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ON HOLOMORPHIC EMBEDDING OF PLANAR DOMAINS INTO $\mathbb{C}^2$

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1. Introduction

The following theorem was proved in [GS].

Theorem 1.1 *Every bounded, finitely connected domain in $\mathbb{C}$ without isolated points in the boundary can be properly holomorphically embedded into $\mathbb{C}^2$.*

At the moment this is the most general result about the yet unsolved problem whether every open Riemann surface admits a proper holomorphic embedding into $\mathbb{C}^2$. Before the paper [GS] the existence of such embeddings was known for discs, punctured discs and annuli. The proof in [GS] is technically complicated. The aim of the present paper is to present a simpler proof of Theorem 1.1 that uses a different approach.

We first recall the main points of the proof in [GS]. Since every finitely connected domain without isolated points in the boundary is conformally equivalent to a circular domain, i. e. to a domain bounded by a finite number of pairwise disjoint circles [Go] it is enough to prove the theorem for circular domains. Given a circular domain D_0, in [GS] one first constructs a function φ_{D_0}, holomorphic in a neighbourhood Ω of $\overline{D_0}$ that satisfies $|\varphi_{D_0}| \approx 1$ on bD_0. If one chooses φ_{D_0} in the right way then the surface $\{(\varphi_{D_0}(\zeta), \zeta)\colon \zeta \in \Omega\}$ intersects $\{(z,w)\colon |z| = 1\}$ transversely and $\{\zeta \in \Omega\colon |\varphi(\zeta)| < 1\}$ is a slight perturbation of D_0. Now one puts $T_0 = 1$, $f_0(\zeta) = (\varphi_{D_0}(\zeta), \zeta)$ and one uses a sequence S_n of shear automorphisms of $\mathbb{C}^2$ of the form

$$\begin{aligned} S_j(z,w) &= \Big(z, w + T_j\Big(\frac{z}{T_{j-1}}\Big)^{N_j}\Big) \quad \text{if } j \text{ is odd} \\ S_j(z,w) &= \Big(z + T_j\Big(\frac{w}{T_{j-1}}\Big)^{N_j}, w\Big) \quad \text{if } j \text{ is even} \end{aligned} \tag{1.1}$$

where $0 < T_j < \infty$ and $N_j \in \mathbb{N}$, j $\in \mathbb{N}$. Letting $f_n = S_n \circ S_{n-1} \circ \cdots \circ S_1 \circ f_0$ one proves that if one chooses T_j and N_j, $j \in \mathbb{N}$, in the right way then the sequence f_n converges, uniformly on compacta on a domain $\Phi(D_0)$ that is a small perturbation of D_0, to a proper holomorphic embedding of $\Phi(D_0)$ to $\mathbb{C}^2$. One then shows that there is a neighbbourhood U of D_0 in the space of circular domains such that for each $D \in U$ one can choose φ_D in such a way that Φ is a continuous map from U to a neighbourhood V of D_0 in the space of Jordan domains, i. e. domains bounded by a finite number of Jordan curves, and Φ can be made arbitrarily close to identity on U provided that T_n and N_n are chosen in the right way. Then one proves that there is a continuous map Ψ mapping V to a neighbourhood W of D_0 in the space of circular domains which is the identity on the subset of circular domains in V such that $\Psi(\Sigma)$ is conformally equivalent to Σ for each $\Sigma \in V$. Since U and W can be viewed as balls in some R^m, $\Psi \circ \Phi$ is a map from the ball U to the ball W which can be made arbitrarily close to the identity on U. A degree argument or the Brouwer fixed point theorem now implies that $D_0 \in (\Psi \circ \Phi)(U)$ provided that $\Psi \circ \Phi$ is sufficiently close to the identity on U which implies that there is a $\Sigma \in U$ such that D_0 is biholomorphically equivalent to a domain $\Phi(\Sigma)$, which admits a proper holomorphic embedding into $\mathbb{C}^2$.

Alternating sequences of shears (1.1) used in [GS] turned out to be quite useful. For instance, T_j and N_j above can be chosen in such a way that if $F_n = S_n \circ \cdots \circ S_1,\ n \in \mathbb{N}$, then the domain of convergence of F_n is a domain with $\mathcal{C}^\infty$-boundary which $F = \lim F_n$ maps biholomorphically onto $\mathbb{C}^2$ [S]. Let Δ be the open unit disc. In [Gl3] it was proved that given a large ball B in $\mathbb{C}^2$ there is a biholomorphic image Ω of $\mathbb{C}^2$ such that $B \cap \Omega$ is arbitrarily small $\mathcal{C}^1$-perturbation of $B \cap (\Delta \times \mathbb{C})$. For an example of such Ω see [Gl4].

We now describe the idea of our proof of Theorem 1.1 whose first part is based on these uses of alternating sequences of shears.

Let D be a circular domain. With no loss of generality assume that $\overline{D} \subset \Delta$. By a theorem of L.Ahlfors [A1, A2] there is a continuous function $g_0: \overline{D} \to \overline{\Delta}$, holomorphic on D and such that $|g_0(\zeta)| = 1$ $(\zeta \in bD)$ with m zeros if D is bounded by m circles. We call such g_0 an *Ahlfors function.* The map $\zeta \to F_0(\zeta) = (g_0(\zeta), \zeta)$ is smooth on $\overline{D}$, holomorphic on D and embeds D holomorphically into Δ^2 in such a way that $F_0(bD) \subset S_0 = b\Delta \times \Delta$. By using alternating sequences of shears we then prove that there are arbitrarily small $\mathcal{C}^3$-perturbations S of S_0 such that if Ω is the bounded domain bounded by S and by $|w| = 1$ then there is a biholomorphic map $\Phi: \Omega \to F(\Omega) \subset \mathbb{C}^2$ such that if $z_n \in \Omega$, $\mathrm{dist}\{z_n, S\} \to 0$, then $|\Phi(z_n)| \to +\infty$. Using some known facts about the Riemann-Hilbert problem for multiply connected domains and the implicit mapping theorem in Banach spaces we show that whenever S is a sufficiently small $\mathcal{C}^3$-perturbation of S_0 there is a smooth function $h: \overline{\Delta} \to \mathbb{C}$, holomorphic on D, with small $\mathcal{C}^1$ norm, such that if $F(\zeta) = (g_0(\zeta) + h(\zeta), \zeta)$ then $F(\zeta) \in S$ $(\zeta \in bD)$. We then use the fact that h is small in $\mathcal{C}^1$ sense to show that $F(\zeta) \in \Omega$ $(\zeta \in D)$. Thus, taking for S a sufficiently small perturbation of S_0 as above the map $\Phi \circ F: D \to \mathbb{C}^2$ is a proper holomorphic embedding.

