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POINT STABILIZERS
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Abstract

In [7] a characterization of transitive permutation groups having a non-self-paired suborbit of length 2 (with respect to which the corresponding orbital graph is connected) was obtained in terms of their point stabilizers. As a consequence, elementary abelian groups were proved to be the only possible abelian point stabilizers arising from such actions, and D_8 was shown to be the only nonabelian group of order 8 with the same property. Constructions of such group actions with point stabilizers isomorphic to D_8 or to $\mathbb{Z}_2^h$, $h \geq 1$, were also given there. These results are extended here to include a more in depth analysis of the structure of point stabilizers of such group actions, resulting in a set of necessary conditions allowing us to obtain a restricted list of 19 possible candidates for point stabilizers of such group actions when the point stabilizers have order 2^h, $h \leq 8$. (For $h \leq 5$, this list gives a complete classification of such point stabilizers.) Furthermore, a construction of a transitive permutation group action with a non-self-paired suborbit of length 2 and point stabilizer isomorphic to $D_8 \times \mathbb{Z}_2^{h-3}$ is given for each $h \geq 3$.

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1 Introduction

Throughout this paper groups are assumed to be finite, unless specified otherwise. By a *graph* we mean an ordered pair (V, E), where V is a finite nonempty set and E is a symmetric irreflexive relation on V whose transitive closure is the universal relation on V. (Graphs are thus assumed to be finite and connected.) For graph-theoretic and group-theoretic terms not defined here we refer the reader to [1, 2, 3, 13]. If a graph X admits a vertex- and edge- but not arc-transitive action of a subgroup G of its automorphism group $\mathrm{Aut}\, X$, in short, a $\frac{1}{2}$-*arc-transitive action* of G. We say that X is $(G, \frac{1}{2})$-*arc-transitive* in this case. If $G = \mathrm{Aut}\, X$ we say that X is $\frac{1}{2}$-arc-transitive.

The work in this paper is motivated by results on point stabilizers of transitive permutation groups having suborbits of small length [8, 9, 11, 14] and by recent research in vertex- and edge- but not arc-transitive graphs

– see [5, 10, 12]. It is easily seen that the class of transitive permutation group having a connected self-paired suborbit of length 2 consists of dihedral groups alone. (A suborbit will be referred to as *connected* if the corresponding orbital graph is connected.) On the other hand, we have the classical results on point stabilizers by Tutte [11] and Wong [14] in the case of self-paired suborbits of length 3, and by Sims [9] in the case of primitive permutation groups and self-paired suborbits of length 4. Some partial results for other lengths are also known (see [8]).

The aim of this paper is to continue the study of the structure of transitive permutation groups having a connected non-self-paired suborbit of length 2 (with the emphasis on their point stabilizers) initiated in [7]. From a graph-theoretic point of view this is equivalent to the study of the structure of connected graphs of valency 4 admitting a vertex- and edge- but not arc-transitive group action, in particular, the study of the corresponding vertex stabilizers.

A lack of general machinery for these sort of problems makes the study of the structure of transitive permutation groups having a (connected) non-self-paired suborbit of length 2 a rather difficult task. As a starting point one would need to have a set of group-theoretic conditions fulfilled by a pair of groups (G, H), where G has a (connected) non-self-paired suborbit of length 2 in its action on the set of left cosets of H. Such a characterization was obtained in [7]. Such pairs are called $\vec{2}$-*creators* (see Section 2 for the definiton). Now when investigating $\vec{2}$-creative pairs of groups, two or three steps are taken, depending on whether the context is group- or graph-theoretic.

Step 1 would consist in the study of "local" structural properties of the group H (in the $\vec{2}$-creative pair (G, H)), that is, those properties of the point stabilizer H that are implicitly subsumed in the definition of a $\vec{2}$-creative pair of groups and can be, at least in theory, extracted from there. Such groups H are called *concentric* (for reasons that will become apparent in Section 2 where the definition is given).

As for *Step 2*, one needs to "imbed" a concentric group H into a supergroup G in such a way that the pair (G, H) is a $\vec{2}$-creator. Note that the difficulties involved in each of these two steps are quite different in nature. The first step is purely group-theoretic, whereas the second presupposes, at least implicitly, a graphical realization of some sort.

Finally, *Step 3* concerns the relative orbital graph corresponding to a (connected) non-self-paired suborbit in the action of G on (left) cosets of H, and is of course, of interest only from a graph-theoretic point of view: is the group G the full automorphism group of this graph, or more precisely, is this graph $\frac{1}{2}$-arc-transitive of is it arc-transitive?

It is the first two steps that we are interested in here. As mentioned above, a characterization of transitive permutation groups having a (connected) non-self-paired suborbit of length 2 in terms of their point stabilizers was obtained in [7]. (Note that the point stabilizer of such an action is necessarily a 2-group.) As a consequence, elementary abelian groups were proved to be the only possible abelian point stabilizers arising from such actions, and D_8 was proved to be the only nonabelian group of order 8 with the same property. Moreover, constructions of such group actions with point stabilizers isomorphic to D_8 or to $\mathbb{Z}_2^h$, $h \geq 1$, were also given there, that is, an infinite family of 4-valent graphs admitting a $\frac{1}{2}$-arc-transitive group action was constructed when the corresponding vertex stabilizer is either elementary abelian or the dihedral group of order 8. For other constructions of such actions with abelian point stabilizers see [4, 6].

