

UNIVERSITY OF LJUBLJANA
INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
DEPARTMENT OF MATHEMATICS
JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

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Boris Zgrablić

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BORIS ZGRABLIĆ
Pedagoška fakulteta
Univerza v Ljubljani
Kardeljeva ploščad 16, SI-1113 Ljubljana
Slovenija
Boris.Zgrablic@uni-lj.si

Abstract

A Cayley graph Γ of a group G is a *graphical doubly regular representation (GDRR) of the group G* if $\text{Aut } \Gamma$ is generated by the left and the right regular representations $L(G)$ and $R(G)$ of G , and by the involution $g \mapsto g^{-1}$ on G .

Examples and properties of GDRRs and their automorphism groups are studied. The problem of determining groups having a GDRR is considered, and some obstructions for a group to have a GDRR are found. Necessary and sufficient conditions for a graph to be a GDRR of two nonisomorphic groups are given. Further, disconnected GDRRs are determined, and imprimitive block systems of GDRRs are characterized.

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1 Introduction

The groups and graphs considered in this paper are finite, the graphs are simple and undirected.

For an arbitrary finite group G and a *Cayley subset* $S \subseteq G$ (i.e. $1 \notin S$ and $S = S^{-1}$), the *Cayley graph* $\Gamma = \text{Cay}(G, S)$ of G relative to S has vertex-set G , and adjacency in Γ is given by $g \sim_{\Gamma} gs$ for all $g \in G$ and all $s \in S$. If besides S is a union of conjugacy classes in G , i.e. S is a normal Cayley subset, then we call Γ a *quasiabelian Cayley graph (of G relative to S)*, according to Wang and Xu [72].

Quasiabelian Cayley graphs have been considered in various contexts: graph automorphisms, Fisk [25], Zgrablić [82]; connectivity of graphs, Imrich [36], Meng and Huang [50]; graph homomorphisms, Larose, Laviolette and Tardif [43], Hahn, Hell and Poljak [30]; graph spectrum, Ito [39]; graph spectrum and counting paths, Zieschang [83]; symmetry in interconnection networks, Lakshmivarahan, Jwo and Dhall [42]; construction of point-color-symmetric picture representations for finite simple nonabelian groups, Marcelo, Ruiz and Shinoda [48]; metrics on groups and their statistical uses, Diaconis [20]; graph spectrum and construction of Ramanujan graphs, Lubotzky [46, 47]; derangements on the n -cube, Chen and Stanley [19]; hamilton-connected derangement graphs, Rasmussen and Savage [60]; induced Cayley graphs and double covers, Schellwat [63]; expanders and random walks, Roichman [61]; equitable partitions, bisection width and eigenvalues, Kalpakis and Yesha [41]; Hopf algebras, Cibils and Rosso [17]; adjacency-transitivity, Zgrablić [80], Pisanski, Tucker and Zgrablić [58]. This list omits the many contributions on Cayley graphs of abelian groups.

The terms used besides *quasiabelian Cayley graph* [72, 80, 58] are *group-graph* [25], *conjugacy (class) graph* [39] and *normal Cayley graph* [43, 61]. (Note that the term *normal Cayley graph* has been used also by Wang, Wang and Xu [71], see also Xu [79], to denote a different graph construction, and the same holds for the *group-graph* in Perkel [57] and the *conjugacy class graph* in Casolo and Dolfi [16]. Yet another *conjugacy graph* concept is introduced by Bowman and Schultz [15], where ties with Perkel's group-graphs and the (*quasiabelian*) *action graph* notion from [58] can be found.)

The automorphism group $\text{Aut } \Gamma$ of a Cayley graph $\Gamma = \text{Cay}(G, S)$ always contains the left regular representation $L(G)$ of the group G . (Here "left"

depends on our definition of a Cayley graph.) If equality holds, that is, if $\text{Aut } \Gamma = L(G)$, then Γ is a *graphical regular representation (GRR)* of the group G (the term GRR is due to Watkins [55]). Equivalently, a graph Δ is isomorphic with a GRR of G if the automorphism group $\text{Aut } \Delta$ is isomorphic with G and acts regularly on the vertex set $V(\Delta)$, that is, transitively such that every nonidentity automorphism fixes no vertices.

The problem of determining all finite groups G having a graphical regular representation has been posed by Watkins [73], with contributions to its solution, in chronological order, by Sabidussi, Chao, McAndrew, Nowitz, Imrich, Watkins, Bannai, Hetzel and Godsil (see [8, 18, 27, 28, 32, 33, 34, 35, 37, 38, 49, 54, 55, 56, 62, 73, 74, 75, 76, 77]). Recently, progresses on an open problem of Godsil about GRRs have been made by Li [44], Li and Sim [45], Fang, Li, Wang and Xu [24].

Wang and Xu [72] and Larose, Laviolette and Tardif [43] noted that the Cayley graph $\Gamma = \text{Cay}(G, S)$ is quasiabelian if and only if the left *and* the right regular representations $L(G)$ and $R(G)$ are contained in $\text{Aut } \Gamma$ (Proposition 2.2).

A natural task of determining those finite groups G admitting a Cayley graph Γ whose automorphism group satisfies $\text{Aut } \Gamma = L(G)R(G)$ was proposed in [72, Problem 3]. The author proved in [82] that the equation is attained if and only if Γ is a GRR of the elementary abelian group $G \simeq (\mathbb{Z}_2)^n$, $n \neq 2, 3, 4$. The proof relies on a strong result about GRRs of abelian groups, due to McAndrew [49] and Imrich [33] (see Theorem 3.4 below), and on the fact that, given a group G and a quasiabelian Cayley graph Γ of G , the inversion bijection $\zeta_G \in \text{Sym } G$, $\zeta_G: x \mapsto x^{-1}$, is an automorphism of Γ (Proposition 2.2).

We say that Γ is a *graphical doubly regular representation (GDRR) of the group G* if Γ is a Cayley graph of G and

$$\text{Aut } \Gamma = L(G)R(G)\langle \zeta_G \rangle.$$

In some sense, a GDRR is a quasiabelian Cayley graph with "as small as possible" automorphism group.

Some basic observations on quasiabelian Cayley graphs are found in Section 2. In Section 3 we describe the structure of the automorphism group of a GDRR (Proposition 3.1) in terms of the central closure of a group. Section 4 is devoted to examples of GDRRs. In these introductory sections, we

often omit proofs. The problem of classifying the groups admitting a GDRR is raised in Section 5 (Question 1), with complete solution for the abelian case (Corollary 5.2), and some partial negative results in the general one. Sections 6 and 7 are devoted to graphs being isomorphic to quasiabelian Cayley graphs or GDRRs of at least two nonisomorphic groups, with the main result in Theorem 7.1. The disconnected GDRRs are determined in Section 8 (Theorem 8.2). We characterize the (im)primitivity of a GDRR in Section 9 (Theorem 9.1), giving also other auxiliary results. The decomposability of the central closure of a group is examined in Section 10. Some further problems are listed in Section 11.

We end this introduction by referring the reader to [21] and [78] for results on permutation groups.

2 General observations on quasiabelian Cayley graphs

We should emphasize that "being quasiabelian Cayley" is not a graph invariant.

Proposition 2.1 *There exists a quasiabelian Cayley graph, isomorphic to a non-quasiabelian Cayley graph of the same underlying group.*

PROOF. Let $G = D_8 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$ be the dihedral group of order 8 and set $S = \{a^2\}$, $T = \{b\}$. Then S, T are Cayley sets in G , the former is a conjugacy class, the latter is not conjugacy closed. One checks that $\text{Cay}(G, S) \simeq 4K_2 \simeq \text{Cay}(G, T)$. Defining $S_1 = G \setminus (S \cup \{1\})$ and $T_1 = G \setminus (T \cup \{1\})$, one gets a connected example. ■

Since the term "quasiabelian" is associated to a graph having vertices coordinatized by group elements, the next equivalent statements by Wang and Xu [72], Larose, Laviolette and Tardif [43] and Zgrablić [82], give a coordinatized characterization of a quasiabelian Cayley graph.