2. Biholomorphic maps of perturbed bidiscs

Given $k \in \mathbb{N}$ and $R > 0$ denote by $\mathcal{C}^k(b\Delta \times R\overline{\Delta})$ the space of all real functions on $b\Delta \times R\overline{\Delta}$ which are continuous together with all their partial derivatives of up to order k. Equipped with the standard norm, $\mathcal{C}^k(b\Delta \times R\overline{\Delta})$ becomes a Banach space.

With each positive function $\phi \in \mathcal{C}^k(b\Delta \times R\overline{\Delta})$ we associate the $\mathcal{C}^k$ surface $S = \{(\phi(\zeta, w)\zeta,\ w): \zeta \in b\Delta, w \in R\overline{\Delta}\}$. We denote the family of all such surfaces by $\mathcal{S}_k(R)$. We shall say that a sequence $S_n \in \mathcal{S}_k(R)$ converges to $S \in \mathcal{S}_k(R)$ provided that the sequence of associated functions ϕ_n converges in $\mathcal{C}^k(b\Delta \times R\overline{\Delta})$ to the function associated with S. Given $S \in \mathcal{S}_k(R)$ we denote by $D(S)$ the bounded domain bounded by S and by $\{(z, w):\ |w| = 1\}$. Let $S = \{(\phi(\zeta, w)\zeta,\ w): \zeta \in b\Delta, w \in R\overline{\Delta}\}$. Given $\varepsilon > 0$ we shall denote the set $\{(t\zeta, w):\ \phi(\zeta, w) - \varepsilon < t < \phi(\zeta, w) + \varepsilon, \zeta \in b\Delta, w \in R\overline{\Delta}\}$ by $B(S, \varepsilon)$ and call it the *ε-belt along S* and call the set $\{((\phi(\zeta, w) - \varepsilon)\zeta, w):\ \zeta \in b\Delta,\ w \in R\overline{\Delta}\}$ *the inner boundary of $B(S, \varepsilon)$*. It is clear that ε-belt along S is a set of the form $\{(z, w) \in U:\ |w| \leq R\}$ where U is an open neighbourhood of the (compact) set S.

If $P, Q \in \mathcal{S}_k(R)$ then under $\mathrm{dist}_k(P, Q)$ we shall understand the distance between the associated functions in $\mathcal{C}^k(b\Delta \times R\overline{\Delta})$.

Lemma 2.1 *Let $S_0 = b\Delta \times \overline{\Delta}$. There are arbitrarily small perturbations S of S_0 in $\mathcal{S}_3(1)$ such that if Ω is the bounded domain bounded by S and by $|w| = 1$ then there is a biholomorphic map $F: \Omega \to F(\Omega) \subset \mathbb{C}^2$ with the property that $|F(z_n, w_n)| \to +\infty$ whenever $(z_n, w_n) \in \Omega$ $(n \in \mathbb{N})$ and dist$\{(z_n, w_n), S\} \to 0$.*

Remark More general results, for $\mathcal{C}^1$-perturbations rather than for $\mathcal{C}^3$-perturbations were proved in [Gl3] and [Gl4]. Since our aim in the present paper is to present a simple and complete proof of Theorem 1.1. we include the proof of Lemma 2.1.

To prove Lemma 2.1 we need

Lemma 2.2 *Let $0 < R < \infty$, $0 < P < \infty$, and let $\varepsilon > 0$. There are arbitrarily large T, $0 < T < \infty$, and arbitrarily large $m \in \mathbb{N}$, such that the following holds:*

(i) the set

$$\Big\{(z,w) \in \mathbb{C}^2\colon\ |z| < P+\varepsilon,\ \Big|w + T\Big(\frac{z}{P}\Big)^m\Big| < T\Big\},$$

a bidisc in the coordinates (Z,W) where $Z = z$, $W = w + T(z/P)^m$, intersects the set $\{(z,w) \in \mathbb{C}^2\colon\ |w| \le R\}$ in a slight $\mathcal{C}^3$-perturbation of $(Pb\Delta) \times (R\overline{\Delta})$, that is,

$$\Big\{(z,w) \in \mathbb{C}^2\colon\ |w| \le R, |z| < P+\varepsilon,\ \Big|w + T\Big(\frac{z}{P}\Big)^m\Big| < T\Big\} =$$
$$= \{(t\zeta, w)\colon\ 0 \le t < \phi(\zeta,w), \zeta \in b\Delta, |w| \le R\},$$

and

$$\{(z,w) \in \mathbb{C}^2\colon\ |w| \le R, |z| < P+\varepsilon,\ |w+T(z/P)^m| = T\} = \{(\phi(\zeta,w)\zeta, w)\colon\ \zeta \in b\Delta, |w| \le R\}$$

where $\|\phi - P\|_{\mathcal{C}^3(b\Delta\times R\overline{\Delta})} < \varepsilon$

(ii) $T(1/2)^m < \varepsilon$

(iii) $|w + T(z/P)^m| < T/2$ *if* $|z| \le P - \varepsilon$, $|w| \le R$

Proof. With no loss of generality assume that $P = 1$. Put $\alpha = 1/T$. If $|w| \le R$ then $1/2 \le |z^m + \alpha w|$ implies that $(1/2 - \alpha R)^{1/m} \le |z|$ so (iii) holds as soon as m is sufficiently large and α is sufficiently small.

Fix $m \in \mathbb{N}$. Assume that $\alpha R = \tau < 1/2$. This implies that for each w, $|w| < R$, and each $\varphi \in \mathbb{R}$ there is a unique $r = r(\varphi,w) > 0$ such that $z = re^{i\varphi}$ satisfies $|z^m + \alpha w| = 1$. Indeed, $|r^m e^{im\varphi} + \alpha w|^2 = 1$ gives $[r^m]^2 + Ar^m + B = 0$ where $A = e^{im\varphi}\alpha\overline{w} + e^{-im\varphi}\alpha w$, $B = \alpha^2 w\overline{w} - 1$. If $|w| \le R$ then

$$|A| < 2\tau, \quad A^2 - 4B > 4(1-2\tau^2) > 1 \tag{2.1}$$

so $(A^2-4B)^{1/2} - |A| \ge 2(1-2\tau^2)^{1/2} - 2\tau > 0$. Since we must have $2r^m = -A \pm (A^2-4B)^{1/2}$ it follows that $r = \big[\big(-A + (A^2-4B)^{1/2}\big)/2\big]^{1/m}$ is the unique $r > 0$ such that $z = re^{i\varphi}$ satisfies $|z^m + \alpha w| = 1$.