In this paper we go a step further by presenting a more detailed analysis of the structure of concentric groups (Step 1), and hence, of the structure of point stabilizers of transitive groups with a (connected) non-self-paired suborbit of length 2, resulting in a restricted list of possible point stabilizers of order 2^h, $h \leq 8$, for such group actions. Namely, in Theorem 3.1 we prove that the point stabilizer is isomorphic to one of the following 19 groups:

(i) $\mathbb{Z}_2^h$ for $1 \leq h \leq 8$;

(ii) $D_8 \times \mathbb{Z}_2^{h-3}$ for $3 \leq h \leq 8$;

(iii) $D_8^2 \times \mathbb{Z}_2^{h-6}$ for $6 \leq h \leq 8$; and

(iv) $\mathcal{H}_7 \times \mathbb{Z}_2^{h-7}$ for $h \in \{7, 8\}$;

where D_8 denotes the dihedral group of order 8 and $\mathcal{H}_7$ denotes a certain 2-extension of the direct product $D_8 \times \mathbb{Z}_2^3$. (For $h \leq 5$, this list gives a complete classification of such stabilizers.) The proof of this result uses three general lemmas about concentric groups (proved in Section 2), Another byproduct of these lemmas is Corollary 2.4 where it is shown that a certain class of concentric groups of order 2^h (notably the one consisting of groups of complexity 1; see Section 2 for the definition) coincides with the direct products of the form

$D_8^{h-d} \times \mathbb{Z}_2^{3d-2h}$, where $d \geq \frac{2}{3}$. Moreover, Theorem 3.2 gives possible defining relations for a group G containing a subgroup H belonging to one of the above four families of groups – for arbitrarily large h – where the pair (G, H) is a $\vec{2}$-creator. (but see also Table 1). Finally, in Section 4 a construction of a transitive permutation group action with a (connected) non-self-paired suborbit of length 2 and point stabilizer isomorphic to $D_8 \times \mathbb{Z}_2^{h-3}$ is given for each $h \geq 3$ (Step 2). As a consequence, we have for every $h \geq 3$ a construction of a connected 4-valent graph admitting a $\frac{1}{2}$-arc-transitive group action with the respective stabilizer isomorphic to the group $D_8 \times \mathbb{Z}_2^{h-3}$, giving us a family of such graphs with arbitrary large nonabelian vertex stabilizers.

For the rest of the paper by a suborbit we shall always mean a connected suborbit.

2 The structure of point stabilizers

An ordered pair of groups (G, H) where H is a subgroup of G is said to be a $\vec{2}$-*creator* (or $\vec{2}$-*creative*) provided the following four conditions are satisfied.

(A1) there are $\tau_1 \in H$ and $a, b \in G \setminus H$ such that $G = \langle a, b \rangle = \langle a, \tau_1 \rangle = \langle b, \tau_1 \rangle$;

(A2) $H = \langle \tau_1, \ldots \tau_h \rangle$ for some positive integer h, where $\tau_i = a^i b^{-1} a^{-i+1}$ is an involution for each $i = 1, \ldots, h$;

(A3) $H_{i,j} = \langle \tau_i, \ldots \tau_j \rangle$, $1 \leq i \leq j \leq h$, has cardinality 2^{j-i+1};

(A4) H is not normal in G.

The following characterization theorem showing the correspondence between $\vec{2}$-creative pairs of groups and transitive permutation groups having a non-self-paired suborbit of length 2 is proved in [7, Theorem 3.1].

Theorem 2.1 *A pair of abstract groups (G, H), where $H \leq G$ and G is not dihedral, is a $\vec{2}$-creator if and only if the action of G on the set of (left) cosets of H has a connected non-self-paired suborbit of length 2.*

We say that a group $H = \langle \tau_1, \ldots, \tau_h \rangle$ is *concentric* if it satisfies the following two conditions.

(C1) the cardinality of $H_{i,j} = \langle \tau_i, \ldots, \tau_j \rangle$ is 2^{j-i+1} for $j > i$;

(C2) there exists a *rotary group isomorphism* $\varphi : H_{1,h-1} \to H_{2,h}$ such that $\varphi(\tau_i) = \tau_{i+1}$ for $i = 1, \ldots, h-1$.

(The name concentric is supposed to reflect the fact that, in view of Lemma 2.2 below, concentric groups have a "large" center, generated by the middle section of the generators $\tau_1, \ldots, \tau_h$.) Clearly, by (C1) each τ_i is an involution. Moreover, it follows from (C1) that every element $x \in H$ can be expressed uniquely as a product $x = \tau_1^{x_1} \tau_2^{x_2} \cdots \tau_h^{x_h}$, where $x_i \in \mathbb{Z}_2$ for $i = 1, \ldots, h$. Such an expression of x will be called the *canonical form* of x. Note that if (G, H) is a $\vec{2}$-creative pair of groups then H is concentric. (In general we do not know if the converse holds.)

A positive integer d is a *noncommuting distance* $H = \langle \tau_1, \ldots, \tau_h \rangle$, if for each $i = 1, \ldots, h$, the elements τ_i and τ_{i+d} do not commute. Clearly, because of the "rotary" property (C2) all possible noncommuting distances d appear if we consider just pairs of generators τ_1 and τ_{d+1}. Let $Dist(H) = (d_1, d_2, \ldots, d_k)$ denote the sequence of all of noncommuting distances in H listed in the increasing order. The shortest noncommuting distance d_1 will be called the *diameter* of H, denoted by $diam(H)$. If all the generators commute (that is, if the group H is elementary abelian) we set $diam(H) = h$. In other words, $d = diam(H)$ is the maximum subscript such that $\tau_1 \tau_d$ is an involution, that is, $d - 1$ is the maximum commuting distance. Furthermore, let $Z(H)$ denote the center of H.

