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# HIGHLY ARC-TRANSITIVE DIGRAPHS WITH NO HOMOMORPHISM ONTO $\mathbb{Z}$

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**Abstract**

In an infinite digraph $D$, an edge $e'$ is *reachable* from an edge $e$ if there exists an alternating walk in $D$ whose initial and terminal edges are $e$ and $e'$. Reachability is an equivalence relation and if $D$ is 1-arc-transitive, then this relation is either universal or all of its equivalence classes induce isomorphic bipartite digraphs. In [1, Question 1.3], Cameron, Praeger and Wormald asked if there exist highly arc-transitive digraphs (apart from directed cycles) for which the reachability relation is not universal and which do not have a homomorphism onto the two-way infinite directed path (a Cayley digraph of $\mathbb{Z}$ with respect to one generator). Evans [2] gave an affirmative answer by constructing a locally infinite example.

For each odd integer $n \geq 3$, a construction of a highly arc-transitive digraph without property $Z$ satisfying the additional properties that its in- and out-degrees are equal to 2 and that the reachability equivalence classes induce alternating cycles of length $2n$, is given. Furthermore, using the line digraph operator, digraphs having the above properties but with alternating cycles of length 4 are obtained.

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# 1 Introduction

We adopt the terminology of [1]. Let $D$ be a digraph, consisting of a set $V(D)$ of vertices and a set $E(D) \subseteq V(D) \times V(D)$ of edges. For a non-negative integer $s$, an *$s$-arc* in $D$ is a sequence $(x_0, \dots, x_s)$ of $s+1$ vertices of $D$ such that, for each $i \in \{0, \dots, s-1\}$ the pair $(x_i, x_{i+1})$ is an edge of $D$, and for each $i \in \{1, \dots, s-1)$, $x_{i-1} \neq x_{i+1}$. We say that $D$ is *$s$-arc-transitive* if its automorphism group $\operatorname{Aut} D$ is transitive on the set of $s$-arcs, and that $D$ is *highly arc-transitive* if it is $s$-arc-transitive for all $s \geq 0$. Note that highly arc-transitive digraphs, with the exception of all directed or two-way directed cycles, are infinite.

An *alternating walk* in $D$ is a walk such that every internal vertex is either the head or the tail of both of its incident edges. Let $e, e' \in E(D)$. The edge $e'$ is *reachable* from $e$ if there exists an alternating walk in $D$ whose initial and terminal edges are $e$ and $e'$. Clearly, reachability is an equivalence relation, which will be denoted by $\mathcal{A}_D$ or just $\mathcal{A}$ when the digraph in question is clear from the context. Let $\mathcal{A}(e)$ denote the equivalence class containing the edge $e \in E(D)$. If $D$ is 1-arc-transitive, then either $\mathcal{A}$ is universal or all of the digraphs $\langle \mathcal{A}(e) \rangle$, $e \in E(D)$, induced by the equivalence classes of $\mathcal{A}$ are isomorphic to a fixed bipartite digraph, denoted by $\Delta(D)$ (see [1, Proposition 1.1]). Given a fixed connected 1-arc-transitive bipartite digraph $\Delta$, we let $\mathcal{D}(\Delta)$ denote the class of all connected 1-arc-transitive digraphs $D$ for which $\Delta(D)$ is isomorphic to $\Delta$.

A digraph has *property $Z$* if it has a digraph homomorphism onto the two-way infinite path $Z$, that is, the digraph with vertex set $\mathbb{Z}$ and edges of the form $(i, i+1)$, $i \in \mathbb{Z}$. Note that property $Z$ is equivalent to having only *balanced* cycles, that is, cycles having the same number of forward and backward edges. It was proved in [1, Lemma 3.2(b)] that a 1-arc-transitive digraph $D$, with $\mathcal{A}_D$ universal, does not have property $Z$. A lack of further examples suggests the following question posed in [1, Question 1.3].

**Question 1.1** *Does there exist a highly arc-transitive digraph $D$ (apart from cycles) in $\mathcal{D}(\Delta)$ for some bipartite digraph $\Delta$, without property $Z$?*

An affirmative answer to this question was given by Evans [2], who constructed an example of such a graph with out-degree 2 and infinite in-degree.

It is the purpose of this paper to construct an infinite familly of locally finite highly arc-transitive digraphs without property $Z$. These are, to the best of our knowledge, the first such examples. More precisely, for each odd integer $n \geq 3$ we give a construction of a 4-valent digraph (with in- and out-degrees 2) satisfying the properties required in Question 1.1 and belonging to $\mathcal{D}(\Delta)$, where $\Delta$ is the alternating cycle of length $2n$.

The construction is suggested by Figure 1, where the case $n = 5$ is shown.

To give an intuitive recursive definition of this digraph, start with an alternating cycle $C$ of length $2n$. At each pair of antipodal vertices $u, v$ of $C$, an alternating cycle $C_{u,v}$ of the same length is glued to $C$, in such a way that $u$ and $v$ are antipodal on $C_{u,v}$, too, and that their in- and out-degrees equal 2. In this way, every vertex on $C$ attains valency 4. The process is recursively repeated in each of the new alternating cycles and at each pair of antipodal vertices of valency 2.