Proposition 2.2 *Let G be a group and let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph of G , where S is a Cayley set. Then the following are equivalent*

- (a) Γ is a quasiabelian Cayley graph (of G relative to S);
- (b) $L(G)R(G) \leq \text{Aut } \Gamma$;

(c) $L(G)R(G)\langle\zeta_G\rangle \leq \text{Aut } \Gamma$.

We remark that the left and right regular representation of a group have been called by Frattini [26, p. 144] *antipotential* and *potential* group, respectively, and that their study goes back to Jordan, who proved in [40, p. 39] that the centralizer of a regular permutation group is itself regular and conjugate to the initial group. See [52, Ch. II, §14, p. 35].

Next proposition provides a coordinateless description of quasiabelian Cayley graphs, similar to the already mentioned fact that a graph Γ is (isomorphic with) a Cayley graph if and only if $\text{Aut } \Gamma$ contains a subgroup acting regularly on the vertex-set $V(\Gamma)$.

Proposition 2.3 *A graph Γ is (isomorphic with) a quasiabelian Cayley graph of a group G if and only if $\text{Aut } \Gamma$ contains two transitive subgroups centralizing each other and isomorphic with G .*

We omit the proof. A classical result in permutation group theory is that two transitive subgroups centralizing each other are both regular and isomorphic. Therefore, the condition "isomorphic with G " in the upper proposition is equivalent to "one of them is isomorphic with G ".

We refer the reader to Bouc [13, Chapter 8] (and also [11, 12]) for a thorough categorial approach to *bisets* or *sets with double action*, i.e. sets with given commuting actions of groups G, H from the left and the right, respectively.

Proposition 2.4 *Let Γ be a quasiabelian Cayley graph of a group G . Then the stabilizer of the vertex $1 \in V(\Gamma)$ in $\text{Aut } \Gamma$ contains the subgroup generated by all inner automorphisms of G and the inverse bijection ζ_G , that is,*

$$(\text{Aut } \Gamma)_1 \geq \text{Inn } G \times \langle\zeta_G\rangle.$$

Equality holds if and only if Γ is a GDRR of G .

Corollary 2.5 (Fisk) *Let Γ be both isomorphic with a GRR of a group G and with a quasiabelian Cayley graph of a group H , with $|V(\Gamma)| > 1$. Then $G \simeq H \simeq (\mathbb{Z}_2)^n$, $n \neq 2, 3, 4$.*

PROOF. See [25, p. 144], or combine Proposition 2.4 and Theorem 3.4. ■

3 The automorphism group of a GDRR

In order to describe the structure of the automorphism group of a GDRR we will need the notion of a central product of groups. Recall that a group A is an *inner central product of the subgroups* B, C [66, p. 137, Def. 4.15] if B and C centralize each other and $A = BC$. See [66, pp. 137-140] for elementary properties.

The group $L(G)R(G)$ is a central product of two isomorphic copies of G . Precisely, let us define the *canonical central product* $G \wr G$ of G by G , or shorter *the canonical central square of* G , to be the group

$$G \wr G = (G \times G)/C, \quad \text{where } C = \{(g, g) \mid g \in Z(G)\}.$$

(The canonical central square $G \wr G$ is the “unique” central product of two copies of G in case $\text{Aut } Z(G) = \text{Aut}_{\text{Aut } G}(Z(G))$ – see [4, p. 32, (11.2)].) Thus we have

$$L(G)R(G) \simeq G \wr G.$$

In the sequel we will denote the coset $(g, h)C \in G \wr G$ by $((g, h))$, and we will often identify $L(G)$ with the subgroup

$$\{((g, 1)) \in G \wr G \mid g \in G\},$$

similarly $R(G)$ with $\{((1, g)) \in G \wr G \mid g \in G\}$, and $\text{Inn } G$ with the diagonal subgroup

$$D = \{((g, g)) \in G \wr G \mid g \in G\}.$$

Note that $L(G)R(G) = L(G) \rtimes \text{Inn } G = R(G) \rtimes \text{Inn } G$, so $L(G)R(G)$ is a subgroup of the holomorph $\text{Hol } G = G \rtimes_{\theta} \text{Aut } G$ of G (here θ denotes the natural action of $\text{Aut } G$ on G).

Define now the *central closure* $G \otimes G$ of the group G as follows: for $G \simeq (\mathbf{Z}_2)^n$, let $G \otimes G = G \wr G \simeq G$; otherwise let $G \otimes G$ be the semidirect product of the canonical central square $G \wr G$ by $\langle \zeta \rangle$, where the involution ζ acts by conjugation on $G \wr G$ as follows:

$$\zeta \cdot ((g, h)) \cdot \zeta^{-1} = ((h, g)).$$

(It is equivalent to say that $G \otimes G$ is the quotient of the wreath product $G \wr S_2$ by its normal subgroup $\{(g, g, 1) \mid g \in Z(G)\}$.) The central closure $G \otimes G$ has the following natural action on G : for every $x \in G$,

$$((g, h))x = gxh^{-1}, \quad \zeta x = x^{-1}.$$

Knowing that

$$L(G)R(G)\langle \zeta_G \rangle \simeq G \otimes G,$$

we now characterize the automorphism group of a GDRR by means of the central closure of a finite group,

Proposition 3.1 *Suppose Γ is a quasiabelian Cayley graph of the group G . Then*

- (a) $\text{Aut } \Gamma$ contains an isomorphic copy of the central closure $G \otimes G$;
- (b) the graph Γ is a GDRR of G if and only if $\text{Aut } \Gamma \simeq G \otimes G$.

We derive some obvious, but worth mentioning numeric necessary and sufficient conditions for a quasiabelian Cayley graph to be a GDRR.

Corollary 3.2 *Let Γ be a quasiabelian Cayley graph of a group G . Then Γ is a GDRR of G if and only if*

- (a) $|\text{Aut } \Gamma| = 2|G|^2/|Z(G)|$ in case $G \not\cong (\mathbb{Z}_2)^n$.
- (b) $|\text{Aut } \Gamma| = 2|G|$ in case G is commutative and $G \not\cong (\mathbb{Z}_2)^n$.
- (c) $|\text{Aut } \Gamma| = |G|$ in case $G \simeq (\mathbb{Z}_2)^n$.
- (d) $|\text{Aut } \Gamma| = 2|G|^2$ in case G is centerless.

For a group G , we denote by π_G the set of prime divisors of the order $|G|$.

Corollary 3.3 *Let Γ be a GDRR of a group G . Then*

- (a) $|\text{Aut } \Gamma| \leq 2|G|^2$.
- (b) The vertex-stabilizers in $\text{Aut } \Gamma$ have order at most $2|V(\Gamma)|$.
- (c) $\pi_{\text{Aut } \Gamma} \subset \pi_G \cup \{2\}$.

Note that in case (c) of Corollary 3.2, Γ is a GRR of G and $n \neq 2, 3, 4$, by the following well known result.

Theorem 3.4 (McAndrew [49], Imrich [33]) *There exists a GRR of the group \mathbb{Z}_2^n if and only if $n \neq 2, 3, 4$.*

4 Examples

Graphs that are (isomorphic to) GDRRs. The justifications that the following graphs are indeed GDRRs mostly rely on Corollary 3.2.