Lemma 2.2 *(The Swinging Lemma). Let $H = \langle \tau_1, \ldots, \tau_h \rangle$ be a nonabelian concentric group of order 2^h and diameter $d = diam(H)$. Then $(\tau_i \tau_{i+d})^2 \in H_{h-d+i, 2d-h+i} \subseteq Z(H)$, for $i \in \{1, \ldots, h-d\}$. In particular, $d \geq \frac{2}{3}h$.*

PROOF. In view of the "rotary"property (C2) it suffices to prove the statement for $i = 1$. Let $x = (\tau_1 \tau_{d+1})^2$ and $m = h - d$. We use induction on m.

Let $m = 1$. Since, by (C1), both $H_{1,h-1}$ and $H_{2,h}$ are index 2 subgroups of H, we have $x = (\tau_1 \tau_{d+1})^2 = (\tau_1 \tau_h)^2 \in H_{1,h-1} \cap H_{2,h} = H_{2,h-1} = H_{2,d}$.

Suppose the statement holds for concentric groups with $h - d = m - 1$. Let H be a concentric group satisfying $h - d = m$. By the induction hypothesis

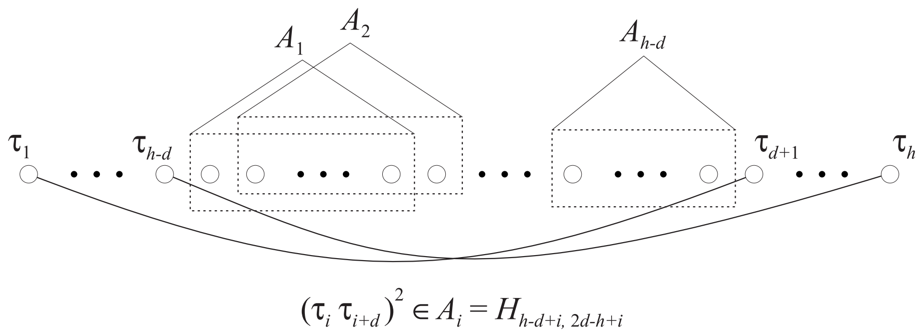

Figure 1: The Swinging Lemma.

the statement holds for the subgroup $H_{1,h-1}$. Consequently, $x \in H_{m,d-m+2}$. We need to see that $x \in H_{m+1,d-m+1}$. Let $z = (\tau_1\tau_h)^2$. Then $z \in H_{2,h-1} \leq H_{1,h-1}$, by (C1). By induction hypothesis x is a central element of $H_{1,h-1}$ and consequently, x and z commute. But z commutes with τ_{d+1}, too. Namely, by induction hypothesis we have $d \geq \frac{2}{3}(h-1)$ and so $h-1 < \frac{2}{3}d+1 < 2d+1$. Hence any $\tau_j \in H_{2,h-1}$ commutes with τ_{d+1} for $j \leq 2d$ in view of the definiton of d, forcing $z \in H_{2,h-1}$ to commute with τ_{d+1}. Using the above facts about z, x and τ_{d+1}, we have,

$$\tau_1 zx = \tau_1 xz = \tau_{d+1}\tau_1\tau_{d+1}z = \tau_{d+1}\tau_1 z\tau_{d+1} =$$

$$\tau_{d+1}\tau_h\tau_1\tau_h\tau_{d+1} = \tau_h\tau_{d+1}\tau_1\tau_{d+1}\tau_h = \tau_h\tau_1 x\tau_h = \tau_1 z\tau_h x\tau_h,$$

forcing $x = \tau_h x\tau_h$. Thus x centralizes τ_h. On the other hand, since d is the diameter of H it follows that x centralizes τ_1. Thus $x \in Z(H)$. Recall that, by induction hypothesis, $x \in H_{m,d-m+2}$. But τ_m, τ_h is a noncommuting pair of involutions. Thus $x \notin \tau_m H_{m+1,d-m+2}$ and so $x \in H_{m+1,d-m+2}$. Similarly, since τ_1, τ_{d-m+2} is a noncommuting pair, we deduce that $x \in H_{1,d-m+1}$. It folows that $x \in H_{m+1,d-m+1} = H_{h-d+1,2d-h+1} \subseteq Z(H)$. Furthermore, since $x \neq 1$ the subgroup $H_{h-d+1,2d-h+1}$ is nontrivial. Therefore $h-d+1 \leq 2d-h+1$, implying that $d \geq \frac{2}{3}h$. ■

Lemma 2.3 *Let H be a concentric group of order 2^h and diameter $d =$ diam(H). Then $H_{h-d+1,d} = Z(H)$ is the center of H.*

PROOF. By the Swinging Lemma we already know that $H_{h-d+1,d} \subseteq Z(H)$. Let us assume, for the contrary, that there exist a central element $x \notin H_{h-d+1,d}$. Let us consider the canonical form $x = \tau_1^{x_1}\tau_2^{x_2}\ldots\tau_h^{x_h}$ of x, where $x_i \in \{0,1\}$. There exists $i \leq h-d$ or $j \geq d+1$ such that either $x_i = 1$ or $x_j = 1$. If $x_i = 1$ we take i to be the minimum subscript with this property. By the Swinging Lemma τ_{i+d} commutes with all τ_k for $k \in \{h-d+1, \ldots h\}$. By the choice of i it commutes with all τ_k for $k \in \{i+1, \ldots, h-d\}$. Thus $\tau_{i+d}x \neq x\tau_{i+d}$. If $x_j = 1$ (for some $j \geq d+1$) we take j to be the maximum subscipt with that property. Using a similar argument as in the previous case we deduce that $\tau_{j-d}x \neq x\tau_{j-d}$. We have thus derived a contradiction in both cases, and consequently $Z(H) = H_{h-d+1,d}$. ■

The number of different noncommuting distances in a concentric group H will be called the *complexity* of the group H with respect to the set of generators $\{\tau_1, \ldots, \tau_h\}$.

