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2-ARC-TRANSITIVE GRAPHS OF  
ORDER A PRODUCT OF TWO  
PRIMES

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## 2-ARC-TRANSITIVE GRAPHS

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### **Abstract**

A classification of all 2-arc-transitive graphs of order a product of two primes is given. Furthermore it is shown that cycles and complete graphs are the only 2-arc-transitive Cayley graph of abelian group of odd order.

# 1 Introductory remarks

Throughout this paper graphs are finite, simple and undirected. By  $p$  and  $q$  we shall always denote prime numbers. A  $k$ -arc in a graph  $X$  is a sequence of  $k + 1$  vertices  $v_1, v_2, \dots, v_{k+1}$  of  $X$ , not necessarily all distinct, such that any two consecutive terms are adjacent and any three consecutive terms are distinct. A graph  $X$  is said to be  $k$ -arc-transitive if the automorphism group of  $X$ , denoted  $\text{Aut } X$ , acts transitively on the  $k$ -arcs of  $X$ .

For a group  $G$  and a generating set  $S$  of  $G$  such that  $1 \notin S = S^{-1}$ , the Cayley graph  $\text{Cay}(G, S)$  of  $G$  relative to  $S$  has vertex set  $G$  and edges of the form  $[g, gs], g \in G, s \in S$ . For any vertex  $g$  of  $\text{Cay}(G, S)$  and any subset  $S'$  of  $S$  an  $S'$ -neighbour of  $g$  is a vertex of the form  $gs', s' \in S'$ .

In [1] the classification of 2-arc-transitive circulants, that is Cayley graphs of cyclic groups, was given.

**Proposition 1.1** *A connected, 2-arc-transitive circulant of order  $n$ ,  $n \geq 3$ , is one of the following graphs:*

- (i) *the complete graph  $K_n$ , which is exactly 2-transitive;*
- (ii) *the complete bipartite graph  $K_{\frac{n}{2}, \frac{n}{2}}$ ,  $n \geq 6$ , which is exactly 3-transitive;*
- (iii) *the complete bipartite graph minus a matching  $K_{\frac{n}{2}, \frac{n}{2}} - \frac{n}{2}K_2$ ,  $\frac{n}{2} \geq 5$  odd, which is exactly 2-transitive; and*
- (iv) *the cycle  $C_n$  of length  $n$ , which is  $k$ -arc-transitive for all  $k \geq 0$ .*

The 2-arc-transitive graphs from the above proposition will be referred to as *trivial* 2-arc-transitive graphs. (For other results on 2-arc-transitive graphs we refer the reader to [10, Sec. 9]).

The object of this paper is to classify 2-arc-transitive graphs whose order is a product of two primes. This will be done by a simple extension of some of the ideas in [1] and by taking into account the known results on vertex-transitive graphs of order a product of two primes [6, 8, 7, 9, 11]. The following is our main result.

**Theorem 1.2** *A connected 2-arc-transitive graph of order  $pq$ , where  $p$  and  $q$  are primes, is isomorphic to one of the graphs in Table 1 below, where the two inside double lines divide the graphs into trivial ones, and nontrivial ones with imprimitive or primitive automorphism group.*

TABLE 1: 2-arc-transitive graphs of order a product of two primes.

row	$qp$	valency	the graph
1	$qp$	$qp - 1$	$K_{qp}$
2	$qp$	2	$C_{qp}$
3	$2p$	$p$	$K_{p,p}$
4	$2p$	$p - 1$	$K_{p,p} - pK_2$
5	$2 \cdot 11$	5	incidence graph of $H(11)$
6	$2 \cdot 11$	6	nonincidence graph of $H(11)$
7	$2 \cdot \frac{k^n - 1}{k - 1}$	$\frac{k^{n-1} - 1}{k - 1}$	incidence graph of $PG(n - 1, k)$
8	$2 \cdot \frac{k^n - 1}{k - 1}$	$k^{n-1}$	nonincidence graph of $PG(n - 1, k)$
9	$2 \cdot 5$	3	Petersen graph ( $O_3$ )
10	$5 \cdot 7$	4	$O_4$
11	$5 \cdot 11$	4	orb. graph $PSL(2, 11)$
12	$11 \cdot 23$	4	orb. graph $PSL(2, 23)$
13	$3 \cdot 19$	6	orb. graph $PSL(2, 19)$ (Perkel)
14	$29 \cdot 59$	6	orb. graph $PSL(2, 59)$
15	$31 \cdot 61$	6	orb. graph $PSL(2, 61)$
16	$7 \cdot 11$	16	orb. graph $M_{22}$

The proof of this theorem will be carried out over the next three sections. In Section 2 we analyze the case  $p = q$ ; as a byproduct we prove that a 2-arc-transitive Cayley graph of an abelian group of odd order is either a cycle or a complete graph (Proposition 2.2). In Sections 3 and 4 the respective cases of  $\text{Aut } X$  imprimitive and primitive are dealt with.