- (a) The complete graph $K_2 = P_2$ is (isomorphic to) a GDRR of \mathbb{Z}_2 .
- (b) The cycle C_n , $n \geq 3$, is (isomorphic to) a GDRR of \mathbb{Z}_n .
- (c) The Möbius ladder M_n on $2n$ vertices is (isomorphic to) a GDRR of \mathbb{Z}_{2n} for $n \geq 4$. The abelian groups admitting a GDRR are classified in Corollary 5.2.
- (d) For every odd integer $n \geq 3$, the graph $2C_n$ is (isomorphic to) a GDRR of the dihedral group D_{2n} . The disconnected GDRRs are determined in Theorem 8.2.
- (e) Let $G = D_{4n} = \langle a, b \mid a^{2n}, b^2, (ba)^2 \rangle$ be a dihedral group of order $4n$, $n \geq 5$, and let C denote the conjugacy class of b in G , that is,

$$C = \{b, a^2b, a^4b, \dots, a^{2n-2}b\}.$$

Then the graph

$$\Gamma = \text{Cay}(D_{4n}, \{a, a^{-1}\} \cup C)$$

is a quasiabelian Cayley graph of the nonabelian group D_{4n} . We will show that Γ is a *GDRR* of D_{4n} . To see this, observe the partition

$$C \cup Cb \cup aC \cup aCb$$

of $V(\Gamma)$, and the graphs induced in Γ by the pairs of parts:

$$\begin{aligned} \Gamma_{C,Cb} &\simeq K_{n,n}, & \Gamma_{C,aC} &\simeq C_{2n}, & \Gamma_{C,aCb} &\simeq 2nK_1, \\ \Gamma_{aC,aCb} &\simeq K_{n,n}, & \Gamma_{Cb,aCb} &\simeq C_{2n}, & \Gamma_{Cb,aC} &\simeq 2nK_1. \end{aligned}$$

Let F be the vertex-set of an induced $2n$ -cycle in Γ . We will prove that either $F = C \cup aC = \langle a \rangle b$ or $F = Cb \cup aCb = \langle a \rangle$, so these two vertex-sets are blocks of imprimitivity for $\text{Aut } \Gamma$. The restriction $n \geq 5$ implies that one of the sets C, Cb, aC, aCb contains at least 3 vertices of F . Assume C does: $m = |F \cap C| \geq 3$. Then $F \cap Cb = \emptyset$. Since there is no edge between C and aCb in Γ it follows also $|F \cap aC| \geq m$. Together with $F \cap aCb = \emptyset$ we conclude $F = C \cup aC$.

The stabilizer in $\text{Aut } \Gamma$ of the vertex $1 \in V(\Gamma)$ is the direct product two groups: one is generated by the reflection of the induced $2n$ -cycle on Cb, aCb that fixes 1, the other is the dihedral group isomorphic with D_{2n} and consisting of those symmetries of the second $2n$ -cycle fixing setwise C and aC . Thus

$$|\text{Aut } \Gamma| = (2 \cdot 2n) \cdot 4n = (4n)^2.$$

The graph Γ is also a quasiabelian graph of the nonabelian group D_{4n} with a center of order 2. By Corollary 3.2, the graph Γ is a GDRR of the dihedral group D_{4n} . This example is considered again in Proposition 6.1.

- (f) If S is the set of all cyclic permutation of order 4 in S_4 , then the Cayley graph $X = \text{Cay}(S_4, S)$ is a GDRR of S_4 (see [58]).

Quasiabelian Cayley graphs that are not GDRRs. The following graphs are not isomorphic to GDRRs, since they have too large vertex-stabilizers, by Corollary 3.3(b) (see also (c)):

- (i) the complete graph K_n for $n \geq 4$;
- (ii) the complete bipartite graph $K_{n,n}$ for $n \geq 3$;
- (iii) the cube graph.

There are also graphs Γ with the following property: Γ is a GDRR of G and isomorphic to a quasiabelian Cayley graph of H , but Γ is not isomorphic to a GDRR of H . Next statement provides examples.

Proposition 4.1 *Every GDRR of the dihedral group D_{2n} is isomorphic to a (quasiabelian) Cayley graph of $\mathbb{Z}_n \times \mathbb{Z}_2$.*

5 (Non)existence results

By the examples above, the cyclic groups \mathbb{Z}_n for $n \geq 3$, and the dihedral group D_{2n} for odd $n \geq 3$ or even $n \geq 10$, have a GDRR.

If a graph Γ were a GDRR of the quaternion group Q_8 , then Corollary 3.2(a) would give $|\text{Aut } \Gamma| = 64$. One can check that every (quasiabelian) Cayley graph of Q_8 has automorphism group of order at least 128 (see also [38, Lemma 2.6] or Corollary 5.7). So Q_8 has no GDRR. We generalize this

fact in Proposition 5.6. Similarly, the alternating group A_4 has no GDRR: the nontrivial quasiabelian Cayley graphs of A_4 are the disconnected graph $3K_4$ and its complement, having a too large automorphism group (see also Theorem 8.2). A generalization is given in Corollary 5.5.

Question 1 (Variant of Problem 3 in [72]) *Which groups G do have a GDRR?*

We will give some partial answers.

In [38], Imrich and Watkins defined the *Cayley index* $c(G)$ of a group G as the minimal quotient $|\text{Aut } \Gamma|/|G|$, where Γ ranges over all Cayley graphs of G . Their determination of the Cayley index of an arbitrary finite abelian group settles our abelian case for GDRRs.

Theorem 5.1 (Imrich, Watkins [38]) *Let G be a finite abelian group. Then $c(G) \leq 2$ unless G is one of the following seven groups: \mathbb{Z}_2^3 , \mathbb{Z}_2^4 , $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2^2$, \mathbb{Z}_3^2 , \mathbb{Z}_3^3 and \mathbb{Z}_4^2 .*

Corollary 5.2 *Every finite abelian group has a GDRR, with exceptions the seven groups of Theorem 5.1 and the group \mathbb{Z}_2^2 .*

PROOF. Let G be an abelian group. By Corollary 3.2, a Cayley graph Γ of G is a GDRR of G if and only if $|\text{Aut } \Gamma|/|G|$ equals 1 (in case G is elementary abelian of exponent 2) or 2 (otherwise). So G has a GDRR unless: G has exponent 2 and has no GRRs, or G has exponent greater than 2 with Cayley index $c(G) > 2$. The result follows from Theorem 3.4 and Theorem 5.1. ■

It is a well known fact that every group automorphism $\phi \in \text{Aut } G$ preserving a Cayley set $S \subseteq G$ induces a graph automorphism of $\text{Cay}(G, S)$. For a group G and a subset C let

$$\text{Aut}(G, C) = \{\phi \in \text{Aut } G \mid \phi(C) = C\}.$$

Godsil [29, Corollary 4.4] gives a construction of GDRRs for two classes of abelian groups.

Proposition 5.3 (Godsil [29]) *Suppose that G is either an abelian p -group with p odd or a cyclic 2-group with order at least four. Let C be a subset of G such that $\text{Aut}(G, C) = \langle \zeta_G \rangle$ and set $\Gamma = \text{Cay}(G, C)$. Then Γ is a GDRR of G .*

We end this section by stating some negative results.

Proposition 5.4 *Let G be a nonabelian group, S a conjugacy closed Cayley set in G and $\phi \in \text{Aut}(G, S) \setminus \text{Inn } G$ an outer automorphism of G preserving S . Then the quasiabelian Cayley graph $\Gamma = \text{Cay}(G, S)$ is not a GDRR. In particular, if all the conjugacy classes in G are ϕ -invariant, then G admits no GDRR.*

PROOF. By assumption, the automorphism ϕ belongs to the stabilizer of 1 in $\text{Aut } \Gamma$. If Γ were a GDRR we would have by Proposition 2.4

$$\phi \in (\text{Inn } G)\langle \zeta_G \rangle.$$

This contradicts the assumptions that ϕ is outer and G is nonabelian, as a straightforward computation shows. ■

Corollary 5.5 *The alternating group A_n , $n \geq 4$, has no GDRR.*

PROOF. Observe the automorphisms in $(\text{Inn } S_n) \setminus (\text{Inn } A_n)$ and apply Proposition 5.4. ■

Proposition 5.6 *Let G be a nontrivial group satisfying the following condition:*

(*) *every nontrivial conjugacy class of G is a coset of the center $Z(G)$.*

Then G has no GDRR.

PROOF. First we show that every quasiabelian Cayley graph of G is isomorphic to a lexicographic product. Denote $g = |G|$ and $z = |Z(G)|$.

By assumption, the group G is not abelian and $z \geq 2$. So $g \geq 8$. Let S be an arbitrary conjugacy closed Cayley set in G , and $\Gamma = \text{Cay}(G, S)$. Denote by $\phi: G \rightarrow G/Z(G)$ the canonical projection, and let $\Omega = \text{Cay}(\phi(G), \phi(S) \setminus \{\phi(1)\})$, $\Pi = \text{Cay}(Z(G), S \cap Z(G))$. One readily checks that Γ is isomorphic with the lexicographic product $\Omega[\Pi]$.