Corollary 2.4 *A concentric nonabelian group $H = \langle\tau_1, \ldots, \tau_h\rangle$ of order 2^h, diameter d and complexity 1 is isomorphic to the direct product $D_8^{h-d} \times \mathbb{Z}_2^{3d-2h}$. Conversely, if $H \cong D_8^{h-d} \times \mathbb{Z}_2^{3d-2h}$, where $d \geq \frac{2}{3}h$, then there exists a set of involutory generators $\{\tau_1, \ldots, \tau_h\}$ of H with respect to which H is concentric with diameter d and complexity 1.*

PROOF. To prove the first part note that $K_i = \langle\tau_i, \tau_{i+d}\rangle \cong D_8$ for $i = 1, \ldots, h-d$. Besides $K_i \cap K_j = 1$ and $x \in K_i$ commutes with $y \in K_j$ for $i \neq j$. The result follows.

As for the second part, it follows from the assumptions that H contains $h-d$ disjoint copies K_i of D_8 generated by pairs of involutions, say ρ_i and σ_i. Furthermore let α_i be the generators of the $3d-2h$ copies of $\mathbb{Z}_2$ in H. Define $\tau_i = \rho_i$ for $i = 1, \ldots, h-d$, $\tau_{h-d+i} = \alpha_i$ for $i = 1, \ldots, 3d-2h$, and $\tau_{d+i} = \sigma_i$ for $i = 1, \ldots, h-d$. Since all the noncommuting pairs of the generators τ_i are at distance d, the group H is concentric having diameter d and complexity 1 with respect to the set $\{\tau_1, \ldots, \tau_h\}$. ■

Let us denote by $\mathcal{H}_7$ the group with 128 elements generated by 7 involutions ω_i, $i \in \{1, 2, \ldots, 7\}$ such that the following irreducible relations hold:

(i) $(\omega_i\omega_j)^2 = 1$ if $|i-j| \leq 4$;

(ii) $(\omega_1\omega_6)^2 = \omega_3$;

(iii) $(\omega_2\omega_7)^2 = \omega_4$;

(iv) $(\omega_1\omega_7)^2 = \omega_5$.

It is not difficult to see that $\mathcal{H}_7$ is uniquelly determined by the above relations (up to group isomorphism).

Lemma 2.5 *Let $H = \langle \tau_1, \ldots, \tau_h \rangle$ be a concentric group of order 2^h, where $h \geq 6$ and with the distance vector $Dist(H) = (h-2, h-1)$. Then $(\tau_i\tau_j)^2 \in H_{3,h-3} = Z(H)$ for all $1 \leq i \leq j \leq h$. Moreover, H is isomorphic to $D_8 \times D_8$ if $h = 6$, and to one of the groups $D_8^2 \times \mathbb{Z}_2^{h-6}$ or $\mathcal{H}_7 \times \mathbb{Z}_2^{h-7}$ if $h \geq 7$.*

Proof. First, we note that by Lemma 2.3, $Z(H) = H_{3,h-2}$. Let $z_1 = (\tau_1\tau_{h-1})^2$, $z_2 = (\tau_2\tau_h)^2$ and $z_3 = (\tau_1\tau_h)^2$. The Swinging Lemma implies that $z_1 \in H_{3,h-3} \subseteq Z(H)$, $z_2 \in H_{4,h-2} \subseteq Z(H)$ and $z_2 = \varphi(z_1)$, where $\varphi : H_{1,h-1} \to H_{2,h}$ is the rotary isomorphism existing by (C2). Moreover, by (C1), $z_3 \in H_{2,h-1}$ which is an elementary abelian group. Thus z_3 centralizes any τ_j for $2 \leq j \leq h-1$. Since, z_3 is the central element of the dihedral group generated by τ_1 and τ_h, it follows that $z_3 \in Z(H) = H_{3,h-2}$. There are two possibilities for the group $Z' = \langle z_1, z_2, z_3 \rangle \subseteq Z(H)$. Either $Z' = \langle z_1, z_2 \rangle$ or $Z' \neq \langle z_1, z_2 \rangle$. Thus either $Z' \cong \mathbb{Z}_2^2$ or $Z' \cong \mathbb{Z}_2^3$.