## 2 The case $p = q$

Let  $X$  be a 2-arc-transitive graph of order  $p^2$ . We observe that every vertex-transitive graph of order  $p^2$ , where  $p$  is a prime, is a Cayley graph [6], and so is either a circulant or a Cayley graph of the group  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Clearly,  $C_4$  and  $K_4$  are the only connected 2-arc-transitive graphs on 4 vertices. Moreover in view of the classification of 2-arc-transitive circulants [1, Theorem 1.1], it follows that a 2-arc-transitive circulant of odd order is either the complete graph or the cycle. As for the analysis of 2-arc-transitivity of Cayley graphs of  $\mathbb{Z}_p \times \mathbb{Z}_p$ , and more generally of abelian groups of odd order, it may be done by extending some of the ideas from [1].

We start by giving the following generalization of [1, Lemma 2.2]

**Lemma 2.1** *Let  $A$  be an abelian group and let  $J$  be the set of involutions in  $A$ . Let  $R \subseteq A \setminus \{0\}$  and let  $i \in A \setminus \{0\}$ . Let  $\rho(i) = |R \cap (-R + i)|$ . Then*

- (i)  $i \notin 2R$  implies that  $\rho(i)$  is even;
- (ii)  $i = 2r$  for some  $r \in R$  and  $|R \cap (J + r)|$  is even imply that  $\rho(i)$  is odd;
- (iii)  $i = 2r$  for some  $r \in R$  and  $|R \cap (J + r)|$  is odd imply that  $\rho(i)$  is even.

PROOF. Note that the elements belonging to the set  $R \cap (-R + i)$  are paired off whenever  $i \notin 2R$ . Namely, if  $r = -r' + i \in R \cap (-R + i)$  then also  $r' = -r + i \in R \cap (-R + i)$  proving (i).

So assume that  $i = 2r$  for some  $r \in R$ . Using the same argument as above, the parity of  $\rho(i)$  now depends solely on the parity of the number  $1 + |R \cap (J + r)|$  of solutions in  $R$  of the equation  $i = 2x$ . This proves both (ii) and (iii). ■

**Proposition 2.2** *Let  $X = \text{Cay}(A, S)$  be a 2-arc-transitive Cayley graph of an abelian group  $A$  of odd order. Then  $X$  is either the complete graph or the cycle on  $|A|$  vertices.*

PROOF. let  $d = |S|$  be the valency of  $X$ . By contradiction assume that  $X$  is neither a cycle nor a complete graph. Then its girth is 4. Namely,  $X$  has no triangles for a 2-arc-transitive graph with girth 3 is necessarily a complete graph. On the other hand, since  $X$  is not a cycle we can choose  $s_1, s_2 \in S$  such that  $s_2 \neq s_1, -s_1$ . Hence  $(0, s_1, s_1 + s_2, s_2, 0)$  is a 4-cycle in  $X$ .

Now, by 2-arc-transitivity any 2-arc of  $X$  is contained on the same number of 4-cycles. Therefore, the number of common neighbors of two vertices of  $X$  at distance 2 is constant, say equal to  $m$ . Since  $X$  has no triangles, it follows that  $t \in N^2(0)$  for any  $t \in (S + S) \setminus \{0\}$ . Of course, by the above comment,  $|N(0, t)| = m$ . But  $N(0, t) = \{S \cap (S + t)\}$  and so

$$|S \cap (S + t)| = m \text{ for each } t \in (S + S) \setminus \{0\}. \quad (1)$$