The group $\text{Aut}(\Omega[\Pi])$ has order at least

$$|\text{Aut } \Pi|^{|\text{V}(\Omega)|} |\text{Aut } \Omega| \geq z^{g/z} g/z. \quad (1)$$

If $g > 8$, then one can show that the right-hand side of (1) is greater than $2g^2/z$. So Γ is not a GDRR of G by Corollary 3.3(a). If $g = 8$, then

$|\phi(G)| = 4$ and $|\text{Aut } \Omega| \geq 8$. Using the left-hand side of (1) one obtains $\text{Aut}(\Omega[\Pi]) \geq 128$. By Corollary 3.3(a) again, Γ is not a GDRR of G . ■

The condition (*) imposed on the group G in Proposition 5.6 is equivalent to the following: for every $a \in G \setminus Z(G)$ holds $Z(G) = \{gag^{-1}a^{-1} \mid g \in G\}$. It is straightforward that G must be a nilpotent p -group of class 2, with a center $Z(G) = [G, G]$ of exponent dividing the order of every noncentral element. Also, if G is a central product of subgroups satisfying (*), then G itself satisfies (*).

Corollary 5.7 *No nonabelian group of order p^3 , p prime, has a GDRR. Also, no extraspecial p -group has a GDRR.*

PROOF. Given an odd prime p , there are two nonabelian groups of order p^3 , with respective presentations

$$\langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1} \rangle,$$

$$\langle a, b, c \mid a^p = b^p = c^p = 1, ab = bac, ac = ca, bc = cb \rangle.$$

One easily checks that each of these groups satisfies the condition (*) of Proposition 5.6. For $p = 2$ we have the dihedral D_8 and the quaternion Q_8 , satisfying (*), too. See also Section 6.

Every extraspecial p -group G is a central product of nonabelian subgroups of order p^3 (see [67, Theorem 4.18]). The latter satisfy condition (*), so by the comment preceding the Corollary and Proposition 5.6, G has no GDRR. ■

Proposition 5.8 *Let G be a nonabelian group in which every element is conjugated to its inverse, that is, all conjugacy classes are inverse-closed. Suppose the nontrivial group H is not elementary abelian of exponent 2. Then $G \times H$ has no GDRR.*

PROOF. By assumption, the groups G, H are not elementary abelian of exponent 2.

Let $\Gamma = \text{Cay}(G \times H, S)$ be a quasiabelian Cayley graph of $G \times H$, with S a conjugacy closed Cayley set. Define the mapping $\zeta_1: G \times H \rightarrow G \times H$ as follows:

$$\zeta_1: (g, h) \mapsto (g^{-1}, h).$$

(One could similarly define ζ_2 as the inversion of the second component in $G \times H$.) A straightforward verification shows that ζ_1 induces an automorphism of Γ fixing $(1, 1) \in V(\Gamma)$, but not belonging to the group $(\text{Inn}(G \times H))\langle \zeta_{G \times H} \rangle$. By Proposition 2.4, the graph Γ is not a GDRR of $G \times H$. ■

6 Recognizing the group – an example

If Γ is isomorphic with a GRR of a group G , then G is uniquely determined up to isomorphism, since $G \simeq \text{Aut } \Gamma$. For GDRRs the situation is different: a graph can be isomorphic with two GDRRs relative to two nonisomorphic groups.

This is related with the fact that a group may have "nonequivalent" central decompositions. The central decomposition of groups has been first studied by Tang [68, 69], and for certain p -groups by Abbasi [1, 2, 3]. Tang [69, Theorem 2.4] proved that if $G/Z(G)$ has trivial center, then the central decomposition of G into centrally indecomposable factors is unique (up to reordering of central factors).

If a graph is isomorphic to two GDRRs of two nonisomorphic groups G and H , then the central closures $G \otimes G$ and $H \otimes H$ of these two groups are isomorphic, by Proposition 3.1. So we shall focus on nonisomorphic groups having isomorphic central closures, or even isomorphic central squares. For instance, the central product of two copies of D_8 is isomorphic with the central product of two copies of Q_8 (see [66, p. 139]). More generally, let

$$D_{2n} = \langle \sigma, \tau \mid \sigma^n = \tau^2 = (\sigma\tau)^2 = 1 \rangle$$

be a dihedral group of order $2n$, $n \geq 3$, and let

$$Q_{4m} = \langle x, y \mid x^{2m} = 1, x^m = y^2, yxy^{-1} = x^{-1} \rangle$$

be the dicyclic group of order $4m$, $m \geq 2$. For $n = 2m = 4k \geq 4$ define two subgroups in the canonical central product $D_{2n} \vee D_{2n}$:

$$Q' = \langle a, b \rangle, \quad Q'' = \langle c, d \rangle,$$

where

$$a = ((\sigma, 1)), \quad b = ((\tau, \sigma^k)), \quad c = ((1, \sigma)), \quad d = ((\sigma^k, \tau)).$$

Then Q' and Q'' are isomorphic with the dicyclic group Q_{4m} . Also, Q' and Q'' centralize each other, they intersect in their center and $Q'Q'' = D_{2n} \wr D_{2n}$. Thus

$$D_{8k} \wr D_{8k} \simeq Q_{8k} \wr Q_{8k}.$$

Moreover, the action of the subgroups Q' and Q'' on D_{2n} is transitive. In view of Proposition 2.3 and Corollary 3.2, we proved the following.

Proposition 6.1 *Let Γ be the GDRR of the dihedral group D_{4n} as described in Section 3, example (e). If $n = 2k$, then Γ is isomorphic to a GDRR of the dicyclic group Q_{8k} , too.*

In the dicyclic group Q_{4m} of order $4m$ with presentation as above, the conjugacy classes are

$$\{1\}, \{x^{2k}\}, \{x^i, x^{-i}\}, \{x^j y \mid j \text{ odd}\}, \{x^j y \mid j \text{ even}\}.$$

They are all closed for taking inverses in case m is even. If $4n = 4m = 8k$ and S is the last conjugacy class in the list above, the graph Γ in Proposition 6.1 is isomorphic with the quasiabelian Cayley graph

$$\Delta = \text{Cay}(Q_{8k}, \{x, x^{-1}\} \cup S). \quad (2)$$

A generalization of Proposition 6.1 follows next lemma.

Lemma 6.2 *A group A is isomorphic with the canonical central square $G \wr G$ if and only if A is an inner central product of two subgroups B, C isomorphic with G , such that*

$$Z(A) = Z(B) = Z(C).$$

Proposition 6.3 *Suppose that the group $G/Z(G)$ is not factorizable into a central product unless a factor is $G/Z(G)$ (i.e. the group $G/Z(G)$ is centrally indecomposable), and let the groups G and H have isomorphic canonical central squares,*

$$G \wr G \simeq H \wr H.$$

Then every quasiabelian Cayley graph of G is isomorphic to a quasiabelian Cayley graph of H , too. In particular, every GDRR of G is also isomorphic to a GDRR of H .

PROOF. Write $Z_1 = Z(G)$. We may assume that the group G (and thus H) is not abelian, the abelian case being trivial. In particular, the group G/Z_1 is not cyclic.

If G/Z_1 were commutative, the decomposition restriction for G/Z_1 would imply G/Z_1 is cyclic, a contradiction. So G/Z_1 is not commutative, too.

Let Γ be a quasiabelian Cayley graph of G . The isomorphism $G\vee G \simeq H\vee H$ implies by Lemma 6.2 that $G\vee G$ is an inner central product of two subgroups K, L , both isomorphic with H and both having center equal to $Z(G\vee G)$. We will show that K and L are regular on Γ . So Γ is isomorphic to a quasiabelian Cayley graph of H by Proposition 2.3.