In the first case $z_3 = z_1^{\delta_1} z_2^{\delta_2}$ where $\delta_1, \delta_2 \in \{0, 1\}$. Replacing τ_1 by $\tau_1' = \tau_1 z_1^{\delta_1}$ and τ_h by $\tau_h' = \tau_h z_2^{\delta_2}$ we get a new generating set $\langle \tau_1', \tau_2, \ldots, \tau_{h-1}, \tau_h' \rangle$ where the only noncommuting pairs are $(\tau_1'\tau_{h-1})^2 = z_1$ and $(\tau_2\tau_h')^2 = z_2$. This can be easily verified using the fact that z_1, z_2 and z_3 are central elements of the group. Since, $z_1 \neq z_2$ it follows that $H \cong D_8^2 \times \mathbb{Z}_2^{h-6}$. If $Z' \neq \langle z_1, z_2 \rangle$, and consequently $Z' \cong \mathbb{Z}_2^3$, we have the isomorphism $H \cong \langle \tau_1, \tau_2, \tau_{h-1}, \tau_h \rangle \times \mathbb{Z}_2^{h-7}$. Now the assignment $\tau_1 \mapsto \omega_1$, $\tau_2 \mapsto \omega_2$, $\tau_{h-1} \mapsto \omega_6$, $\tau_h \mapsto \omega_7$ extends to a group isomorphism $\langle \tau_1, \tau_2, \tau_{h-1}, \tau_h \rangle \to \mathcal{H}_7$. ■

Let us remark that the groups $D_8^2 \times \mathbb{Z}_2^{h-6}$, $\mathcal{H}_7 \times \mathbb{Z}_2^{h-7}$ are nonisomorphic for any $h \geq 7$. Both have centers isomorphic to $\mathbb{Z}_2^{h-4}$, however the subgroup $Z'(H)$ of the center of minimum order and such that the factor group $H/Z'(H)$ is elementary abelian, has order 4 for $H = D_8^2 \times \mathbb{Z}_2^{h-6}$, while $Z'(H)$ has order 8 for $H = \mathcal{H}_7 \times \mathbb{Z}_2^{h-7}$. Using the same argument we can even deduce that $\mathcal{H}_7$ cannot be a subgroup of the direct product $D_8^2 \times \mathbb{Z}_2^{h-6}$.

3 Point stabilizers of order $2^h, h \leq 8$

Using the lemmas of the previous section we are now able to give a restricted list of possible point stabilizers of order $2^h \leq 8$ for transitive permutation groups having a non-self-paired suborbit of length 2.

Theorem 3.1 *Let G be a transitive permutation group having a (connected) non-self-paired suborbit of length 2 and let H be a point stabilizer of order $2^h, h \leq 8$. Then H is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_2^h$ *for* $1 \leq h \leq 8$;

(ii) $D_8 \times \mathbb{Z}_2^{h-3}$ *for* $3 \leq h \leq 8$;

(iii) $D_8^2 \times \mathbb{Z}_2^{h-6}$ *for* $6 \leq h \leq 8$;

(iv) $\mathcal{H}_7 \times \mathbb{Z}_2^{h-7}$ *for* $h \in \{7, 8\}$.

PROOF. By Theorem 2.1, H is a concentric group. If H is abelian it must be elementary abelian (see Section 1 and also [7, Theorem 4.1]). Assume H is nonabelian. By the Swinging Lemma we have that $diam(H) \geq \frac{2}{3}h$, and consequently, the complexity of H is either 1 or 2; and if it is 2 then the distance vector $Dist(H) = (h-2, h-1)$. Now the statement follows combining together Corollary 2.4 and Lemma 2.5. ■

It follows from Theorem 3.1 that concentric groups of order $2^h, h \leq 8$ split into four families characterized by the nonabelian factors in their direct product decomposition. Constructions of $\vec{2}$-creators of the form $(G, \mathbb{Z}_2^h)$ are known for every $h \geq 1$ (see [7]). In fact an even stronger result holds: for every $h \geq 1$ there are infinitely many $\frac{1}{2}$-arc-transitive graphs whose vertex stabilizer is isomorphic to $\mathbb{Z}_2^h$ see [6]. Point stabilizers isomorphic to the direct products of one copy of D_8 with an elementary abelian group are investigated in the subsequent section where $\vec{2}$-creators of the form $(G, D_8 \times \mathbb{Z}_2^h)$ are given. As concerns the other two families, let us remark that we know no example of a transitive permutation group having a non-self-paired suborbit of length 2 whose point stabilizer is neither elementary abelian nor a product of D_8 with an elementary abelian group. However, checking all possible relations between the involutory generators of H, a list of h relations in terms of the generators a, b of the group G can be produced (see part (A1) in the definiton of a $\vec{2}$-creative pair of groups for the definiton of a and b). To find an imbedding of a particular concentric group H into a (finite) group G such that the pair is $\vec{2}$-creative, it amounts to finding a finite quotient of the group determined by the above mentioned list of relations in which the relations are irreducible. To be more precise we have the following characterization theorem.

Theorem 3.2 *Let G be a group. Given elements a, b of G let $\tau_i = a^i b^{-1} a^{-i+1}$ for each $i \geq 1$. The following statements hold:*

(i) A pair of groups (G, H), where $H \cong \mathbb{Z}_2^h$, is a $\vec{2}$-creator if and only if there are elements $a, b \in G$ such that $G = \langle a, b \rangle$, $(a^i b^{-i})^2 = 1$ for $i = 1, \ldots, h$ are irreducible relations and $a\tau_h a^{-1} \notin H = \langle ab^{-1}, \ldots, a^h b^{-h} \rangle$;

(ii) A pair of groups (G, H), where $H \cong D_8 \times \mathbb{Z}_2^{h-3}$, $h \geq 3$, is a $\vec{2}$-creator if and only if there are elements $a, b \in G$ and $h \geq 3$ such that $G = \langle a, b \rangle$, and $(a^i b^{-i})^2 = 1$ for $i = 1, \ldots, h-1$, $(\tau_1 \tau_h)^2 = z$ for some non-trivial $z \in \langle \tau_2, \ldots, \tau_{h-1} \rangle$ are irreducible relations, and $a\tau_h a^{-1} \notin H = \langle \tau_1, \ldots, \tau_h \rangle$;