Recall that  $S = -S$  and apply Lemma 2.1. By assumption  $A$  has no involutions and so the parity of  $m$  depends solely on whether  $t$  belongs to  $2S$  or not, forcing  $m$  to be odd and  $S + S \setminus \{0\}$  to coincide with  $2S$ . It follows that  $|N^2(0)| = d = |N(0)|$ . We deduce that the bipartite graph induced

by all the edges having one endvertex in  $N(0)$  and the other in  $N^2(0)$  is isomorphic to the graph  $K_{d,d} - dK_2$ . This would then imply the existence of a vertex  $v \in N^3(0)$  adjacent to the whole of  $N^2(0)$ , which is impossible for  $X$  has odd order. ■

**Corollary 2.3** *A 2-arc-transitive Cayley graph of order  $p^2$ , where  $p$  is a prime, is isomorphic to  $C_{p^2}$  or to  $K_{p^2}$ .*

### 3 Aut $X$ is imprimitive

Let  $X$  be a graph admitting a complete system of imprimitivity  $\mathcal{B}$  with respect to some subgroup  $G \leq \text{Aut } X$ . Let  $K$  be the kernel of the action of  $G$  on  $\mathcal{B}$ . The quotient group  $\bar{G} = G/K$  acts faithfully on the set  $\mathcal{B}$  and can be viewed as a subgroup of the symmetric group  $S_n$ . The quotient graph  $X_{\mathcal{B}}$  is defined as a graph with vertex set  $\mathcal{B}$  and edges of the form  $BB'$ ,  $B, B' \in \mathcal{B}$ , whenever there is an edge of  $X$  with one endvertex in  $B$  and the other endvertex in  $B'$ . Clearly, the group  $\bar{G}$  is a subgroup of  $G$  and if  $G$  acts vertex-, edge-, arc-, or 2-arc-transitively on  $X$ , then  $\bar{G}$  acts vertex-, edge-, arc-, or 2-arc transitively on  $\bar{X}$ , respectively.

The following result may be extracted from [3, Theorem 1.1].

**Proposition 3.1** *Let  $p$  be a prime and  $X$  be a connected 2-arc-transitive regular  $p$ -fold cover of a complete graph  $K_n$ ,  $n \geq 3$ . Then either  $X \cong C_{3p}$  or  $X \cong K_{n,n} - nK_2$ .*

In the analysis of the graphs of order  $2p$  the following extension of [5, Theorem 6.2] will be used. The proof, which is only outlined here, can be obtained with a slight modification of the proof of the above mentioned theorem.

**Proposition 3.2** *Let  $X$  be a connected vertex-transitive graph of order  $2p$ ,  $p$  a prime, and let  $\mathcal{B}$  be a complete system of imprimitivity for the automorphism group  $\text{Aut } X$  with blocks of length 2. Then either*

- (i)  $X$  is a circulant or
- (ii)  $\text{Aut } X$  has also a complete system of imprimitivity with blocks of length  $p$ .



PROOF. Since  $G = \text{Aut } X$  is a transitive group of degree  $2p$ , there is an element  $\pi$  of  $G$  of order  $p$ . Let  $K$  denote the kernel of the action of  $G$  on  $\mathcal{B}$ . If  $K$  is not trivial, then it contains an involution  $\tau$ , which clearly commutes with  $\pi$  and thus together with  $\pi$  generates a cyclic group of order  $2p$ , forcing  $X$  to be a circulant. Assume now that  $K$  is trivial and thus the action of  $G$  on  $\mathcal{B}$  faithful. Note that since  $|G| \geq 2p$ , the action of  $G$  on  $\mathcal{B}$  is not regular. If this action is simply transitive (not 2-transitive) then by the theorem of Burnside [12, Theorem 11.6] we have that  $G$  is a Frobenius group. In this case a Sylow  $p$ -subgroup of  $G$  is normal and its orbits form a complete system of imprimitivity. We are now left with the case when the action of  $G$  on  $\mathcal{B}$  is 2-transitive, which is taken care of by the last paragraph in the proof of [5, Theorem 6.2]. ■

In the proof of Proposition 3.5 the following two lemmas will be used.

**Lemma 3.3** *Let  $X$  be a connected 2-arc-transitive graph and let  $\mathcal{B}$  be a complete system of imprimitivity of  $\text{Aut } X$ . Then either*

- (i)  $X$  is bipartite and  $\mathcal{B}$  is a bipartition of  $X$ , or
- (ii)  $|\mathcal{B}| > 2$  and for arbitrary distinct blocks  $B, B' \in \mathcal{B}$  and a vertex  $v \in B$ , we have  $N(v) \cap B = \emptyset$  and  $|N(v) \cap B'| \leq 1$ .