Write $D_1 = \frac{\partial}{\partial\varphi}$, $D_2 = \frac{\partial}{\partial w}$, $D_3 = \frac{\partial}{\partial\overline{w}}$. Computing the derivatives we see that there is a constant $c_1 < \infty$ depending only on R such that if $|w| \le R$ then for all $i,j,k \in \{1,2,3\}$ and all φ

$$|(D_iA)(\varphi,w)| < c_1\alpha.m, \quad |(D_iD_jA)(\varphi,w)| < c_1\alpha.m^2, \quad |(D_iD_jD_kA)(\varphi,w)| < c_1\alpha.m^3 \tag{2.2}$$

and

$$|(D_iB)(\varphi,w)| < c_1\alpha, \quad |(D_iD_jB)(\varphi,w)| < c_1\alpha, \quad (D_iD_jD_kB)(\varphi,w)| = 0. \tag{2.3}$$

Note that $|z^m + \alpha w| = 1$ implies that if $|w| \le R$ then $1-\tau < r^m < 1+\tau$ so

$$1-\tau < r < 1+\tau. \tag{2.4}$$

Using (2.1), (2.2), (2.3) and (2.4) we get a constant $c < \infty$ depending only on R such that for every φ and every w, $|w| \le R$,

$$|(D_ir)(\varphi,w)| < c.\alpha, \quad |(D_iD_jr)(\varphi,w)| < c.\alpha.m, \quad |(D_iD_jD_kr)(\varphi,w)| < c.\alpha.m^2 \tag{2.5}$$

The inequalities (2.2), (2.3) and (2.4) show that if $\phi(e^{i\varphi},w) = r(\varphi,w)$ then the norm $\|\phi - 1\|_{\mathcal{C}^3(b\Delta\times R\overline{\Delta})}$ is arbitrarily small provided that $m^2\alpha < \eta$ where η is sufficiently small. To get (ii) we must have $(1/\alpha)(1/2)^m < \varepsilon$. Thus, α must satisfy

$$\frac{1}{\varepsilon}\Big(\frac{1}{2}\Big)^m < \alpha < \frac{\eta}{m^2}. \tag{2.8}$$

It is clear that there are arbitrarily large m and arbitrarily small α such that (2.8) holds. This completes the proof.

Proof of Lemma 2.1. Let $S_{-1} = S_0 = (b\Delta)\times\overline{\Delta}$, let $F_0 = (\Phi_0,\Psi_0) = \mathrm{id}$, let $T_0 = 1$, $\delta_{-1} = 3/4$. We have $|\Phi_0| = T_0$ on S_0 and

$$|\Phi_0| < \frac{T_0}{2} \quad \text{on} \quad D(S_0)\setminus B(S_{-1},\delta_{-1}).$$

Define

$$D_0 = \big\{(z,w)\in D(S_0)\colon\ |\Phi_0(z,w)| < \frac{T_0}{2}\big\}.$$

Since $S_0 \subset B(S_{-1},\delta_{-1}/2)$ and since $|\Phi_0| = T_0$ on S_0 one can choose $\delta_0 > 0$ such that $B(S_0,\delta_0) \subset B(S_{-1},\delta_{-1})$ and such that

$$|\Phi_0| > \frac{3T_0}{4} \quad \text{on} \quad B(S_0,\delta_0).$$

Consider the map

$$F_1 = \Big(\Phi_0, \Psi_0 + T_1\Big(\frac{\Phi_0}{T_0}\Big)^{N_1}\Big) = (\Phi_1,\Psi_1).$$

By Lemma 2.2 one can choose $T_1 > 1$, $N_1 \in \mathbb{N}$ in such a way that there is a surface $S_1 \in \mathcal{S}_3(1)$ such that

(a) $|\Psi_1| = T_1$ on S_1, $|\Psi_1| < T_1$ on $D(S_1)$
(b) $S_1 \subset B(S_0,\delta_0/2)$
(c) $\mathrm{dist}_3(S_1,S_0) < (1/2)^1$
(d) $|\Psi_1| < T_1/2$ on $D(S_1)\setminus B(S_0,\delta_0)$

(e) $T_1(1/2)^{N_1} < (1/2)^1$
which implies that $|F_1 - F_0| = |T_1(\Phi_0/T_0)^{N_1}| < (1/2)^1$ whenever $|\Phi_0| < T_0/2$. In particular
(f) $|F_1 - F_0| < (1/2)^1$ on D_0.
Define $D_1 = \{(z,w) \in D(S_1)\colon\ |\Psi_1(z,w)| < T_1/2\}$.

Since $S_1 \subset B(S_0, \delta_0/2)$ and since $|\Psi_1| = T_1$ on S_1 one can choose $\delta_1,\ 0 < \delta_1 < \delta_0/2$, such that $B(S_1,\delta_1) \subset B(S_0,\delta_0)$ and such that

$$|\Psi_1| > \frac{3T_1}{4} \quad \text{on} \quad B(S_1,\delta_1).$$

Consider the map

$$F_2 = \Big(\Phi_1 + T_2\Big(\frac{\Psi_1}{T_1}\Big)^{N_2}, \Psi_1\Big) = (\Phi_2, \Psi_2).$$

Using Lemma 2.2 again one can choose $T_2 > 2,\ N_2 \in \mathbb{N}$ in such a way that there is a surface $S_2 \in \mathcal{S}_3(1)$ such that
(a) $|\Phi_2| = T_2$ on S_2, $|\Phi_2| < T_2$ on $D(S_2)$
(b) $S_2 \subset B(S_1, \delta_1/2)$
(c) $\text{dist}_3(S_2, S_1) < (1/2)^2$
(d) $|\Phi_2| < T_2/2$ on $D(S_2) \setminus B(S_1,\delta_1)$
(e) $T_2(1/2)^{N_2} < (1/2)^2$
which implies that $|F_2 - F_1| = |T_2(\Psi_1/T_1)^{N_2}| < (1/2)^2$ whenever $|\Psi_1| < T_1/2$. In particular
(f) $|F_2 - F_1| < (1/2)^2$ on D_1.
Define $D_2 = \{(z,w) \in D(S_2)\colon\ |\Phi_2(z,w)| < T_2/2\}$.