(iii) A pair of groups (G, H), where $H \cong D_8^2 \times \mathbb{Z}_2^{h-6}$, $h \geq 6$, is a $\vec{2}$-creator if and only if there are elements $a, b \in G$ and $h \geq 6$ such that $G = \langle a, b \rangle$, and $(a^i b^{-i})^2 = 1$ for $i = 1, \ldots, h-2$, $(\tau_1 \tau_{h-1})^2 = z_1$ for some non-trivial $z_1 \in \langle \tau_3, \ldots, \tau_{h-3} \rangle$, $(\tau_1 \tau_h)^2 = z_2$ for some $z_2 \in \langle z_1, az_1 a^{-1} \rangle$ are irreducible relations, and $a\tau_h a^{-1} \notin H = \langle \tau_1, \ldots, \tau_h \rangle$;

(iv) A pair of groups (G, H), where $H \cong \mathcal{H}_7 \times \mathbb{Z}_2^{h-7}$, $h \geq 7$, is a $\vec{2}$-creator if and only if there are elements $a, b \in G$ and $h \geq 7$ such that $G = \langle a, b \rangle$, and $(a^i b^{-i})^2 = 1$ for $i = 1, \ldots, h-2$, $(\tau_1 \tau_{h-1})^2 = z_1$ for some non-trivial $z_1 \in \langle \tau_3, \ldots, \tau_{h-3} \rangle$, $(\tau_1 \tau_h)^2 = z_2$ for some non-trivial $z_2 \in \langle \tau_3, \ldots, \tau_{h-2} \rangle \setminus \langle z_1, az_1 a^{-1} \rangle$ are irreducible relations, and $a\tau_h a^{-1} \notin H = \langle \tau_1, \ldots, \tau_h \rangle$.

Proof. Let us note that part (i) of the theorem was proved in [7, Theorem 4.1]. The arguments used in the proof of each of (ii) -(iv) are similar. We show here only the proof of part (iii).

Assume that the pair (G, H) is a $\vec{2}$-creator and that $H \cong D_8^2 \times \mathbb{Z}_2^{h-6}$. Let $d = diam(H)$. We first prove that $d = h - 2$. If $d \geq h - 1$ then either H

is elementary abelian or, by Corollary 2.4, H is isomorphic to $D_8 \times \mathbb{Z}_2^{h-3}$. Assume $d < h-2$. The complexity of the subgroup $H_{1,d+2} \leq H$ is either 1 or 2. If it is 1 then $H_{1,d+2} \cong D_8^2 \times \mathbb{Z}_2^{d-4}$, by Corollary 2.4. Thus, in view of the assumption on H, we must have $H = H_{1,d+2} \times H_{d+3,h}$. But τ_{d+3} does not commute with $\tau_3 \in H_{1,d+2}$, a contradiction. If the complexity of $H_{1,d+2}$ is 2, then we have $Dist(H_{1,d+2}) = (d, d+1)$ for the distance vector. Now Lemma 2.5 applies and we deduce that either $H_{1,d+2} \cong D_8^2 \times \mathbb{Z}_2^{d-4}$, or $H_{1,d+2} \cong \mathcal{H}_7 \times \mathbb{Z}_2^{d-5}$. The latter case is impossible since $\mathcal{H}_7$ cannot be a subgroup of $D_8^2 \times \mathbb{Z}_2^{h-6}$ (as was remarked at the end of Section 2). If $H_{1,d+2} \cong D_8^2 \times \mathbb{Z}_2^{d-4}$, then as above we derive a contradiction from the fact that τ_{d+3} and $\tau_3 \in H_{1,d+2}$ do not commute. We have therefore proved that the diameter of H is $h-2$. Now the relations $(a^i b^{-i})^2 = 1$ for $i = 1, \ldots, h-2$ follow from part (i) and the fact that $H_{1,h-2}$ is an abelian group. The Swinging Lemma gives us $(\tau_1\tau_{h-1})^2 = z_1 \in H_{3,h-3}$. As for $(\tau_1\tau_h)^2 = z_2$ the analysis goes as in the proof of Lemma 2.5. First, we deduce that $z_2 \in H_{3,h-2} = Z(H)$. Second, either $z_2 \in \langle z_1, az_1a^{-1}\rangle$ or $z_2 \notin \langle z_1, az_1a^{-1}\rangle$. But the second case yields $\mathcal{H}_7 \leq H$, a contradiction. We have thus established that G satisfies the required set of relations. Finally, assume for the contrary that $a\tau_h a^{-1} \in H$. Since $aH_{1,h-1}a^{-1} = H_{2,h}$ this means that a normalizes H. But $G = \langle a, H\rangle$ forcing H to be a normal subgroup of G, a contradiction.

