in X with $A_0 = K$ and $A^l = \bigcup_{0 \le j \le l} A_j = L$. It suffices to apply our proof just on this string.

&7. Sections of vector bundles avoiding analytic subsets.

In this section we prove theorems 1.5 and 1.6.

We begin with the proof of parts (b) and (c) in theorem 1.5. To prove part (b) we recall the following result of Rosay and Rudin [RRu]: *For each positive integer q there exists a discrete set $\Sigma \subset \mathbf{C}^q$ such that any entire holomorphic map $f: \mathbf{C}^n \to \mathbf{C}^q\backslash\Sigma$ into the complement of Σ has complex rank at most $q-1$ at each point.* Of course this is true for all n once it holds for $n = q$, and the same is then true for holomorphic maps $f: X \to \mathbf{C}^q\backslash\Sigma$ from a Stein manifold X whose universal cover is a Euclidean space, for instance for $X = (\mathbf{C}^*)^n$. The rank condition on f implies that its image $f(X)$ is contained in a countable union $A = \cup_{j=1}^{\infty} A_j \subset \mathbf{C}^q\backslash\Sigma$ where each A_j is a compact subset contained in a local analytic set of complex dimension $\le q-1$ in $\mathbf{C}^q$ (Chirka [Chi]). Given a point $z \in \mathbf{C}^q$ we let $C_z(A_j)$ denote the real cone on A_j with vertex at z, i.e., the union of segments from z to points in A_j. For dimension reasons the set of z's for which $C_z(A_j)$ avoids Σ is open and everywhere dense in $\mathbf{C}^q$ (one can apply the transversality theorem for analytic sets). By Baire's theorem we can choose $z \in \mathbf{C}^q\backslash\Sigma$ such that the cone $C_z(A)$ with vertex at z avoids Σ. Since $f(X) \subset A$, it follows that we can contract f inside $\mathbf{C}^q\backslash\Sigma$ to the constant map $X \to z$. Thus each holomorphic map $X \to \mathbf{C}^q\backslash\Sigma$ is holomorphically contractible inside $\mathbf{C}^q\backslash\Sigma$ for such manifolds X. On the other hand when $n = 2q-1$ there exist continuous (even real-analytic) maps $f: X = (\mathbf{C}^*)^n \to \mathbf{C}^q\backslash\Sigma$ which are not homotopic to constant: it suffices to contract $(\mathbf{C}^*)^n$ onto the torus T^n and then embed T^n

as a real hypersurface in $\mathbf{C}^q \backslash \Sigma$ so that at least one point of Σ is contained in the bounded component of $\mathbf{C}^q \backslash T^n$.

A similar proof can be given for part (c) of theorem 1.5 by using proper holomorphic embeddings $g: \mathbf{C}^k \to \mathbf{C}^q$ for any $1 \le k < q$ such that *every entire map* $f: \mathbf{C}^n \to D = \mathbf{C}^q \backslash g(\mathbf{C}^k)$ *has complex rank* $< q - k$ *at each point.* Such embeddings were constructed in [BFo] for $k = 1, q = 2$ and in [For] in the general case. As before this implies that any holomorphic map of a Stein manifold X which is covered by a Euclidean space into $D = \mathbf{C}^q \backslash g(\mathbf{C}^k)$ is contractible to a point while for some such X there exist nontrivial smooth maps into D. For instance we can embed the torus T^n with $n = 2(q-k) - 1$ into the normal plane $\Lambda \subset \mathbf{C}^q$ to the submanifold $g(\mathbf{C}^k)$ at some point $z \in g(\mathbf{C}^k)$ so that z is contained in the bounded component of $\Lambda \backslash T^n$, and then we extend this to a map $f: (\mathbf{C}^*)^n \to D$. Clearly f is not homotopic to a constant map in D.

This proves parts (b) and (c) of theorem 1.5. To prove the other results we need the following lemma on the existence of sprays in the complements of subvarieties. Since the required result is local with respect to the base, we shall assume that the base is an open set $U \subset \mathbf{C}^n$ and the bundle is trivial over U. For each nonzero vector $v \in \mathbf{C}^q$ we denote by $[v] \in \mathbf{CP}^{q-1}$ the complex line $\mathbf{C}v \subset \mathbf{C}^q$ determined by v, considered as a point in the projective space $\mathbf{CP}^{q-1}$. We denote by $\pi_v: \mathbf{C}^q \to v^\perp = \mathbf{C}^{q-1}$ the linear projection with kernel spanned by v.