We continue in the same way. At each step we use Lemma 2.2 in a suitable bidisc in the coordinates (Φ_n, Ψ_n). By induction, we get

- sequences T_n and N_n such that $T_n > n$ for each n
- sequences $F_n = (\Phi_n, \Psi_n)$ of maps such that

$$\Phi_{n+1} = \Phi_n + T_{n+1}\Big(\frac{\Psi_n}{T_n}\Big)^{N_{n+1}}, \quad \Psi_{n+1} = \Psi_n \qquad \text{if } n \text{ is odd}$$
$$\Phi_{n+1} = \Phi_n, \quad \Psi_{n+1} = \Psi_n + T_{n+1}\Big(\frac{\Phi_n}{T_n}\Big)^{N_{n+1}} \qquad \text{if } n \text{ is even}$$

- a sequence of surfaces $S_n \in \mathcal{S}_3(1)$ such that

$$\begin{aligned} &|\Phi_n| = T_n \text{ on } S_n,\ |\Phi_n| < T_n \text{ on } D(S_n) \text{ if } n \text{ is even} \\ &|\Psi_n| = T_n \text{ on } S_n,\ |\Psi_n| < T_n \text{ on } D(S_n) \text{ if } n \text{ is odd} \end{aligned} \tag{2.9}$$

$$\text{dist}_3(S_{n+1}, S_n) < \Big(\frac{1}{2}\Big)^{n+1} \tag{2.10}$$

- a sequence δ_n, $0 < \delta_{n+1} < \delta_n/2$, such that $B(S_{n+1},\delta_{n+1}) \subset B(S_n,\delta_n)$, and such that

$$\begin{aligned} &|\Psi_n| > \frac{3T_n}{4} \text{ on } B(S_n,\delta_n) \text{ if } n \text{ is odd} \\ &|\Phi_n| > \frac{3T_n}{4} \text{ on } B(S_n,\delta_n) \text{ if } n \text{ is even} \end{aligned} \tag{2.11}$$

$$
\begin{aligned}
|\Psi_n| &< \frac{T_n}{2} \text{ on } D(S_n)\setminus B(S_{n-1},\delta_{n-1}) \text{ if } n \text{ is odd}\\
|\Phi_n| &< \frac{T_n}{2} \text{ on } D(S_n)\setminus B(S_{n-1},\delta_{n-1}) \text{ if } n \text{ is even.}
\end{aligned} \tag{2.12}
$$

Moreover, if

$$
D_n = \{(z,w)\in D(S_n)\colon\ |\Phi_n(z,w)| < \frac{T_n}{2}\} \ \text{ if } n \text{ is even}
$$

$$
D_n = \{(z,w)\in D(S_n)\colon\ |\Psi_n(z,w)| < \frac{T_n}{2}\} \ \text{ if } n \text{ is odd}
$$

then

$$
|F_{n+1}-F_n| < \left(\frac{1}{2}\right)^n \text{ on } D_n. \tag{2.13}
$$

By (2.10) S_n converges in $\mathcal{S}_3(1)$. Denote the limit by S and let $\Omega = D(S)$. Suppose that $n\in\mathbb{N}$ is even. Since $S_n \subset B(S_n,\delta_n)\subset B(S_{n-1},\delta_{n-1})$ and since $|\Phi_n| < T_n/2$ on $D(S_n)\setminus B(S_{n-1},\delta_{n-1})$ it follows that D_n consists of $D(S_n)\setminus B(S_{n-1},\delta_{n-1})$ and of all those points between the inner boundaries of $B(S_{n-1},\delta_{n-1})$ and $B(S_n,\delta_n)$ at which $|\Phi_n| < T_n/2$. By (2.11), $|\Phi_n|\geq 3T_n/2$ on $B(S_n,\delta_n)$ so D_n is contained in $D(S_n)\setminus\overline{B(S_n,\delta_n)}$. In fact, D_n is at positive distance from $B(S_n,\delta_n)$. On the other hand, $D(S_n)\setminus\overline{B(S_n,\delta_n)}\subset D_{n+1}$ since $B(S_{n+1},\delta_{n+1})\subset B(S_n,\delta_n)$ and since $|\Psi_{n+1}| < T_{n+1}/2$ on $D(S_{n+1})\setminus B(S_n,\delta_n)$. This implies that $D_n\subset D_{n+1}$. The same holds for odd n. Since $\delta_n\to 0$ it follows also that $\cup_{n=0}^{\infty}D_n = \Omega$ and that each D_n is at a positive distance from S.

Since each compact subset of Ω is contained in some D_n, (2.13) implies that F_n converges, uniformly on compacta in Ω, to a limit F. If J denotes Jacobian we have $JF_n\equiv 1$ for all n so $JF\equiv 1$. A well known argument [BM] gives that F is one to one on Ω.

Suppose that n is even and let $(z,w)\in D_{n+1}\setminus D_n$. Since $(z,w)\notin D_n$ we have $|\Phi_n(z,w)|\geq T_n/2$. Since $\Phi_{n+1}=\Phi_n$ it follows that $|F_{n+1}(z,w)|\geq T_n/2$. Since $(z,w)\in D_{n+1}$ it follows that $(z,w)\in D_j$ for each $j\geq n+1$. By (2.13) it follows that

$$
\begin{aligned}
|F(z,w)| &\geq |F_{n+1}(z,w)| - |F_{n+2}(z,w)-F_{n+1}(z,w)| - |F_{n+3}(z,w)-F_{n+2}(z,w)| - \cdots\\
&\geq \frac{T_n}{2} - \left(\frac{1}{2}\right)^{n+1} - \left(\frac{1}{2}\right)^{n+2} - \cdots\\
&\geq \frac{n}{2} - 1.
\end{aligned}
$$

Thus $|F(z,w)|\geq n/2-1$ $((z,w)\in D_{n+1}\setminus D_n)$ if n is even. In the same way we get this for odd n. Since S is at positive distance from D_n for each n it follows that $|F(z_n,w_n)|\to+\infty$ whenever $(z_n,w_n)\in\Omega$ $(n\in\mathbb{N})$ and $\mathrm{dist}\{(z_n,w_n),S\}\to 0$. This completes the proof.

3. Holomorphic perturbations on finitely connected planar domains

Let D, $\overline{D}\subset\Delta$ be a circular domain bounded by m circles, and let $\Sigma_0 = b\Delta\times bD$. Suppose that $g_0\colon\overline{D}\to\overline{\Delta}$ is an Ahlfors' function. Given a small smooth perturbation $\Sigma\subset\mathbb{C}\times bD$ of Σ_0 denote by $\Sigma(\zeta)=\{z\in\mathbb{C}\colon(z,\zeta)\in\Sigma\}$ the fiber above $\zeta\in bD$. By transversality, each $\Sigma(\zeta)$ is a small smooth perturbation of $b\Delta$.