PROOF. Since  $X$  is connected and edge-transitive the endvertices of an edge of  $X$  must belong to different blocks in  $\mathcal{B}$ . Assume that there are distinct blocks  $B, B' \in \mathcal{B}$  and  $v \in B$  having at least two neighbours  $x, y$  in  $B'$ . Then  $(x, v, y)$  is a 2-arc and in view of 2-arc-transitivity the endvertices of any 2-arc of  $X$  must belong to the same block. But then the connectedness of  $X$  implies that  $V(X) = B \cup B'$  and so  $X$  is bipartite. ■

**Lemma 3.4** *Let  $p$  be a prime, let  $X$  be a vertex-transitive graph and let  $\mathcal{B}$  be a complete system of imprimitivity of blocks of length  $p$  for the automorphism group  $G = \text{Aut } X$ . If  $p > |\mathcal{B}|$  then the kernel  $K$  of the action of  $G$  on  $\mathcal{B}$  is transitive on each block in  $\mathcal{B}$ .*

PROOF. Let  $n = |\mathcal{B}|$ . Since  $|V(X)| = pn$  and  $G$  acts transitively on  $V(X)$ , there is an automorphism  $\pi \in G$  of order  $p$ . Let  $\bar{\pi}$  denote its image under the homomorphism  $G \rightarrow G/K \leq S_n$ . The order of  $\bar{\pi}$  is either  $p$  or 1. Since  $n < p$  it cannot be  $p$  implying that  $\pi \in K$  and that  $K$  acts transitively on at least one (and therefore on each) block in  $\mathcal{B}$ . ■

**Proposition 3.5** *Let  $X$  be a connected 2-arc-transitive graph of order  $pq$ , where  $p > q$  are primes, such that  $\text{Aut } X$  is imprimitive on  $V(X)$ . Then  $X$  is isomorphic to one of the following graphs.*

- (i)  $C_{pq}$ ,
- (ii)  $K_{p,p}$ ,
- (iii)  $K_{p,p} - pK_2$ ,
- (iv) *the incidence or the non-incidence graph of the Hadamard design  $H(11)$ , with  $pq = 22$ ,*
- (v) *the incidence or the non-incidence (?) graph of the projective geometry  $PG(n-1, k)$ , with  $pq = 2^{\frac{k^n-1}{k-1}}$ .*

PROOF. Let  $G = \text{Aut } X$ . Note that the quotient graph corresponding to a complete system of imprimitivity of  $G$  is, on the one hand, a circulant, and on the other hand, a 2-arc-transitive graph, and therefore, in view of Proposition 1.1, either a cycle or a complete graph.

We are going to distinguish two different cases.

CASE 1:  $q > 2$ .

Suppose first that  $G = \text{Aut } X$  contains a complete system of imprimitivity  $\mathcal{B}$  such that the kernel  $K$  of the action of  $G$  on  $\mathcal{B}$  is transitive on each block in  $\mathcal{B}$ . Of course,  $K$  is cyclic (of prime order). By Lemma 3.3 we have that  $X$  is a regular cover of  $X_{\mathcal{B}}$  (the latter being either a cycle or a complete graph of prime order) with a cyclic group of covering transformations. By Proposition 3.1, it follows that  $X \cong C_{pq}$  is the only possibility.

Suppose now that  $G = \text{Aut } X$  does not have a complete system of imprimitivity  $\mathcal{B}$  such that the kernel  $K$  of the action of  $G$  on  $\mathcal{B}$  is transitive on each block in  $\mathcal{B}$ . By Lemma 3.4 it follows that  $G$  has only blocks of length  $q$ . Using [7] it follows that  $p = 2^{2^s} + 1$  is a Fermat prime,  $q$  divides  $p - 2$ , and  $X$  is isomorphic to an orbital graph arising from the unique imprimitive representation (with blocks of length  $q$ ) of  $SL(2, 2^{2^s})$  with  $PG(1, 2^{2^s})$  as the complete system of imprimitivity. But it may be seen that these  $q$ -fold covers of  $K_p$  have triangles [7, p.378], and therefore they are not 2-arc-transitive graphs. Namely, a 2-arc-transitive graph with triangles must be a complete graph.

CASE 2:  $q = 2$ .