7.1 Lemma. (Existence of sprays.) *Let* $U \subset \mathbf{C}^n$ *be an open set* ($n \ge 1$) *and let* $\Sigma \subset U \times \mathbf{C}^q$ *for* $q \ge 2$ *be a closed analytic subset such that each fiber* $\Sigma_x = \{w \in \mathbf{C}^q : (x,w) \in \Sigma\}$ *has complex codimension at least two in* $\mathbf{C}^q$ *(it may be empty). Assume that there exists a nonempty open set* $\Omega \subset \mathbf{CP}^{q-1}$ *such that for each* $[v] \in \Omega$ *the linear projection* $\tilde{\pi}_v: U \times \mathbf{C}^q \to U \times \mathbf{C}^{q-1}$ *defined by*

$$\tilde{\pi}_v(x,w) = \big(x, \pi_v(w)\big) \quad (x \in U,\ w \in \mathbf{C}^q) \tag{7.1}$$

is proper when restricted to Σ. *Then the projection* $h: (U \times \mathbf{C}^q) \backslash \Sigma \to U$ *given by* $h(x,w) = x$ *admits a spray.*

Proof. We denote the coordinates on $U \times \mathbf{C}^q$ by $z = (x,w)$. Fix a point $z_0 = (x_0, w_0) \in (U \times \mathbf{C}^q) \backslash \Sigma$. For each line $[v] \in \Omega$ such that the affine complex line $\{w_0 + tv : t \in \mathbf{C}\} \subset \mathbf{C}^q$ does not intersect Σ_{x_0} the projection $\tilde{\pi}_v$ (7.1) (which is proper on Σ) satisfies $\tilde{\pi}_v(z_0) \notin \tilde{\pi}_v(\Sigma)$. By the codimension condition on Σ this is the case for all directions $[v]$ outside a proper subvariety of Ω. For each such direction $[v]$ there exists a holomorphic function $g: U \times \mathbf{C}^{q-1} \to \mathbf{C}$ which vanishes on the subvariety $\tilde{\pi}_v(\Sigma)$ and equals one at the point $\tilde{\pi}_v(z_0)$. The holomorphic vector field $V(z) = g\big(\tilde{\pi}_v(z)\big)v$ is $\mathbf{C}$-complete and vertical on $U \times \mathbf{C}^q$, with the flow $\theta_t(z) = z + tg\big(\tilde{\pi}_v(z)\big)v$ $(t \in \mathbf{C})$. By the choice of g the field V vanishes on Σ and hence its restriction to $(U \times \mathbf{C}^q) \backslash \Sigma$ is also a complete field. Moreover we have $V(z_0) = v$.

This shows that the complete vertical fields on $(U \times \mathbf{C}^q) \backslash \Sigma$ generate the vertical tangent space at each point. It remains to see that we can do the same already with finitely many fields of this type. We may assume that U is connected. We begin by choosing complete vertical fields $V_1, \dots, V_q$ which generate $VT_z(U \times \mathbf{C}^q)$ at one point of $(U \times \mathbf{C}^q)\backslash\Sigma$; hence they generate at each point outside a proper analytic subset $A \subset (U \times \mathbf{C}^q)\backslash\Sigma$. Let $A = \cup_j A_j$ be the (finite or countable) decomposition of A into irreducible components. Choose a point $z_j \in A_j$ in each component and consider the set $\Omega_j \subset \Omega$ of all complex directions $[v] \in \Omega$ for which v belongs to the linear span of the vectors $V_k(z_j)$ $(1 \le k \le q)$ or $\tilde\pi_v(z_j) \in \tilde\pi_v(\Sigma)$. Clearly Ω_j is a proper analytic subset of Ω and hence $\cup_j \Omega_j$ is a set of the first category in Ω. Choose any direction $[v] \in \Omega\backslash \cup_j \Omega_j$. By construction we then have $\tilde\pi_v(z_j) \notin \tilde\pi_v(\Sigma)$ for each j. Let $g_1, \dots, g_k$ be holomorphic functions in $U \times \mathbf{C}^{q-1}$ whose common zero set is precisely $\tilde\pi_v(\Sigma)$. If we add the corresponding complete vertical vector fields $W_i(z) = g_i\big(\tilde\pi_v(z)\big)v$ for $i = 1, \dots, k$ to the previous fields $V_1, \dots, V_q$, we increase the dimension of the linear span by at least one at each point z_j, and hence at each point outside a proper subvariety of each irreducible component A_i of A. An induction on dimensions of the span and of the exceptional set completes the proof of lemma 7.1. ♠