Lemma 3.1 *If $\Sigma \subset \mathbb{C} \times bD$ is a sufficiently small $\mathcal{C}^3$-perturbation of Σ_0 then there is a function $g: \overline{D} \to \mathbb{C}$, holomorphic on D such that $g(\zeta) \in \Sigma(\zeta)$ $(\zeta \in bD)$. The function g can be chosen arbitrarily close to g_0 in $\mathcal{C}^1$-sense provided that Σ is sufficiently close to Σ_0 in $\mathcal{C}^3$-sense.*

Granted the lemma we complete the proof of Theorem 1.1 as follows. By Lemma 2.1 and by the last paragraph of Section 1 it remains to prove that whenever S is a sufficiently small $\mathcal{C}^3$-perturbation of $S_0 = b\Delta \times \Delta$ there is a smooth function $g: \overline{D} \to \mathbb{C}$, holomorphic on D, such that $(g(\zeta), \zeta) \in S$ $(\zeta \in b\Delta)$ and $(g(\zeta), \zeta) \in \Omega$ $(\zeta \in D)$ where Ω is the domain bounded by S and by $\{(z, w): |w| = 1\}$. We shall write $F_0(\zeta) = (g_0(\zeta), \zeta)$, $F(\zeta) = (g(\zeta), \zeta)$. Since $\Sigma = \{(z, w) \in S: w \in bD\}$ is arbitrarily small $\mathcal{C}^3$-perturbation of Σ_0 provided that S is a sufficiently small $\mathcal{C}^3$-perturbation of S_0, Lemma 3.1 provides a smooth function $g: \overline{D} \to \mathbb{C}$, holomorphic on D, arbitrarily close to g_0 in $\mathcal{C}^1$ sense, such that $F(\zeta) \in S$ $(\zeta \in bD)$, provided that S is a sufficiently small $\mathcal{C}^3$-perturbation of S_0. To prove that $F(\zeta) \in \Omega$ $(\zeta \in D)$ if S is a sufficiently small $\mathcal{C}^3$-perturbation of S_0 we use a transversality argument. Given $a \in \mathbb{C}^2$, $b \in \mathbb{C}^2$, $|b| = 1$, and δ, $0 < \delta < \pi/2$, we call the set $P = \{a + tb': 0 < t < r, \ \cos\delta < \mathrm{Re} < b|b' > \leq 1, \ |b'| = 1\}$ *the δ-cone of length r in the direction of b and with vertex at a.* P is the union of all open segments J of length r with the initial point a such that the angle between J and b is smaller than δ. Suppose that $\delta > 0$, $r > 0$ and let K be a compact subset of S_0. It is easy to see that if S is a sufficiently small $\mathcal{C}^1$-perturbation of S_0 and $\varepsilon > 0$ is small enough then the following holds: Let P be a 2δ-cone of length $2r$ that has vertex at a point $a \in K$ and is contained in $\Delta \times \Delta$. Then the δ-cone of length r that has vertex at a point $w \in S$, $|w - a| < \varepsilon$, and has the same direction as P, is contained in Ω.

Since $|g_0(\zeta)| = 1$ $(\zeta \in bD)$ the function g_0 extends holomorphically to a neighbourhood $U \subset \mathbb{C}$ of $\overline{D}$ and g_0' has no zero on bD. This implies that for each $\zeta \in bD$, the derivative $d_0(\zeta)$ of $F_0(\zeta)$ at ζ in the outward direction perpendicular to bD is transverse to S_0 at $F_0(\zeta)$. By compactness it follows that there are $\delta > 0$ and $r > 0$ such that for each $\zeta \in bD$, the 4δ-cone of length $4r$ in the direction of $-d_0(\zeta)$ and vertex at $F_0(\zeta)$, is contained in $\Delta \times \Delta$. The preceding discussion now shows that provided that S is a sufficiently small $\mathcal{C}^3$-perturbation of S_0, $F(\zeta)$ is an arbitrarily small $\mathcal{C}^1$-perturbation of $F_0(\zeta)$, and consequently for each $\zeta \in bD$, the 2δ-cone of length $2r$ in direction $-d_0(\zeta)$ and with vertex $F(\zeta)$ is contained in Ω. There is a $\tau > 0$ such that if $J(\zeta)$, $\zeta \in bD$, is the open segment in $\mathbb{C}$ of length τ with one endpoint at ζ, and pointing into D, then $J(\zeta) \subset D$ and $F_0(J(\zeta))$ is contained in the δ-cone of length r in direction $-d_0(\zeta)$ with vertex $F_0(\zeta)$. Provided that S is a sufficiently small $\mathcal{C}^1$-perturbation of S_0 and F is a sufficiently small $\mathcal{C}^1$-perturbation of F_0, $F(J(\zeta))$ is contained in the 2δ-cone of length $2r$ in direction of $-d_0(\zeta)$ with vertex $F(\zeta)$ for each $\zeta \in bD$. In particular, $F(J(\zeta)) \subset \Omega$ $(\zeta \in bD)$. Since $Q = D \setminus \cup_{\zeta \in b\Delta} J(\zeta)$ is a compact set it follows that $F_0(Q)$ is a compact subset of $\Delta \times \Delta$. So, in addition to the fact that $F(D \setminus Q) \subset \Omega$ we have $F(Q) \subset \Omega$ provided that F is a sufficently small perturbation of F_0 and S is a sufficiently small perturbation of S_0. This completes the proof of Theorem 1.1.

To prove Lemma 3.1 we shall use some known facts about the (linear) Riemann-Hilbert problem on multiply connected domains. Let $0 < \alpha < 1$ and let $\mathcal{A}^{1,\alpha}(D)$ be the Banach algebra of $\mathcal{C}^{1,\alpha}$ functions on bD which extend holomorphically to D equipped with the standard $\mathcal{C}^{1,\alpha}$ norm. Suppose that $a: bD \to \mathbb{C} \setminus \{0\}$ is a function of class $\mathcal{C}^{1,\alpha}$. Given a real function c on bD of class $\mathcal{C}^{1,\alpha}$ the *Riemann - Hilbert problem* is to find a holomorphic function $k \in \mathcal{A}^{1,\alpha}(D)$ such that

$$\mathrm{Re}\left(\overline{a(\zeta)}k(\zeta)\right) = c(\zeta) \quad (\zeta \in bD).$$

The existence of solutions depends on the *index* of the function a. We equip D with the usual orientation in the complex plane $\mathbb{C}$. The orientation of D induces the orientation of the boundary bD. For each (oriented) boundary component of D we can compute the winding number of a about 0. The index κ of a is then defined as the sum of all winding numbers of a over all m boundary components of D (the corresponding *Maslov index* is 2κ).