It suffices to prove that the vertex $1 \in V(\Gamma)$ has trivial stabilizer in K and in L . The regularity of the subgroups K and L then follows from the equation $|H| = |G|$ in Lemma 7.3. Suppose an element $x = ((a, b)) \in K \cup L$ stabilizes the vertex $1 \in V(\Gamma)$. Then $a \cdot 1 \cdot b^{-1} = 1$, so $b = a$ and $x = ((a, a))$. We claim $a \in Z_1$.

To prove our claim, define the subgroup K_1 in G as the "group of first components of elements of K ". Exactly,

$$K_1 = \{g \in G \mid ((g, h)) \in K \text{ for some } h \in G\}.$$

We define the subgroups K_2, L_1, L_2 in a similar fashion. Then the subgroups K_1 and L_1 both contain Z_1 , and they together generate G : $G = K_1 L_1$. Denote by $\phi: G \rightarrow G/Z_1$ the canonical quotient homomorphism. Then $\phi(K_1)$ and $\phi(L_1)$ centralize each other in G/Z_1 .

By the assumption on the group G/Z_1 , one of the subgroups K_1, L_1 equals G , and the other is contained in the second center $Z_2 = \phi^{-1}(Z(G/Z_1))$. The same holds for the pair K_2, L_2 . We may assume that $K_1 = G$ and $L_1 \leq Z_2$.

Suppose $K_2 = G$ and $L_2 \leq Z_2$ hold. The group L is then contained in $\{((l_1, l_2)) \mid l_1, l_2 \in Z_2\}$. So $L/Z(G\vee G)$ is isomorphic to a subgroup in the commutative subgroup $(Z_2/Z_1) \times (Z_2/Z_1)$. Since

$$L/Z(G\vee G) = L/Z(L) \simeq H/Z(H) \simeq \text{Inn } H \simeq \text{Inn } G \simeq G/Z_1$$

and G/Z_1 is not commutative, a contradiction occurs.

Thus $K_2 \leq Z_2$ and $L_2 = G$. So $a \in Z_2$. We still have to prove $a \in Z_1$. We may assume $x \in K$. By the above it follows that for every $h \in G$ there exists $b \in Z_2$ such that the element $y = ((b, h))$ belongs to L . Since $x \in K$

and $y \in L$ commute in $G \vee G$, there exists $c \in Z_1$ such that the following two equations hold,

$$ab = bac, \quad ah = hac.$$

It is easy to verify that a and $b^{-1}h$ commute. In other words, $b^{-1}h$ belongs to the centralizer $Z_G(a)$ of a in G . Thus $h \in Z_2 Z_G(a)$. Since h was arbitrary in G , we have

$$G = Z_2 Z_G(a).$$

By projecting this equation in G/Z_1 we obtain

$$G/Z_1 = (Z_2/Z_1)(Z_G(a)/Z_1).$$

Since Z_2/Z_1 is the center of G/Z_1 , the last equality implies that the quotient G/Z_1 is an inner central product of Z_2/Z_1 with $Z_G(a)/Z_1$. From the decomposition restriction for G/Z_1 it follows that either $Z_2/Z_1 = G/Z_1$ or $Z_G(a)/Z_1 = G/Z_1$. The former leads to a contradiction as above, whereas the latter implies $Z_G(a) = G$. So $a \in Z_1 = Z(G)$. ■

The assumption in Proposition 6.3 that $G/Z(G)$ be centrally indecomposable needs an explanation. If $G/Z(G)$ admits a nontrivial central decomposition, the following situation may occur: $G \vee G$ is an inner central product of K and L with $K \simeq L$ and $Z(G \vee G) = Z(K) = Z(L)$, but K and L are not transitive in the natural action of the central square $G \vee G$ on G . For example, let $G = D_8$ with presentation as above, and observe in $G \vee G$ the subgroups

$$K = \langle \langle (\sigma, \tau), (\tau, \tau) \rangle \rangle, \quad L = \langle \langle (\sigma\tau, \sigma), (\sigma\tau, \sigma\tau) \rangle \rangle.$$

These two subgroups centralize each other, they are isomorphic with G , and their product equals $G \vee G$. But the orbit of K containing 1, in the natural action on $G = D_8$, is

$$K(1) = \{1, \sigma^2, \sigma\tau, \sigma^2\tau\}.$$

So K is not transitive on G , and neither is L . The author doesn't know if there are examples of nontransitive action of K with $K \neq G$. See Problem 11.2.

7 Recognizing the group – the theorem

We will prove that, given a graph Γ which is a GDRR, one can recover from $\text{Aut } \Gamma$ the center $Z(G)$ and the inner automorphism group $\text{Inn } G$ of the relative group G .

Theorem 7.1 *Let Γ be a GDRR of two nonisomorphic groups G and H . Then G and H are nonabelian with nontrivial isomorphic centers and isomorphic central quotients. Also, the normal subgroup lattices $\mathcal{N}(G)$ and $\mathcal{N}(H)$ of G and H , respectively, are isomorphic.*

REMARK. The existence and uniqueness of groups with preassigned central and central quotient group (i.e. center and inner automorphism group) has been first approached by Baer [6, 7] for the abelian case. Recently, the term *capable group* is used to describe a group which is isomorphic to the inner automorphism group of some group. Capability is still investigated, see [9, 10, 64, 23]. Also, groups G and H having the same order and isomorphic normal subgroup lattices are said to be *in the same genus* by Hall and Senior [31] if an additional condition is fulfilled.

Before analyzing isomorphic central closures and proving Theorem 7.1 we discuss the easier case: isomorphic central squares.

Lemma 7.2 $Z(G\curlywedge G) \simeq Z(G)$ and $(G\curlywedge G)/Z(G\curlywedge G) \simeq \text{Inn } G \times \text{Inn } G$.

PROOF. Suppose $((a, b))$ is a central element of $G\curlywedge G$. Then for every $((x, y)) \in G\curlywedge G$ there is some $c \in Z(G)$ such that

$$ax = xac, \quad by = ybc.$$

In particular, $y = 1$ implies $c = 1$, and since x is an arbitrary element of G , one infers $a \in Z(G)$. Similarly $b \in Z(G)$. So

$$Z(G\curlywedge G) = \{((a, b)) \mid a, b \in Z(G)\} \simeq Z(G).$$

The homomorphism defined by $((g, h)) \mapsto (gZ(G), hZ(G))$ induces an isomorphism between $(G\curlywedge G)/Z(G\curlywedge G)$ and $G/Z(G) \times G/Z(G)$, proving the second statement. ■

The center $Z(G)$, the inner automorphism group $\text{Inn } G$ and the order of G can be recognized from $G\curlywedge G$ up to isomorphism.

Proposition 7.3 *Suppose $G\vee G \simeq H\vee H$. Then $Z(G) \simeq Z(H)$, $\text{Inn } G \simeq \text{Inn } H$ and $|G| = |H|$.*

PROOF. By Lemma 7.2,

$$Z(G) \simeq Z(G\vee G) \simeq Z(H\vee H) \simeq Z(H).$$

One then infers from the same Lemma that $\text{Inn } G \simeq \text{Inn } H$, and consequently $|G| = |H|$. ■

The analysis of isomorphic central closures relies on the following upper bound for the order the centralizer of a certain element in $G\otimes G$.