In view of Proposition 3.2 we may assume that either  $X$  is a circulant or  $G$  has a complete system of imprimitivity  $\mathcal{B} = \{B, B'\}$  with blocks of length  $p$ . The first case is done by Proposition 1.1 and we may therefore assume that the latter occurs.

By the Burnside theorem [12, Theorem 11.6] a simply transitive group of prime degree is a subgroup of the automorphism group of the field  $GF(p)$ . Thus the 2-arc-transitivity of  $X$  implies that the restrictions of  $G_B = G_{B'}$  to  $B$  and  $B'$  are 2-transitive groups. There are three possibilities for the restriction of  $G_B$ . If it is unfaithful, then  $X \cong K_{p,p}$ . If it is faithful and acts equivalently on  $B$  and  $B'$  then  $X \cong K_{p,p} - pK_2$ . Finally, if it is faithful and inequivalent then  $X$  must be one of the incidence / nonincidence graphs in (iv) and (v). ■

## 4 Aut $X$ is primitive

A connected graph with a 2-transitive automorphism group is a complete graph, which is clearly 2-arc-transitive. On the other hand, using the classification of finite simple groups we have that the only simply primitive groups of degree  $2p$  are  $A_5$  and  $S_5$  acting on the set of unordered pairs of a 5-element set [2]. Of the corresponding orbital graphs only the Petersen graph is 2-arc-transitive.

**Proposition 4.1** *The Petersen graph is the only nontrivial 2-arc-transitive graph of order  $2p$ ,  $p$  a prime, with a primitive automorphism group.*

For the rest of the section we will assume that  $q > 2$ . We have the following result.

**Proposition 4.2** *A nontrivial 2-arc-transitive graph of order  $pq$ , where  $3 \leq q < p$  are primes, is isomorphic to one of the graphs from rows 4, 11, 13–16 and 18 in Table 2, with respective valencies indicated in boldface.*

PROOF. Our proof is based on Table 2 below, extracted from ([9]), which lists all arc-transitive graphs of order  $pq$ ,  $3 \leq q < p$ , with a primitive automorphism group  $G$ . Note that  $T = \text{soc}G$  denotes the socle of  $G$ . We have added a column giving the order of the vertex stabilizer when needed in

our analysis of 2-arc-transitivity (it can easily be computed from the lemmas in sections 2 and 3 in [9]).

TABLE 2 : Vertex-primitive symmetric  $qp$ -graphs, where  $3 \leq q < p$  are primes.

row	$T = \text{soc } G$	$pq$	valency	$ G_v $	comment
1	$A_{pq}$				$K_{pq}$ or $K_{pq}^C$
2	$A_p$	$p \frac{p-1}{2}$			$T_n$ or $T_n^C$
3	$A_{p+1}$	$p \frac{p+1}{2}$			$T_n$ or $T_n^C$
4	$A_7$	$5 \cdot 7$	<b>4</b> , 12, 18		if $d = 4$ , $X \cong O_4$
5	$PSL(4, 2)$	$5 \cdot 7$	16, 18	$2^6 \cdot 3^2$	
6	$PSL(5, 2)$	$5 \cdot 31$	42, 112	$2^{10} \cdot 3^2 \cdot 7$	
7	$\Omega^\pm(2d, 2)$	$(2^d \mp 1)(2^{d-1} \pm 1)$			see [4]
8	$PSp(4, k)$	$(k+1)(k^2+1)$	$k+k^2, k^3$		see [8]
9a	$PSL(2, q^2)$	$q \frac{q^2+1}{2}$	$q^2-1, \frac{q^2-q}{2},$ $q^2-q, q^2+q$	$2q(q^2-1)$	$q \equiv 1(4)$
9b	$PSL(2, q^2)$	$q \frac{q^2+1}{2}$	$q^2-1, \frac{q^2+q}{2},$ $q^2-q, q^2+q$	$2q(q^2-1)$	$q \equiv 3(4)$
10	$PSL(3, 2)$	$3 \cdot 7$	4, 8	$2^4$	see [11, Ex. 2.3]
11	$PSL(2, 19)$	$3 \cdot 19$	<b>6</b> , 20, 30	$G_v \cong A_5$	see [11, Ex. 2.4]
12	$PSL(2, 29)$	$2 \cdot 29$	12, 20, 30, 60	$G_v \cong A_5$	
13	$PSL(2, 59)$	$29 \cdot 59$	<b>6</b> , 10, 12, 20, 30, 60	$G_v \cong A_5$	
14	$PSL(2, 61)$	$31 \cdot 61$	<b>6</b> , 10, 12, 20, 30, 60	$G_v \cong A_5$	
15	$PSL(2, 11)$	$5 \cdot 11$	<b>4</b> , 6, 8, 12, 24	$G_v \cong S_4$	
16	$PSL(2, 23)$	$11 \cdot 23$	<b>4</b> , 6, 8, 12, 24	$G_v \cong S_4$	
17a	$PSL(2, p)$	$p \cdot \frac{p \pm 1}{2}$	$\frac{p \pm 1}{2}, p \pm 1, 2(p-1)$	$2(p \pm 1)$	$p \equiv \pm 3$ or $\mp 1(8)$ , $p \geq 11$
17b	$PSL(2, p)$	$p \cdot \frac{p+1}{2}$	$\frac{p+1}{4}$ iff $p \equiv -1(8)$ , $\frac{p+1}{2}, p+1$	$p+1$	$p \equiv 3$ or $-1(8)$ , $p \geq 11$
17c	$PSL(2, p)$	$p \cdot \frac{p-1}{2}$	$\frac{p-1}{4}$ iff $p \equiv 1(8)$ , $\frac{p-1}{2}, p-1$	$p-1$	$p \equiv -3$ or $1(8)$ , $p \geq 11$
18	$M_{22}$	$7 \cdot 11$	<b>16</b> , 60	$G_v \cong \mathbf{Z}_2^4 \cdot S_6$	