The following is a part of a theorem from [Ga, p. 347] (see also [V, page 257]):

Theorem 3.1 *If the index κ of the function a satisfies $\kappa > m-2$, then the Riemann-Hilbert boundary value problem*

$$\mathrm{Re}(\overline{a(\zeta)}\,k(\zeta)) = c(\zeta)$$

is always solvable and the corresponding homogeneous problem ($c=0$) has $2\kappa-(m-2)$ linearly independent solutions.

In [Ga] and [V] the theorem is actually proved for the $\mathcal{C}^\alpha$ case, that is, the functions a and c are assumed to be of class $\mathcal{C}^\alpha$ and solutions belong to the space $\mathcal{A}^\alpha(D)$. However, using an argument similar to the argument in the proof of Theorem 4.1 in [V,pp. 231-232] (containing the fact that the Hilbert transform maps the space $\mathcal{C}^{1,\alpha}(b\Delta)$ into itself and the Cauchy formula), one can also prove Theorem 3.1 in the $\mathcal{C}^{1,\alpha}$ case.

Proof of Lemma 3.1. Start with the Ahlfors function g_0 and the fact that $g_0(\zeta) \in \Sigma_0(\zeta)$ $(\zeta \in bD)$. Given a sufficiently small $\mathcal{C}^3$-perturbation $\Sigma \subset \mathbb{C} \times bD$ of Σ_0 we are looking for $g \in \mathcal{A}^{1,\alpha}$, close to g_0, such that $g(\zeta) \in \Sigma(\zeta)$ $(\zeta \in bD)$. Our problem is local so for each $\zeta \in bD$ one needs to look only at a portion of $\Sigma_0(\zeta)$ near $g_0(\zeta)$. Choose a small disc Δ_0 and choose a real function $\rho_0 \in \mathcal{C}^3(\Delta_0 \times bD)$ such that for each $\zeta \in bD$,

(a) $\Sigma_0 \cap \{g_0(\zeta)+\Delta_0\} = \{g_0(\zeta)+z: z \in \Delta_0, \rho_0(z,\zeta)=0\}$

(b) $d_z\rho_0(z,\zeta) \neq 0$ on Δ_0.

Obviously, $\rho_0(0,\zeta)=0$, i.e. $g_0(\zeta) \in \Sigma_0(\zeta)$. In our case one can take $\rho_0(z,\zeta) = |z+g_0(\zeta)|^2-1$. Let $G \subset \mathcal{A}^{1,\alpha}(D)$ be the open set of all functions h such that $h(bD) \subset \Delta_0$ and consider the map

$$\Phi\colon\ G \times \mathcal{C}^3(\Delta_0 \times bD) \to \mathcal{C}^{1,\alpha}(bD)$$

defined as

$$\Phi(h,\rho)(\zeta) = \rho(h(\zeta),\zeta) \quad (\zeta \in bD).$$

If $\Sigma(\zeta) \cap [g_0(\zeta)+\Delta_0] = \{g_0(\zeta)+z\colon\ \rho(z,\zeta)=0, z \in \Delta_0\}$ then $g = g_0+h$, $h \in G$, solves our problem if and only if $\Phi(h,\rho)=0$. We know that the pair $(0,\rho_0)$ solves this equation. For each $\rho \in \mathcal{C}^3(\Delta_0 \times bD)$, sufficently close to ρ_0, we must find $h \in G$ such that $\Phi(h,\rho)=0$ and show that one can choose h arbitrarily small provided that ρ is sufficiently close to ρ_0. We shall apply the implicit mapping theorem.

First one needs to check that Φ is of class $\mathcal{C}^1$. Since Φ is linear in the ρ variable, it is $\mathcal{C}^\infty$ for each fixed $h \in G$:

$$(D_\rho \Phi(h, \rho)\tau)(\zeta) = \tau(h(\zeta), \zeta).$$

On the other hand Lemma 5.1 in [HT] implies that for each fixed $\rho \in \mathcal{C}^3(\Delta_0 \times bD)$ the mapping $\Phi(\cdot, \rho) : G \to \mathcal{C}^{1,\alpha}(bD)$ is of class $\mathcal{C}^1$:

$$(D_h \Phi(h, \rho)k)(\zeta) = 2\mathrm{Re}(\partial_z \rho(h(\zeta), \zeta)k(\zeta)).$$

To prove that Φ is actually $\mathcal{C}^1$ in both variables one has to show that $\Phi(h, \rho)$ and partial derivatives $D_\rho \Phi(h, \rho)$ and $D_h \Phi(h, \rho)$ are continuous in the variables h and ρ, [L]. For the interested reader this is proved in Lemma 4.1 and Lemma 4.2 in the appendix.

We denote the nowhere zero $\mathcal{C}^{1,\alpha}$ function $2\,\partial_{\overline{z}}\rho(h(\zeta), \zeta)$ on bD by $a_{h,\rho}(\zeta)$. An easy computation shows that the partial derivative of the mapping Φ with respect to h is the bounded real-linear map $A_{h,\rho} : \mathcal{A}^{1,\alpha}(D) \to \mathcal{C}^{1,\alpha}(bD)$ of the form

$$(A_{h,\rho}k)(\zeta) = \mathrm{Re}(\overline{a_{h,\rho}(\zeta)}\, k(\zeta)) \quad (\zeta \in bD.$$