Now let us assume that G satisfies the required conditions. We are going to prove that the pair (G, H) is a $\vec{2}$-creator and that $H \cong D_8^2 \times \mathbb{Z}_2^{h-6}$. The relations together with part (i) imply that $H \cong \langle \tau_1, \tau_2, \tau_{h-1}, \tau_h\rangle \times \mathbb{Z}_2^{h-6}$. The analysis of the relations gives $\langle \tau_1, \tau_2, \tau_{h-1}, \tau_h\rangle \cong D_8 \times D_8$. We conclude that $H \cong D_8^2 \times \mathbb{Z}_2^{h-6}$. Of course, the properties (A1) and (A2) are satisfied by the pair of groups (G, H). Combining the facts that $H_{1,j}$ is elementary abelian for $j \leq h-2$ and that $H_{1,h-1}$ is a 2-extension of $H_{1,h-2}$ (for $(\tau_1\tau_{h-1})^2 \in H_{1,h-2}$), it follows that the order of $H_{1,j}$ is 2^j if $j \leq h-1$. We already know that $|H| = 2^h$. Since $H_{i,j} = a^{i-1}H_{1,j-i+1}a^{-i+1}$, we get $|H_{i,j}| = 2^{j-i+1}$, and consequently, (A3) holds. Finally, the condition $a\tau_h a^{-1} \notin H$ implies that H is not a normal subgroup of G, thus (A4) holds too. ∎

The table below gives a set of possible relations of the group $G = \langle a, b\rangle$ for each of the four families of concentric groups $H \leq G$ in Theorem 3.2. Note that they arise from a particular choice of the elements z, z_1 and z_2.

TABLE 1: Possible relations for $G = \langle a, b\rangle$ with (G, H) a $\vec{2}$-creator and $|H| \le 2^8$.

stabilizer H	relations for $G = \langle a, b\rangle$
$\mathbb{Z}_2^h$	$(a^i b^{-i})^2 = 1$ for $i = 1, \ldots, h$
$D_8 \times \mathbb{Z}_2^{h-3}$	$(a^i b^{-i})^2 = 1$ for $i = 1, \ldots, h-1$, $a^h b^{-h} a^h b^{-1} a^{-1} b^{-h+2} = 1$
$D_8^2 \times \mathbb{Z}_2^{h-6}$	$(a^i b^{-i})^2 = 1$ for $i = 1, \ldots h-2, h$, $a^{h-1} b^{-h+1} a^{h-1} b^{-1} a^{-1} b^{-h+3} = 1$
$\mathcal{H}_7 \times \mathbb{Z}_2^{h-7}$	$(a^i b^{-i})^2 = 1$ for $i = 1, \ldots, h-2$, $a^{h-1} b^{-h+1} a^{h-1} b^{-2} a^{-1} b^{-h+4} = 1$, $(ab^{-1} a^h b^{-1} a^{-h+1})^2 a^4 b a^{-5} = 1$

If h is not too large it is not difficult to find all possible sets of relations depending on the choices of the elements z, z_1 and z_2 in Theorem 3.2. For instance, if $H \cong D_8 \times D_8$ then part (iii) of Theorem 3.2 applies. We have no choice for $z_1 = \tau_3$, while we have four choices for $z_2 = \langle \tau_3, \tau_4 \rangle$. Thus we have the following corollary characterizing $\vec{2}$-creative pairs of the form $(G, D_8 \times D_8)$. (Note that $D_8 \times D_8$ is the smallest concentric group H for which a $\vec{2} - (G, H)$-creator is not known.)

Corollary 3.3 *Let $(G, D_8 \times D_8)$ be a $\vec{2}$-creative pair of groups. Then G is a finite quotient of one of the following four groups:*

$$G_j = \langle\, a, b;\ R_1, R_2, R_3, R_4, R_5, Rel_j \,\rangle,\ j = 1, 2, 3, 4,$$

where

(i) $R_i \equiv (a^i b^{-i})^2 = 1$, *for* $i = 1, 2, 3, 4$, *and* $R_5 \equiv a^5 b^{-1} a^5 b^{-2} a^{-1} b^{-2} = 1$, *while*

(ii) $Rel_1 \equiv a^6b^{-6}a^6b^{-2}a^{-2}b^{-2} = 1$, $Rel_2 \equiv a^6b^{-6}a^6b^{-3}a^{-1}b^{-2} = 1$, $Rel_3 \equiv a^6b^{-6}a^6b^{-2}a^{-1}b^{-3} = 1$, *and* $Rel_4 \equiv a^6b^{-6}a^6b^{-6} = 1$.

Moreover, any finite quotient G *of one of the groups* $G_1, \ldots, G_4$ *for which* $a^7b^{-1}a^{-6} \notin \langle ab^{-1}, a^2b^{-1}a^{-1}, \ldots, a^6b^{-1}a^{-5}\rangle$ *and the respective relations above are irreducible, determines a* $\vec{2}$*-creative pair of groups* $(G, D_8 \times D_8)$.

Since we know no $\vec{2}$-creators of the form $(G, D_8 \times D_8)$, the groups G_j, $j = 1, 2, 3, 4$, may not exist. However, if they exist one can prove that they are necessarily infinite.

4 Constructing actions with point stabilizers $D_8 \times \mathbb{Z}_2^{h-3}$

Let $h \geq 3$ and $H = D_8 \times \mathbb{Z}_2^{h-3}$. By Corollary 2.4 there exists a set of involutory generators $\{\tau_1, \ldots, \tau_h\}$ of H with respect to which H is concentric, of complexity 1 and diameter $d = h - 1$. Note that in any representation of H as a concentric group the complexity is 1 and diameter is $h - 1$. Basically, all such representations differ only in the choice of the central element $z = (\tau_1\tau_h)^2 \in H_{2,h-1}$. We are now going to prove that for any concentric representation of H there exists a supergroup G such that the pair (G, H) is a $\vec{2}$-creator.