Lemma 7.4 *Suppose that for every $k \geq 0$, $Z(G) \not\cong (\mathbb{Z}_2)^k$, $Z(G) \not\cong \mathbb{Z}_4 \times (\mathbb{Z}_2)^k$. Let $x \in (G\otimes G) \setminus (G\vee G)$. Then*

$$|Z_{G\otimes G}(x)| < |G|.$$

PROOF. We have $x = ((g, h))\zeta$ for some $g, h \in G$. Write $C = Z_{G\otimes G}(x)$. It is easy to check that $C = C' \cup C''$, where

$$C' = C \cap (G\vee G) = \{((a, hah^{-1}c)) \mid a(gh)a^{-1}(gh)^{-1} = c^2, c \in Z(G)\},$$

$$C'' = C \setminus (G\vee G) = \{((a, hag^{-1}d))\zeta \mid a(hg)a^{-1}(gh)^{-1} = d^2, d \in Z(G)\}.$$

Denoting $Z(G)_2 = \{z \in Z(G) \mid z^2 = 1\}$, we have that

$$|C'| \leq |\text{Inn } G| \cdot |Z(G)_2|,$$

and the same estimate holds for $|C''|$. The assumption on the structure of $Z(G)$ implies $2|Z(G)_2| < |Z(G)|$, whence

$$|Z_{G\otimes G}(x)| = |C| = |C'| + |C''| \leq 2|\text{Inn } G| \cdot |Z(G)_2| < |\text{Inn } G| \cdot |Z(G)| = |G|.$$

■

Proposition 7.5 *Suppose $G\otimes G \simeq H\otimes H$, where $|G| \leq |H|$ and for every $k \geq 0$, $Z(G) \not\cong (\mathbb{Z}_2)^k$, $Z(G) \not\cong \mathbb{Z}_4 \times (\mathbb{Z}_2)^k$. Then $G\vee G \simeq H\vee H$.*

PROOF. Since $G\otimes G \simeq H\otimes H$ and G is not elementary abelian of exponent 2, the group $G\otimes G$ contains two subgroups K, L , isomorphic with H , centralizing each other and intersecting in their center, such that KL has index 2 in $G\otimes G$. For each $x \in K$ we have $Z_{G\otimes G}(x) \geq L \simeq H$, so $|Z_{G\otimes G}(x)| \geq |H|$. The same inequality holds for $x \in L$. If $KL \neq G\vee G$, then there would exist some $x \in (K \cup L) \setminus G\vee G$. But then by Lemma 7.4, $|Z_{G\otimes G}(x)| < |G| \leq |H|$, a contradiction. So $KL = G\vee G$, thus $H\vee H \simeq G\vee G$. ■

The next result was pointed out also by Serge Bouc [14].

Proposition 7.6 *Suppose $G \otimes G \simeq H \otimes H$ and G has no subgroup of index 2 (for instance: G has odd order). Then $G \vee G \simeq H \vee H$.*

PROOF. We may assume G is not trivial. By the assumptions the group $G \otimes G$ contains a pair of subgroups K, L with common center, centralizing each other and isomorphic with H , such that the subgroup $KL \simeq H \vee H$ has index 2 in $G \otimes G$.

The intersection $T = (G \vee G) \cap KL$ is normal in $G \otimes G$. If $G \vee G \neq KL$, then T has index 2 in $G \vee G$. It follows in this case that G contains a subgroup E of index 2, contradiction. ■

Finally, let us characterize the center of a central closure.

Lemma 7.7 $Z(G \otimes G) \simeq \{c \in Z(G) \mid c^2 = 1\}$.

PROOF. For $G \simeq (\mathbb{Z}_2)^n$ we have $G \otimes G = G$ and the Lemma holds. Assume now $G \not\simeq (\mathbb{Z}_2)^n$. Every element in $G \otimes G$ has either the form $((a, b))$ or $((a, b)) \cdot \zeta$. A computation shows that the latter does not belong to $Z(G \otimes G)$ unless G is abelian of exponent 2, which is not the case. The former does belong to $Z(G \otimes G)$ if and only if $a, b \in Z(G)$ and $a^2 = b^2$. So

$$\begin{aligned} Z(G \otimes G) &= \{((a, b)) \in G \vee G \mid a, b \in Z(G), a^2 = b^2\} \\ &= \{((c, 1)) \in G \vee G \mid c \in Z(G), c^2 = 1\} \\ &\simeq \{c \in Z(G) \mid c^2 = 1\}. \end{aligned}$$

■

PROOF (of Theorem 7.1). Suppose Γ is isomorphic to two GDRRs of two nonisomorphic groups G and H : $\Gamma \simeq \text{Cay}(G, S)$ and $\Gamma \simeq \text{Cay}(H, R)$, where S and R are conjugacy closed Cayley subsets of G and H , respectively. We can assume G and H are not elementary abelian of exponent 2 (otherwise Γ is isomorphic to a GRR of G and $G \simeq \text{Aut } \Gamma \simeq H$, contradiction – see also Corollary 2.5).

By Proposition 2.4, the stabilizer in $\text{Aut } \Gamma$ of the vertex 1 is isomorphic to both $\text{Inn } G \times \mathbb{Z}_2$ and $\text{Inn } H \times \mathbb{Z}_2$, so $\text{Inn } G \simeq \text{Inn } H$. Of course, $|G| = |V(\Gamma)| = |H|$, implying $|Z(G)| = |Z(H)|$.

If at least one of the centers of G and H is neither isomorphic with $(\mathbb{Z}_2)^k$ nor $\mathbb{Z}_4 \times (\mathbb{Z}_2)^k$, $k \geq 0$, Proposition 7.5 yields $G \vee G \simeq H \vee H$. Then Proposition 7.3 gives $Z(G) \simeq Z(H)$ (and again $\text{Inn } G \simeq \text{Inn } H$).

Let now $Z(G) \simeq (\mathbb{Z}_2)^k$. By the characterization of the center of a central closure in Lemma 7.7, $Z(\text{Aut } \Gamma) \simeq Z(G \otimes G) \simeq Z(G)$. If $Z(H) \simeq (\mathbb{Z}_2)^l$, we have $Z(\text{Aut } \Gamma) \simeq Z(H)$, too. Thus $Z(G) \simeq Z(H)$. If $Z(H) \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^l$, then $Z(\text{Aut } \Gamma) \simeq (\mathbb{Z}_2)^{l+1}$, so $k = l + 1$. But then $|Z(G)| = 2^k \neq 2^{l+2} = |Z(H)|$, a contradiction. The cases when $Z(G) \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^k$ are treated similarly, ending either with a contradiction or with $Z(G)$ and $Z(H)$ isomorphic.

Suppose G and H are abelian. Then

$$\text{Aut } \Gamma \simeq G \rtimes \langle \zeta_G \rangle \simeq H \rtimes \langle \zeta_H \rangle,$$

where ζ_G and ζ_H denote the (nontrivial) inversion automorphism of G and H , respectively. We may assume that G and H are contained in $\text{Aut } \Gamma$, and suppose $G \neq H$. For every $x \in \text{Aut } \Gamma \setminus G$ and every $g \in G$ holds the equality $xgx^{-1} = \zeta_G g \zeta_G^{-1} = g^{-1}$. In particular, the last equality is valid for every $x \in H \setminus G$. Since H is abelian, every element in the intersection $G \cap H$ equals its inverse. Hence $G \cap H \simeq (\mathbb{Z}_2)^n$ for some $n \geq 0$. The index $[\text{Aut } \Gamma : G \cap H] = 4$ and the assumption that G and H are abelian of exponent greater than 2 gives

$$G \simeq \mathbb{Z}_4 \times \mathbb{Z}_2^{n-1} \simeq H, \quad n \geq 1.$$

If G is abelian and H is not, a contradiction occurs by Corollary 3.2.

The isomorphism of the lattices of normal subgroups in G and H is an immediate consequence of Theorem 9.1. ■

8 Connectedness of a GDRR

The disjoint union of two cycles of odd length, already met as example (d) in Section 4, is isomorphic to a disconnected GDRR. We will prove that this graph is close to the general disconnected case.

Lemma 8.1 *Let Γ be a quasiabelian Cayley graph of a group G : $\Gamma = \text{Cay}(G, S)$. If Γ is not connected, then Γ is a disjoint union of $[G : \langle S \rangle]$ isomorphic copies of the connected quasiabelian Cayley graph $H = \text{Cay}(\langle S \rangle, S)$.*

Theorem 8.2 *Let Γ be a disconnected GDRR of a group G . Then Γ is isomorphic with one of $2K_1$, $3K_1$, $2K_2$ or 2Δ , where Δ is a connected GDRR of an abelian group H of odd order and of index 2 in the nonabelian group $G \simeq H \rtimes \langle \zeta_H \rangle$.*

PROOF. We have $\Gamma = \text{Cay}(G, S)$ for a conjugacy closed Cayley set $S \subseteq G$. Since Γ is not connected, it is by Lemma 8.1 a disjoint union $\Gamma = r\Delta$ of, say, $r \geq 2$ isomorphic copies of the connected quasiabelian Cayley graph $\Delta = \text{Cay}(H, S)$, where $H = \langle S \rangle$. We will denote

$$g = |G| = |V(\Gamma)|, \quad h = |H| = |V(\Delta)|, \quad a = |\text{Aut } \Gamma|, \quad b = |\text{Aut } \Delta|, \quad z = |Z(G)|.$$

By Corollary 3.2,

$$2r^2h^2 = 2g^2 \geq a = r!b^r \geq r!h^r, \quad (3)$$

or in short

$$2r^2 \geq r!h^{r-2}. \quad (4)$$

Thus $r \leq 4$.