In our case $\rho_0(z, \zeta) = |z + g_0(\zeta)|^2 - 1 = (z + g_0(\zeta))\overline{(z + g_0(\zeta))} - 1$ so $a_{0,\rho_0}(\zeta) = 2(z + g_0(\zeta)) \,|_{z=0}= 2g_0(\zeta)$. Since g_0 has m zeros in D the corresponding index κ of the function a_{0,ρ_0} equals $m > m - 2$, so Theorem 3.1 implies that the partial derivative of the map Φ with respect to h at the point $(0, \rho_0)$ is a surjective operator from $\mathcal{A}^{1,\alpha}(D)$ onto $\mathcal{C}^{1,\alpha}(bD)$ with $2\kappa - (m - 2)$ dimensional kernel. Hence the kernel of $D_h\Phi(0, \rho_0)$ is complemented in $\mathcal{A}^{1,\alpha}(D)$, i.e., $\mathcal{A}^{1,\alpha}(D) = \mathrm{Ker}(D_h\Phi(0, \rho_0)) \oplus \mathcal{A}$, where $\mathcal{A}$ is a closed subspace of $\mathcal{A}^{1,\alpha}(D)$. The partial derivative of the mapping Φ with respect to the variable in $\mathcal{A}$ at the point $(0, \rho_0)$ is now a linear isomorphism between the spaces $\mathcal{A}$ and $\mathcal{C}^{1,\alpha}(bD)$. In this case one can use the implicit mapping theorem [C] and conclude that there exist a neighbourhood U of $0 \in \mathcal{A}$, a neighbourhood V of ρ_0 in $\mathcal{C}^3(\Delta_0 \times bD)$, a neighbourhood W of $0 \in \mathrm{Ker}(D_h\Phi(0, \rho_0))(\approx \mathbb{R}^{2\kappa-(m-2)})$ and a $\mathcal{C}^1$ mapping $\varphi : W \times V \longrightarrow U$ such that the pair $(h, \rho) \in (W \oplus U) \times V$ solves the equation $\Phi(h, \rho) = 0$ if and only if $h = t \oplus \varphi(t, \rho)$ for some $t \in W$. Since $\varphi(0, \rho_0) = 0$ it follows that choosing t sufficiently small and ρ sufficiently close to ρ_0 the function h has all the required properties. This completes the proof of Lemma 3.1

Remark Results on holomorphic perturbations of finitely connected planar domains along maximally real surfaces are not new. For the linear case, as already quoted, they are proved in [Ga] and [V]. For general finite Riemann surfaces they are proved in [HLS] (see also [Gr]). However, all these results are stated and proved only for a *fixed* maximally real surface Σ_0 along which the initial mapping F_0 is perturbed. Because of this and since it was our intention to make this paper as much self-contained as possible, we included and proved some results on holomorphic perturbations needed for our purposes.

4. Appendix

For the interested reader we give the details of the proof that the map Φ in the proof of Lemma 3.1 is of class $\mathcal{C}^1$.

Lemma 4.1 *Bounded linear maps*

$$D_\rho\Phi(h,\rho):\mathcal{C}^3(\Delta_0\times bD)\longrightarrow \mathcal{C}^{1,\alpha}(bD)\qquad D_\rho\Phi(h,\rho)\tau=\tau(h(\cdot),\cdot),$$

$$D_h\Phi(h,\rho):\mathcal{A}^{1,\alpha}(D)\longrightarrow \mathcal{C}^{1,\alpha}(bD)\qquad D_h\Phi(h,\rho)k=2\mathrm{Re}((\partial_z\rho)(h(\cdot),\cdot)k(\cdot))$$

depend continuously on $(h,\rho)\in G\times\mathcal{C}^3(\Delta_0\times bD)$.

Proof. Let $\{(h_n,\rho_n)\}_{n=1}^\infty$ be a sequence of points in $G\times\mathcal{C}^3(\Delta_0\times bD)$ which converge to (h,ρ). We have to show that the sequences of bounded linear maps $\{D_\rho\Phi(h_n,\rho_n)\}_{n=1}^\infty$ and $\{D_h\Phi(h_n,\rho_n)\}_{n=1}^\infty$ converge to $D_\rho\Phi(h,\rho)$ and $D_h\Phi(h,\rho)$ respectively. Let us estimate the norm of the differences

$$||D_\rho\Phi(h_n,\rho_n)-D_\rho\Phi(h,\rho)||=\sup_{||\tau||_3=1}||D_\rho\Phi(h_n,\rho_n)\tau-D_\rho\Phi(h,\rho)\tau||_{1,\alpha}$$

$$=\sup_{||\tau||_3=1}||\tau(h_n(\cdot),\cdot)-\tau(h(\cdot),\cdot)||_{1,\alpha}.$$

and

$$||D_h\Phi(h_n,\rho_n)-D_h\Phi(h,\rho)||=\sup_{||k||_{1,\alpha}=1}||D_h\Phi(h_n,\rho_n)k-D_h\Phi(h,\rho)k||_{1,\alpha}$$

$$=\sup_{||k||_{1,\alpha}=1}2||\mathrm{Re}((\partial_z\rho_n(h_n(\cdot),\cdot)-\partial_z\rho(h(\cdot),\cdot))k(\cdot))||_{1,\alpha}\le$$

$$\le 2||\partial_z\rho_n(h_n(\cdot),\cdot)-\partial_z\rho(h(\cdot),\cdot)||_{1,\alpha}.$$

Since the unit ball in $\mathcal{C}^3(\Delta_0\times bD)$ is precompact in $\mathcal{C}^2(\Delta_0\times bD)$, the proof of Lemma 4.1 will be finished once we prove the following

Lemma 4.2 *The mapping*

$$\Psi:G\times\mathcal{C}^2(\Delta_0\times bD)\to\mathcal{C}^{1,\alpha}(b\Delta)\qquad \Psi(h,\rho)=\rho(h(\cdot),\cdot)$$

is continuous.

Proof. Let $\{(h_n,\rho_n)\}_{n=1}^\infty$ be a sequence of points in $G\times\mathcal{C}^2(\Delta_0\times bD)$ which converge to (h,ρ). Then

$$||\Psi(h_n,\rho_n)-\Psi(h,\rho)||_{1,\alpha}=||\rho_n(h_n(\cdot),\cdot)-\rho(h(\cdot),\cdot)||_{1,\alpha}\le$$

$$\le||\rho_n(h_n(\cdot),\cdot)-\rho_n(h(\cdot),\cdot)||_{1,\alpha}+||\rho_n(h(\cdot),\cdot)-\rho(h(\cdot),\cdot)||_{1,\alpha}.$$

We will show that each term converges to 0 as $n\to\infty$. The $\mathcal{C}^1$ convergence is obvious, since $\{\rho_n\}_{n=1}^\infty$ converges to ρ in the $\mathcal{C}^2$ sense and $\{h\}_{n=1}^\infty$ to h in the $\mathcal{C}^{1,\alpha}$ sense. To prove the $\mathcal{C}^{1,\alpha}$ convergence we have to show that the derivatives of the functions $\{\rho_n(h_n(\cdot),\cdot)\}_{n=1}^\infty$ converge in the $\mathcal{C}^\alpha$ sense to the derivative of the function $\rho(h(\cdot),\cdot)$.