Part (a) of theorem 1.5 and the first part of theorem 1.6 now follow from theorem 1.2 by observing that the conditions in these results imply that Σ satisfies the condition in lemma 7.1 over small subsets in X and hence the complement admits a spray locally over X. In part (a) of theorem 1.5 it is clear that any linear projection $\pi\colon \mathbf{C}^q \to \mathbf{C}^{q-1}$ with kernel sufficiently close to $\{0\} \times \mathbf{C}$ is proper on Σ, and this proves part (a) in theorem 1.5.

As for theorem 1.6 we observe the following. If $\Sigma \subset \mathbf{CP}^q = \mathbf{C}^q \cup \Lambda$ is a closed algebraic subset (where $\Lambda \cong \mathbf{CP}^{q-1}$ is the hyperplane at infinity), then for each $v \in \mathbf{C}^q\backslash\{0\}$ such that the line $[v] \in \Lambda$ does not belong to $\Sigma \cap \Lambda$ the linear projection $\pi_v\colon \mathbf{C}^q \to \mathbf{C}^{q-1}$ with kernel $\mathbf{C}v$ is proper when restricted to $\Sigma \cap \mathbf{C}^q$. The conditions in theorem 1.6 imply that $\Sigma\backslash V$ is a closed analytic subset of $\overline{V}\backslash V$ which does not contain the hyperplane at infinity $\Lambda_x = \overline{V}_x\backslash V_x$ for any $x \in X$. Hence each point $x_0 \in X$ has an open neighborhood $U \subset X$ such that $\overline{V}|U$ is trivial and such that there is a nonempty open set $\Omega \subset \Lambda_x$ independent of $x \in U$, satisfying $\Sigma_x \cap \Omega = \emptyset$ for all $x \in U$. (Here we have identified the fibers $\overline{V}_x$ for $x \in U$ using the triviality of $\overline{V}|U$.) For each $v \in \Omega$ the projection $\tilde\pi_v$ (7.1) is proper on $\Sigma \cap \mathbf{C}^q$ and hence lemma 7.1 applies. This proves the first part of theorem 1.6.

It remains to prove the second part of theorem 1.6 with interpolation of the given section on a subvariety $X_0 \subset X$. This cannot be done directly by the procedure that was used to prove theorem 1.4. We begin choosing a holomorphic section $\phi\colon X \to V$ which matches f_0 to order k along X_0 (but whose graph need not avoid Σ). We introduce new holomorphic coordinates on V by $(x, v') = (x, v - \phi(x))$ $(x \in X,\ v \in V_x)$. In these coordinates the section f_0 satisfies the same properties as before and in addition it vanishes to order k on X_0. It is evident that the set Σ satisfies the required condition also in these new coordinates. By Cartan's Theorem A and B we can write $f_0 = \sum_{j=1}^m h_j\, g_j^0$ where the h_j's are holomorphic function on X which vanish to order k on X_0 and whose common zero set is precisely X_0 and where $g_j^0\colon X \to V$ for $1 \le j \le m$ are continuous sections of $V \to X$ which are holomorphic in a neighborhood of $X_0 \cup K$. We can view $G^0 = (g_1^0, g_2^0, \dots, g_m^0)$ as a section of the vector bundle $V^m \to X$ obtained by taking the Whitney (direct) sum of m copies of $V \to X$.