If $h = 1$, then S is empty and $\Gamma \simeq rK_1$. Besides, G must be abelian since it has order $g = hr = r \leq 4$. For $r = 4$, we have either $G \simeq (\mathbb{Z}_2)^2$ or $G \simeq \mathbb{Z}_4$, contradicting through Corollary 3.2 the order $|\text{Aut}(4K_1)| = 24$. Thus $r \leq 3$. One concludes that Γ is either the GDRR $2K_1$ of the group \mathbb{Z}_2 or the GDRR $3K_1$ of the group \mathbb{Z}_3 .

So we may assume $h \geq 2$. Then $r \leq 3$ by the estimate (4).

Suppose $r = 3$. The inequalities (4) and (3) give $h \leq 3$ and $(b/h)^2b \leq 3$. The integer h divides b since Δ is vertex-transitive. One infers that $b/h = 1$ and $b \leq 3$. In particular, $b = h$, i.e. Δ is a GRR of H . Since Δ is also a quasiabelian Cayley graph of H , Corollary 2.5 and the order of Δ , as well as connectedness of Δ imply $\Delta \simeq K_2$, giving $\Gamma \simeq 3K_2$. On one hand, $|\text{Aut}(3K_2)| = 48$. On the other hand, $g = 6$ forces either $G \simeq \mathbb{Z}_6$ or $G \simeq D_6$, whence $a = 2g^2/z \leq 2 \cdot 6^2/2 = 36$, a contradiction. Hence $r \neq 3$.

Assume now $r = 2$. Then $\Gamma = 2\Delta$ and $a = 2b^2$. Also, the subgroup H has index 2 in G and is thus normal. If $b > 2h$ then

$$2g^2 \geq a = 2b^2 > 8h^2 = 2g^2,$$

a contradiction. So $b \leq 2h$. Besides, Δ is a quasiabelian Cayley graph of H , so $\text{Aut } \Delta$ contains by Corollary 3.1 the central closure of H , so $b \geq 2h^2/|Z(H)|$. One concludes that H is abelian.

The inequality $b \leq 2h$ implies either (a) $b/h = 1$ (in this case Δ is a GRR of H), or (b) $b/h = 2$.

Suppose (a) holds. Again by Corollary 2.5 we have $H \simeq (\mathbb{Z}_2)^n$ for some $n \neq 2, 3, 4$. From $g = 2h$ it follows that G is a 2-group and $a = 2h^2 = g^2/2$. If G is abelian, then $G \simeq (\mathbb{Z}_2)^{n+1}$ or $G \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^{n-1}$. If the latter holds, one computes $g = 4$ from the additional equality $a = 2g$ in Corollary 3.2, so $G \simeq \mathbb{Z}_4$ and $\Gamma \simeq 2K_2 \simeq \text{Cay}(\mathbb{Z}_4, \{2\})$. If the former occurs, the graph Γ would be a GRR, that is, $a = g$, a contradiction to the order $a = g^2/2$ and the assumption $h \geq 2$. Assume now G is not abelian. For every $x \in G \setminus H$, the inner automorphism $\phi_x \in \text{Aut } H$ induced by x is nontrivial and preserving S . Thus ϕ_x induces a nontrivial automorphism of Δ fixing the identity 1, a contradiction to the assumption (a) that Δ is a GRR.

We proceed with case (b). The exponent of the abelian group H is by Corollary 3.2 greater than 2. If G were abelian, then on one hand, by Corollary 3.2,

$$a = 2g = 2 \cdot 2h = 4h,$$

and on the other hand, $\Gamma = 2\Delta$ gives

$$a = 2b^2 = 8h^2,$$

a contradiction. So G is not abelian. As above, for all $x \in G \setminus H$ the inner automorphism ϕ_x of G induces a nontrivial automorphism of Δ fixing the vertex 1, so $\phi_x = \zeta_H$. This implies $G \simeq \text{Aut } \Delta \simeq H \rtimes \langle \zeta_H \rangle$. Since on one hand we have $a = 2b^2 = 2g^2$ and on the other $a = 2g^2/z$, we conclude $Z(G) = 1$, so H has no element of order 2. The abelian group H has therefore odd order. ■

9 Imprimitivity

We dedicate a few words to the characterization of the (im)primitivity of the automorphism group of a GDRR. If \mathcal{B} is an imprimitivity block system for the action of a group G on a set Ω , we call \mathcal{B} *normal* if it consists of orbits of some normal subgroup in G .

Theorem 9.1 *Let Γ be a GDRR of a group $G \neq (\mathbb{Z}_2)^n$. Then there is a bijective correspondence between the set \mathcal{N} of normal subgroups of G and the set \mathcal{S} of imprimitivity block systems in $V(\Gamma)$, inducing an isomorphism of the respective lattices. Moreover, every imprimitivity block system for the action of $\text{Aut } \Gamma$ on $V(\Gamma)$ is normal.*

PROOF. We will identify $\text{Aut } \Gamma$ and $G \otimes G$. If H is normal in G then the subgroup

$$\phi(H) = \{((h, k)) \in G \otimes G \mid h, k \in H\}$$

is normal in $G \otimes G$. Thus the orbits of $\phi(H)$ on $V(\Gamma)$ form a (maybe trivial) imprimitivity block system \mathcal{B}_H for the action of $G \otimes G$. We claim the mapping

$$\psi: H \mapsto \mathcal{B}_H$$

is a bijection from \mathcal{N} to \mathcal{S} .

Every imprimitivity block system $\mathcal{B} \in \mathcal{S}$ is determined by its block $\mathcal{B}(1)$ containing $1 \in V(\Gamma)$. Since $\mathcal{B}_H(1) = H$ for every $H \in \mathcal{N}$, the mapping ψ is injective.

On the other hand, let $\mathcal{B} \in \mathcal{S}$ be arbitrary and define $K = \mathcal{B}(1)$. We first prove that K is a normal subgroup of G . Observe that $L(K)$ moves 1 within K , so $L(K)K \subseteq K$, giving K is closed for multiplication. The stabilizer of $1 \in V(\Gamma)$ in $G \otimes G$ stabilizes setwise the block $\mathcal{B}(1) = K$. We apply Proposition 2.4 and infer from $(\text{Inn } G)(K) = K$ and $\zeta K = K$ that K is conjugacy and inverse closed. Hence K is a normal subgroup in G .

The normal subgroup $\phi(K)$ in $G \otimes G$ satisfies

$$(\phi(K))(1) = K = \mathcal{B}(1).$$

Thus $\psi(K) = \mathcal{B}$, the mapping ψ is surjective. Also, every imprimitivity block system $\mathcal{B} \in \mathcal{S}$ is normal, relative to the normal subgroup $\phi(\mathcal{B}(1))$ in $G \otimes G$.

It is obvious that ψ induces an isomorphism of the respective lattices. ■

Corollary 9.2 *The center $Z(G)$ is maximal among the normal subgroups H in G inducing a semiregular action of $\phi(H)$ on G . (i.e. without fixed points).*

Note that Theorem 9.1 holds as well in case "GDRR" is substituted with "quasiabelian Cayley graph", provided \mathcal{S} denotes the set of imprimitivity block systems for the action of $L(G)R(G)\langle\zeta_G\rangle$ on $V(\Gamma)$. Compare also Theorem 9.1 with the following Thévenaz' result on subgroups of direct products [70, Lemma 1.1]. For results on lattices of normal subgroups of groups see Schmidt's book [65, Section 9.1].