For each boundary component $\xi=a+r\,e^{i\theta}$ $(\theta\in\mathbb{R})$ of D the derivative with respect to θ is

$$\frac{d}{d\theta}\rho(h(\xi),\xi)=2\mathrm{Re}(ir\,e^{i\theta}(\partial_z\rho(h(\xi),\xi)h'(\xi)+\partial_w\rho(h(\xi),\xi)))$$

and similarly for other functions. Hence

$$||D(\rho_n(h_n(\cdot),\cdot)) - D(\rho_n(h(\cdot),\cdot))||_\alpha \leq$$

$$\leq ||(\partial_z\rho_n(h_n(\cdot),\cdot)h_n' + \partial_w\rho_n(h_n(\cdot),\cdot)) - (\partial_z\rho_n(h(\cdot),\cdot)h' + \partial_w\rho_n(h(\cdot),\cdot))||_\alpha \leq \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,$$

where

$\mathrm{I}_1 = ||(\partial_z\rho_n(h_n(\cdot),\cdot) - \partial_z\rho_n(h(\cdot),\cdot))h_n'||_\alpha \leq ||\partial_z\rho_n(h_n(\cdot),\cdot) - \partial_z\rho_n(h(\cdot),\cdot)||_\alpha ||h_n'||_\alpha,$
$\mathrm{I}_2 = ||\partial_z\rho_n(h(\cdot),\cdot)(h_n' - h')||_\alpha \leq ||\partial_z\rho_n(h(\cdot),\cdot)||_\alpha ||h_n' - h'||_\alpha,$
$\mathrm{I}_3 = ||\partial_w\rho_n(h_n(\cdot),\cdot) - \partial_w\rho_n(h(\cdot),\cdot)||_\alpha.$

Also

$$||D(\rho_n(h(\cdot),\cdot)) - D(\rho(h(\cdot),\cdot))||_\alpha \leq$$

$$\leq ||(\partial_z\rho_n(h(\cdot),\cdot)h' + \partial_w\rho_n(h(\cdot),\cdot)) - (\partial_z\rho(h(\cdot),\cdot)h' + \partial_w\rho(h(\cdot),\cdot))||_\alpha \leq \mathrm{I}_4 + \mathrm{I}_5,$$

where

$\mathrm{I}_4 = ||(\partial_z\rho_n(h(\cdot),\cdot) - \partial_z\rho(h(\cdot),\cdot))h'||_\alpha \leq ||\partial_z\rho_n(h(\cdot),\cdot) - \partial_z\rho(h(\cdot),\cdot)||_\alpha ||h'||_\alpha,$
$\mathrm{I}_5 = ||\partial_w\rho_n(h(\cdot),\cdot) - \partial_w\rho(h(\cdot),\cdot)||_\alpha.$

Observe that factors $||h_n'||_\alpha$, $||\partial_z\rho_n(h(\cdot),\cdot)||_\alpha$ $(n \in \mathbb{N})$ are uniformly bounded. Sequences $\{\partial_z\rho_n(h_n(\cdot),\cdot)\}_{n=1}^\infty$ and $\{\partial_z\rho_n(h(\cdot),\cdot)\}_{n=1}^\infty$ converge in the $\mathcal{C}^1$ sense to $\partial_z\rho(h(\cdot),\cdot)$. Thus terms I_1 and I_4 tend to 0 as $n \to \infty$. Similarly we have that terms I_3 and I_5 converge to 0 as $n \to \infty$ since $\{\partial_w\rho_n(h_n(\cdot),\cdot)\}_{n=1}^\infty$ and $\{\partial_w\rho_n(h(\cdot),\cdot)\}_{n=1}^\infty$ converge in the $\mathcal{C}^1$ sense to $\partial_w\rho(h(\cdot),\cdot)$. Finally, terms I_2 converge to 0 since $\{h_n\}_{n=1}^\infty$ converges to h in the $\mathcal{C}^{1,\alpha}$ sense. This completes the proof of Lemma 4.2 and thus also the proof of Lemma 4.1.

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References

[A1] L. Ahlfors: Bounded analytic functions.
Duke Math. J. 14 (1947) 1-11

[A2] L. Ahlfors: Open Riemann surfaces and extremal problems in compact subregions.
Comment. Math. Helv. 24 (1950) 100-134

[BM] S. Bochner, T. W. Martin: *Several complex variables.* Princeton Univ. Press 1948

[C] H. Cartan: *Calcul différentiel.* Hermann, Paris 1967

[Č] M. Černe: Analytic discs attached to a generating CR-manifold.
Ark. Mat. 33 (1995) 217–248

[F] F. Forstnerič: Analytic discs with boundaries in a maximal real submanifolds of $\mathbb{C}^2$.

Ann. Inst. Fourier 37 (1987) 1–44

[Ga] F. D. Gakhov: *Boundary value problems.* Pergamon Press 1966

[Gl1] J. Globevnik: Perturbation by analytic discs along maximal real submanifolds of $\mathbb{C}^N$. Math. Z. 217 (1994) 287-316

[Gl2] J. Globevnik: Perturbing analytic discs attached to maximal real submanifolds of $\mathbb{C}^N$.
Indag. Math. N. S. 7 (1996) 37-46

[Gl3] J. Globevnik: A bounded domain in $\mathbb{C}^N$ which embeds holomorphically into $\mathbb{C}^{N+1}$.
Ark. Mat. 35 (1997) 313–325

[Gl4] J. Globevnik: On Fatou-Bieberbach domains.
Math. Z. 229 (1998) 91-106

[GS] J. Globevnik, B. Stensones: Holomorphic embeddings of planar domains into $\mathbb{C}^2$.
Math. Ann. 303 (1995) 579–597

[Go] G. M. Goluzin: *Geometrische Funktionentheorie.* VEB Deutscher Verlag der Wissenschaften, Berlin 1957

[Gr] M. Gromov: Pseudoholomorphic curves in symplectic manifolds.
Invent. Math. 81 (1985) 307–347

[HT] D. C. Hill, D. Taiani: Families of analytic discs in $\mathbb{C}^n$ with boundaries in a prescribed CR manifold.
Ann. Sc. Norm. Super. Pisa 5 (1978) 327–380

[HLS] H. Hofer, V. Lizan, J.-C. Sikorav: On genericity for holomorphic curves in four-dimensional almost-complex manifolds.
J. Geom. Anal. 7 (1997) 149–159

[L] S. Lang: *Differential and Riemannian manifolds.* GTM 160, Springer-Verlag, New York 1995

[S] B. Stensones: Fatou-Bieberbach domains with $\mathcal{C}^\infty$-smooth boundary.
Ann. Math. 145 (1997) 365–377

[V] I. N. Vekua: *Generalized analytic functions (Russian).* Fizmatgiz, Moskow 1959.

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