To prove theorem 1.6 it suffices to modify the section G^0 by a homotopy into a holomorphic section $G = (g_1, \dots, g_m)\colon X \to V^m$ such that the corresponding section $f =$

$\sum_{j=1}^{m} h_j\, g_j$ of $V \to X$ will satisfy the required properties. Note that the interpolation condition along X_0 will be automatically satisfied for each choice of G, and to insure that f approximates f_0 on K it suffices to insure that G approximates G^0 on K. What is more difficult to insure is that the graph of G (and of the entire homotopy from G^0 to G) would miss the set $\Sigma' \subset V^m$ which is the inverse image of $\Sigma \subset V$ under the projection $\Theta\colon V^m \to V$ defined by

$$\Theta(x; v_1, \ldots, v_m) = \sum_{j=1}^{m} h_j(x) v_j \qquad (x \in X,\ v_1, \ldots, v_m \in V_x).$$

By construction G^0 avoids Σ' over $X_0 \cup K$. Note that Θ is a linear surjection of holomorphic vector bundles over the set $X \backslash X_0$ while for each $x \in X_0$ it maps the entire fiber V_x^m to the point $0_x \in V_x$.

It is easy to verify that locally over $X \backslash X_0$ the set Σ' satisfies the condition in theorem 1.6; for this it suffices to observe that any vector bundle surjection is locally equivalent to a standard projection $(x; v, w) \to (x; v)$ of trivial bundles, and hence in such coordinates Σ' is defined by the same equations as Σ where the additional coordinates are not present.

The situation is more complicated near points $x \in X_0$: the fiber Σ'_x is empty since $(x, 0_x) \in V$ does not belong to the closure of Σ, but the behavior of nearby fibers of Σ' over points outside X_0 might be complicated. In particular we do not know whether there is a spray on $V^m \backslash \Sigma'$ over a neighborhood of $x \in X_0$, and this prevents us from patching sections when the set that we wish to attach to the previous set intersects X_0.

The idea is to carry out the modification procedure described in the proof of theorem 1.4 (section 5) so that we only have to patch sections over Cartan pairs (A, B) in X for which $B \cap X_0 = \emptyset$ (so we have a spray over a neighborhood of B as required). We need the following

7.2 Lemma. *Let X be a Stein manifold, X_0 a closed analytic subvariety of X, $\mathcal{U} = \{U_j\}_{j=1}^{\infty}$ an open covering of $X \backslash X_0$, $K \subset X$ a compact holomorphically convex subset and $U_0 \subset X$ an open set containing $X_0 \cup K$. Then for any compact set $L \subset X$ containing K there is a Cartan string $(A_0, A_1, \ldots, A_n)$ in X such that A_0 is holomorphically convex in X and we have*

(i) $K \cup (X_0 \cap L) \subset A_0 \subset U_0$,

(ii) *for* $j = 1, 2, \ldots, n$ *we have* $A_j \cap (X_0 \cup K) = \emptyset$ *and* $A_j \subset U_k$ *for some* $k = k(j) \geq 1$,

(iii) $L \subset A^n = \bigcup_{0 \leq j \leq n} A_j$.

Proof. We may (and will) assume that L is holomorphically convex in X. Choose a compact holomorphically convex set $K' \subset X$ which contains L (and hence K) in its interior. Then the set $S = (X_0 \cap K') \cup K$ is holomorphically convex in X (an easy excercise for the reader). By theorem 4.6 there exists a Cartan string $(A'_0, A'_1, \ldots, A'_n)$ in X satisfying the following conditions:

(a) $S = (X_0 \cap K') \cup K$ is contained in the interior of A'_0 and $A'_0 \subset U_0$,

(b) for each $j = 1, \ldots, n$ we have $A'_j \cap S = \emptyset$,

(c) if $A'_j \cap X_0 = \emptyset$ then $A'_j \subset U_k$ for some $k = k(j) \geq 1$,

(d) if $A'_j \cap X_0 \neq \emptyset$ for some $1 \leq j \leq n$ then $A'_j \cap L = \emptyset$, and

(e) $L \subset \cup_{0 \leq j \leq n} A'_j$.