Proposition 9.3 (Thévenaz) *Let $\phi: G \rightarrow H$ be a group isomorphism and $\Delta_\phi = \{(g, \phi(g)) \mid g \in G\}$ be the graph of ϕ . Then the lattice of subgroups of $G \times H$ containing Δ_ϕ is isomorphic to the lattice of normal subgroups in G . In particular Δ_ϕ is maximal if and only if G is simple (and hence H too).*

We omit the proof of the following corollary of Theorem 9.1.

Corollary 9.4 *Let Γ be a GDRR of a group $G \neq (\mathbb{Z}_2)^n$. Then the following are equivalent:*

- (a) $\text{Aut } \Gamma$ is primitive on $V(\Gamma)$;
- (b) G is simple;
- (c) $\text{Inn } G \times \langle \zeta \rangle$ is maximal in $\text{Aut } \Gamma$;
- (d) $\text{Inn } G$ is maximal in $L(G)R(G)$.

REMARK. If Γ is a GDRR of G and $\text{Aut } \Gamma$ is primitive on $V(\Gamma)$ then $\text{Aut } \Gamma \simeq G \wr S_2$. Relative to the classification of primitive permutation groups into eight disjoint classes by Praeger [59], the action of the automorphism group $\text{Aut } \Gamma$ on the vertices of Γ is of *simple diagonal type (SD)*.

Let us end this section with a result on transitive normal subgroups in the automorphism group of a GDRR.

Proposition 9.5 *Suppose H is a transitive normal subgroup in the action of the central closure $G \otimes G$ on G . Then the quotient $(G \otimes G)/H$ is abelian.*

PROOF. The transitivity of H implies

$$G = \{x(1) \mid x \in H\} = \{ab^{-1} \mid ((a, b)) \in H \text{ or } ((a, b))\zeta \in H\}. \quad (5)$$

We claim that

$$\{((g, g^{-1})) \mid g \in G\} \subseteq H. \quad (6)$$

Indeed, $[x, \zeta] = x\zeta x^{-1}\zeta^{-1} \in H$ for every $x \in H$, so by writing $x = ((a, b))$ or $x = ((a, b))\zeta$ one gets

$$((ab^{-1}, ba^{-1})) \in H.$$

Our claim follows from (5).

For arbitrary $a, b \in G$ let $x = ((a, 1)) \in G \otimes G$ and $y = ((b, b^{-1})) \in H$. Then $[x, y] = (([a, b], 1)) \in H$, and also $\zeta[x, y]\zeta^{-1} = ((1, [a, b])) \in H$. So the commutator subgroup

$$[G \curlywedge G, G \curlywedge G] = \{((h, k)) \mid h, k \in [G, G]\}$$

is contained in H .

To prove the commutativity of the quotient $(G \otimes G)/H$ it thus suffices to check that ζH commutes with every coset $((a, b))H$, where $a, b \in G$:

$$[\zeta H, ((a, b))H] = \zeta((a, b))\zeta^{-1}((a, b))^{-1}H = ((ba^{-1}, ab^{-1}))H = H,$$

the last equation following from (6). ■

10 Decomposition of the central closure of a group

The following result generalizes the fact that the dihedral group D_{2n} is decomposable if and only if n is twice an odd number.

Proposition 10.1 *These are equivalent:*

- (a) $G \otimes G$ is decomposable;
- (b) $G \otimes G$ has a direct factor isomorphic with \mathbb{Z}_2^k ;
- (c) G has a direct factor isomorphic with \mathbb{Z}_2^k .

PROOF. The Proposition is obvious in case G is elementary abelian of exponent 2. So we proceed with $G \not\cong \mathbb{Z}_2^n$.

Suppose $G \otimes G = HK$ is a nontrivial direct product. Then $\zeta = xy$ for some $x \in H, y \in K$. We may assume that $x = ((c^{-1}, d^{-1}))$, $y = ((c, d))\zeta$ for some $c, d \in G$. We will show that H is elementary abelian of exponent 2.

One infers from $xy = yx$, $\zeta^2 = 1$ and $H \cap K = 1$ that

$$c = dz, \quad z \in Z(G), \quad z^2 = 1, \quad d^2 \in Z(G).$$

For every $h = ((a, b)) \in H \cap (G \curlywedge G)$, where $a, b \in G$, the commutativity $hy = yh$ gives $a = dbd^{-1}e$ for some $e \in Z(G)$, $e^2 = 1$. In particular,

$$a^2 = db^2d^{-1} \quad \text{for all } ((a, b)) \in H \cap (G \curlywedge G). \quad (7)$$

The normality of $H \cap (G \vee G)$ implies that $((gag^{-1}, b)) \in H \cap (G \vee G)$ for all $g \in G$. By (7) one gets $ga^2g^{-1} = db^2d^{-1} = a^2$, so the element a^2 belongs to the center $Z(G)$, and from (7) again also $a^2 = b^2$. We conclude $h^2 = 1$ for all $h = ((a, b)) \in H \cap (G \vee G)$. So $H \cap (G \vee G)$ is elementary abelian of exponent 2.

On the other hand, for every $h = ((a, b))\zeta \in H \setminus (G \vee G)$ one proves, after conjugating h by $((a, 1))\zeta$, that $((ab, 1))\zeta \in H$. The latter commutes with $y \in K$, giving $ab \in Z(G)$ and $(ab)^2 = 1$. So $h^2 = ((ab, ba)) = 1$. Hence H is elementary abelian of exponent 2. We proved the implication (a) \Rightarrow (b). Note that, since H is an abelian direct factor, we have $H \leq Z(G \otimes G)$, whence $H \leq L(G)$ by Lemma 7.7. So $L(G) = H(K \cap L(G))$ is a direct product, i.e. G has a direct factor isomorphic with \mathbb{Z}_2^k , proving (b) \Rightarrow (c).

If $G \simeq \mathbb{Z}_2^k \times K$, then one easily checks

$$G \otimes G \simeq \mathbb{Z}_2^k \times (K \otimes K),$$

proving the implication (c) \Rightarrow (a,b). ■

Corollary 10.2 *Suppose Γ is a GDRR having a nontrivial cartesian decomposition with at least two nonisomorphic factors. Then Γ has a cartesian factor which is (isomorphic with) a GRR of \mathbb{Z}_2^k .*

11 Further problems

We end this paper by formulating some problems and questions.

Problem 11.1 *Find more examples of nonisomorphic groups having isomorphic canonical central squares.*

Problem 11.2 *If the canonical central squares of the groups G and H are isomorphic, is every quasiabelian Cayley graph of G isomorphic to a quasiabelian Cayley graph of H ? (That is, is the assumption " $G/Z(G)$ centrally indecomposable" in Proposition 6.3 obsolete?)*

Problem 11.3 *The Cayley graph $\text{Cay}(S_4, S)$ from Section 3, example (f), is a GDRR which is not isomorphic with a Cayley graph of an abelian group. Find the smallest graph relative to this property.*

The original Problem 3 of Wang and Xu [72] becomes a challenge in the context of directed graphs. Let us call a Cayley digraph D of a group G a *digraphical doubly regular representations (DDRR)* of the group G if $\text{Aut } D = L(D)R(D)$.

Problem 11.4 (Wang, Xu) *For which groups G there exists a DDRR of G , that is, a Cayley digraph of G with property $\text{Aut } D = L(G)R(G)$.*

Of course, a necessary condition for a group G (which is not elementary abelian group of exponent 2) to have a DDRR is that not all conjugacy classes in G are inverse-closed. For instance, if $m \geq 5$ is odd, the conjugacy class $S \subseteq Q_{4m}$ defined in the paragraph following Proposition 6.1 is not closed for taking inverses. One can easily verify that in this case, the Cayley digraph Δ defined in (2) is a DDRR of Q_{4n} .

Infinite digraphical regular representations have been also considered [5] and revisited recently [51, 53].

Problem 11.5 *Analyze infinite GDRRs and DDRRs.*

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