A connected 2-arc-transitive graph  $X$  of order  $pq$  must be isomorphic to one of the graphs in Table 2. If graph  $X$  contains triangles it must be isomorphic to  $K_{pq}$ . As a consequence, graph  $X$  is not isomorphic to any of the graphs from rows 2, 3, 7 and 8. Namely, the graphs in the rows 2 and 3 are the so called triangle graphs, that is, the line graphs of the complete graphs, and their complements. Clearly, they all contain triangles. Similarly, the graphs from rows 7, known as  $C.4.$ <sup>±</sup> in Hubaut's list ([4]), are strongly regular graphs with parameter  $\lambda$  different from 0. As for the two graphs in row 8, it was proved in [8] that they are unions of orbital graphs arising from an imprimitive action of the group  $SL(2, k^2)$  with  $PG(1, k^2)$  as the complete system of imprimitivity, and therefore they contain triangles (see the last paragraph in Case 1 of the proof of Proposition 3.5).

To analyze the remaining graphs in Table 2 we check the vertex stabilizers  $G_v$ . Namely, since  $X$  is 2-arc-transitive the vertex stabilizer  $(\text{Aut } X)_v$  must act 2-transitively on the set of neighbours  $N(v)$  of the vertex  $v \in V(X)$ . In particular, if  $d = \text{val}X$  is the valency of  $X$ , then  $d(d-1)$  divides  $|(\text{Aut } X)_v|$ . A straightforward computation shows that the only graphs from rows 4, 5, 6, 9–18 satisfying this condition are the graph in rows 4, 11, 13, 14, 15, 16 and 18 with respective valencies indicated in bold face. Observe, that the 4-valent graph in the row 4 is in fact odd graph  $O_4$  (see [8, section 3]), which is easily seen to be 2-arc-transitive. Next, vertex stabilizers of the 4-valent graphs in rows 15 and 16 are isomorphic to  $S_4$  and since every transitive permutation representation of  $S_4$  of degree 4 is 2-transitive, the result follows. Similarly, the 2-arc-transitivity of the 6-valent graphs in rows 11, 13 and 14, with vertex stabilizers isomorphic to  $A_5$ , is obtained. We are left with the orbital graph of valency 16 arising from the action of  $M_{22}$  on 77 blocks of the Steiner triple system  $S(3,6,22)$ . The corresponding vertex stabilizer is isomorphic to  $(\mathbb{Z}_2)^4 \rtimes S_6$  and it may be easily seen that it acts 2-transitively on the set of neighbours, completing the proof of Proposition 4.2.

## 5 Conclusion

The proof of Theorem 1.2 now follows combining together Corollary 2.3, Proposition 3.5, Proposition 4.1 and Proposition 4.2.

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