Set $A_j = A'_j \cap L$ for $0 \leq j \leq n$. Since L is holomorphically convex and hence a Stein compactum, proposition 4.3 implies that $(A_0, A_1, \ldots, A_n)$ is a Cartan string in X. (Some sets A_j in this string may be empty and those we simply delete.) It is easily seen that this string satisfies lemma 7.2. Part (i) is clear, and (ii) holds because no A'_j for $j \geq 1$ intersects both X_0 and L at the same time. We also have (iii) with equality. This proves lemma 7.2. ♠

We continue with the proof of theorem 1.6. We start with the section $G^0 \colon X \to V^m \backslash \Sigma'$ obtained above which is holomorphic in an open set $U_0 \subset X$ containing $X_0 \cup K$. Fix a larger compact holomorphically convex set $L \subset X$ containing K in its interior. Our goal is to modify G^0 by a homotopy of sections which avoids Σ' into another section $G^1 \colon X \to V^m \backslash \Sigma'$ which is holomorphic over a neighborhood of the larger set $X_0 \cup L$ and approximates G^0 uniformly on K. Once this is done we can finish the proof by an obvious induction as in theorem 1.4.

Choose an open locally finite cover $\{U_j\}_{j=1}^{\infty}$ of $X \backslash X_0$ such that the submersion $V^m \to X$ admits a spray over each set U_j. Furthermore we may assume that the sets U_j for $j \geq 1$ are sufficiently small such that we can apply proposition 4.7 to the section G^0 over the cover $\{U_0, U_1, U_2, \ldots\}$ and modify it into a holomorphic complex. Thus, if $\mathcal{A} = (A_0, A_1, \ldots, A_n)$ is a Cartan string given by lemma 7.2, we can deform G^0 over a neighborhood of $\cup_{j=0}^{n} A_j = L$ into a holomorphic $\mathcal{K}(\mathcal{A})$-complex H_*. Since A_0 is contained in the set U_0 where G^0 is holomorphic, we may (and do) take the section $H_{(0)}$ in the complex H_* to be G^0 restricted to a suitable open neighborhood of A_0. By the process described in section 5 we can modify the complex H_* into a holomorphic section $H^1 \colon U' \to V^m \backslash \Sigma'$ over an open neighborhood U' of $A^n = L$ so that H^1 approximates G^0 on K and such that there is a homotopy H^t $(0 \leq t \leq 1)$ from G^0 to H^1 over L which avoids Σ' and which is holomorphic in a neighborhood of A_0.

The only remaining problem is that H^1 is not globally defined on X and so we cannot continue the induction yet. However, since $V^m \to X$ is a vector bundle and L is holomorphically convex in X, we can approximate H^1 uniformly over L by a globally defined holomorphic section, still denoted H^1, whose graph will also avoid Σ' over a neighborhood of L (but perhaps not elsewhere). By the same argument we can approximate the homotopy H^t (which connects G^0 and H^1) uniformly on A_0 (where it is holomorphic) by a global holomorphic homotopy. Finally we can modify the homotopy H^t outside a smaller neighborhood of $X_0 \cup K$ (by patching it with the initial homotopy) to get a new homotopy, still denoted H^t, which avoids Σ' over L and which is holomorphic in a neighborhood of $X_0 \cup K$.

Since Σ' has no points over X_0, it now comes for free that the graph of H^1 and of the entire homotopy H^t also avoids Σ' over X_0 and hence over an open neighborhood $U'_1 \subset X$ of $X_0 \cup L$. This enables us to modify H^1 outside a smaller neighborhood of $X_0 \cup L$ into a section $G^1 \colon X \to V^m \backslash \Sigma'$ which is holomorphic near $X_0 \cup L$. To do this we choose a smooth cut-off function $\chi\colon X \to [0,1]$ which equals one in a smaller neighborhood $U_1 \subset U_1'$ of $X_0 \cup L$ and equals zero outside U_1'. The section $G^1\colon X \to V^m$ defined by

$$G^1(x) = H^{\rho(x)}(x) \qquad (x \in X)$$

satisfies all required properties: it is holomorphic in U_1 since it coincides with H^1 there, it approximates G^0 on K, and its graph avoids Σ' everywhere since G^1 coincides with the section G^0 outside U_1'. In the same way we can modify the homotopy H^t. The induction may now continue and theorem 1.6 is proved. ♠

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