Theorem 4.1 *Let* $h \geq 3$ *be an integer and let* $H = D_8 \times \mathbb{Z}_2^{h-3}$. *Then for any concentric representation of* H *there exists a supergroup* G *such that the pair* (G, H) *is a* $\vec{2}$*-creator.*

Proof. In view of Theorem 2.1 it is sufficient to find a supergroup $G \geq H$ for H such that (G, H) is a $\vec{2}$-creative pair of groups. In what follows we assume that $H = \langle \tau_1, \ldots, \tau_h\rangle \cong D_8 \times \mathbb{Z}_2^{h-3}$, where all τ_i are involutions, and $\tau_i\tau_j = \tau_j\tau_i$ unless $|j - i| = h - 1$. Since τ_1 and τ_h are involutory generators of a dihedral subgroup of order 8, there is an involution $z \in H_{2,h-1}$ such that $(\tau_1\tau_h)^2 = z$. Since every element x of the concentric group $H = \langle \tau_1, \ldots, \tau_h\rangle$ can be expressed in a unique way in the canonical form $x = \tau_1^{x_1}\tau_2^{x_2} \cdots \tau_h^{x_h}$, where $x_i \in \mathbb{Z}_2$ we may think of the group H as a "sqewed" vector space $V = \mathbb{Z}_2^h$ (so to speak) with the unit vectors e_i as generators. Note, however that the binary operation $\oplus$ induced by H is not commutative in this case.

We now define a regular permutation representation $\bar{H}$ of H on V by first taking the right regular representation of H (on itself) and second, by making an appropriate identification of H with V given via the above canonical form of its elements. In view of the defining relations for H we have that $\bar{\tau}_i(x) = x + \epsilon_1 + x_h z$ if $i = 1$ and $\bar{\tau}_i(x) = x + \epsilon_i$ for $i = 2, \ldots, h$, where $\bar{\tau}_i \in \bar{H}$ denotes the permutation acting on V, corresponding to $\tau_i \in H$.

Now we find an appropriate permutation φ of V such that the pair of groups $(G, \bar{H})$ is a $\vec{2}$-creator, where $G = \langle \varphi, \bar{H} \rangle$. To this end let φ act by the following rule $\varphi(x) = R(x + x_1 x_h z)$, where R is the linear transformation cyclically permuting the unit vectors, that is, $R(e_i) = e_{i+1}$, and $x = (x_1, x_2, \ldots, x_h)$, $x_i \in \mathbb{Z}_2$. It is easy to verify that φ is injective.

To see that $G = \langle \bar{\tau}_1, \varphi \rangle$ gives rise to a $\vec{2}$-creative pair $(G, \bar{H})$ of groups, it is sufficient to prove that $\varphi(\bar{\tau}_i)\varphi^{-1} = \bar{\tau}_{i+1}$ for $i \in \{1, \ldots, h-1\}$ and that $\varphi(\bar{\tau}_h)\varphi^{-1} \notin \bar{H}$. We distinguish two cases.

Case 1: $2 \leq i \leq h - 1$.

We have by computation,

$$\varphi\bar{\tau}_i(x) = \varphi(x + e_i) = Rx + x_1 x_h Rz + e_{i+1} = \bar{\tau}_{i+1}(R(x + x_1 x_h z)) = \bar{\tau}_{i+1}\varphi(x).$$

Case 2: $i = 1$.

In this case the definition of $\bar{\tau}_1$ implies

$$\varphi\bar{\tau}_1(x) = \varphi(x + e_1 + x_h z) = Rx + e_2 + x_h Rz + (x_1 + 1)x_h Rz =$$

$$= Rx + e_2 + x_1 x_h Rz = \bar{\tau}_2(Rx + x_1 x_h Rz) = \bar{\tau}_2\varphi(x).$$

Thus the conjugation by φ sends $\bar{H}_{1,h-1}$ onto $\bar{H}_{2,h}$. We complete the proof by checking that $\varphi(\bar{\tau}_h)\varphi^{-1} \notin \bar{H}$. Assume for the contrary that $\varphi(\bar{\tau}_h)\varphi^{-1} \in \bar{H}$. Since $\varphi\bar{\tau}_h\varphi^{-1}(0) = e_1 = \bar{\tau}_1(0)$, the regularity of the action of $\bar{H}$ on V forces $\varphi(\bar{\tau}_h)\varphi^{-1} = \bar{\tau}_1$, which is equivalent to the identity $\varphi\bar{\tau}_h = \bar{\tau}_1\varphi$. However, $\varphi\bar{\tau}_h(e_{h-1}) = e_1 + e_h$ and $\bar{\tau}_1\varphi(e_{h-1}) = e_1 + e_h + z$, a contradiction. ■

By restating the above theorem in graph-theoretic terms we have, in view of Theorem 2.1, the following corollary.

Corollary 4.2 *Let $h \geq 3$ be an integer. Then there exist a connected 4-valent graph X and a group $G \leq Aut X$ such that X is $(G, \frac{1}{2})$-arc-transitive with vertex stabilizer isomorphic to $D_8 \times \mathbb{Z}_2^{h-3}$).*

Remark 4.3 The binary operation on vectors considered above can be defined by means of h Boolean functions in variables $x_1, \ldots, x_h, y_1, \ldots, y_h$, for each coordinate of $x \oplus y$ we have one function. Thus the problem of characterization of concentric groups is related to the problem which sets of h Boolean functions of $2h$ variables define a binary group operation on vectors over $\mathbb{Z}_2$. For instance, for $h = 3$ the respective binary operation used in the above proof is $(x_1, x_2, x_3) \oplus (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2 + x_3 y_1, x_3 + y_3)$.

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