It can be seen that any two adjacent alternating cycles in the above construction intersect in precisely two antipodal vertices. By taking one half of each of these two alternating cycles, we obtain an unbalanced cycle. Therefore the constructed digraph does not have property $Z$.

A more precise definition of the above digraphs is given in Section 2. Their high arc-transitivity is proved in Theorem 3.3. Furthermore, using the line digraph operator, digraphs having the above properties but with alternating cycles of length 4 are obtained (Corollary 3.4).

## 2 Formal constructions

We are now going to formalize the intuitive definition from Section 1. There are two ways of doing it; each of these has its advantages and disadvantages (which is why both are introduced here).

If we endow the arcs in Figure 1 with labels $R$ and $L$ as shown in Figure 2, the alternating cycles of length $2n$ give rise to the "relation" $(RL^{-1})^n$. The cycles of length $2n$, consisting of two halfs of adjacent alternating cycles (these are the unbalanced cycles mentioned in Section 1), lead to the "relation" $((RL^{-1})^{(n-1)/2}R)^2$.

This suggests that the digraphs given intuitively in the introduction could be defined in the following way. First, for every odd positive integer $n \geq 3$ let $G_n$ be the group with the presentation

$$G_n = \langle L, R \mid (RL^{-1})^n,\ ((RL^{-1})^{(n-1)/2}R)^2 \rangle,$$

and let $X_n$ be the Cayley digraph $\mathrm{Cay}(G_n, \{L, R\})$. It follows immediately from the defining relations that $X_n$ contains unbalanced cycles and therefore it does not have property $Z$. As for the proof that $X_n$ indeed corresponds to the given intuitive picture and that it is highly arc-transitive, it is of crucial importance that the intersection graph of alternating cycles (that is, the graph whose vertices are alternating cycles of $X_n$ with adjacency defined by nontrivial intersection) is a tree. To this end we now introduce the second formalization, from which the underlying tree-like structure is transparent.

As a starting point we take the infinite $n$-valent (undirected) tree $T_n$ as the Cayley graph of the free product of $n$ copies of $\mathbb{Z}_2$,

$$F_n = \langle a_0 \rangle * \cdots * \langle a_{n-1} \rangle, \quad a_i^2 = 1, \ i \in \mathbb{Z}_n,$$

with respect to the generating set $A^{(n)} = \{a_0, \ldots, a_{n-1}\}$. Next we truncate $T_n$, that is, we replace each vertex by an $n$-cycle in a natural way according to the cyclic order of the generating set $A^{(n)}$, obtaining a cubic graph $T_n^*$. As a second step, we construct the directed tensor product $T_n^* \otimes \vec{K}_2$ of $T_n^*$ by the directed digraph $\vec{K}_2$ with two vertices and one edge, that is, the (directed) canonical double cover of $T_n^*$, where all edges are oriented from level 0 to level 1. Observe that the $n$-cycles in $T_n^*$ have been transformed into alternating cycles of length $2n$ in the digraph $T_n^* \otimes \vec{K}_2$. Finally, the digraph $Y_n$ is obtained by contracting all the edges in $T_n^* \otimes \vec{K}_2$ arising from edges in $T_n$. Of course, the digraph $Y_n$ has valency 4 with in-degree and out-degree 2. By construction, the intersection graph of alternating cycles in $Y_n$ is isomorphic to the tree $T_n$. Similarly, by construction $Y_n$ corresponds to the intuitive picture. In Proposition 3.2 we prove that $X_n$ and $Y_n$ are isomorphic.

Note that every vertex $v$ in $T_n$ may be written in a canonical form as a reduced word in the alphabet $A^{(n)}$ with all exponents equal to 1 (this will be referred to as the *canonical expression of* $v$). After the truncation, every vertex $v$ in $T_n$ gives rise to the cycle $v_0 v_1 \ldots v_{n-1}$ in such a way that each vertex $v_i$ is adjacent to $(va_i)_i$. This cycle lifts to the alternating cycle $C_v = v_0^+ v_1^- v_2^+ \ldots v_{n-1}^+ v_0^- v_1^+ v_2^- \ldots v_{n-1}^-$ in the tensor product $T_n^* \otimes \vec{K}_2$ with the (directed) edges $(v_i^+, v_{i+1}^-)$ and $(v_i^+, v_{i-1}^-)$ for all $i \in \mathbb{Z}_n$. Finally, the vertices $v_i^\varepsilon$ and $(va_i)_i^{-\varepsilon}$ of $T_n^* \otimes \vec{K}_2$ are identified in $Y_n$ for all $v \in T_n$, $i \in \mathbb{Z}_n$, $\varepsilon \in \{+, -\}$. The vertices of $Y_n$ will be denoted by $[v_i^\varepsilon] = \{v_i^\varepsilon, \ (va_i)_i^{-\varepsilon}\}$.

A thorough analysis of the digraphs $X_n$ and $Y_n$ leading to an affirmative answer to Question 1.1 is done in the subsequent section.

## 3 High arc-transitivity

For each vertex $v \in V(T_n)$ let $\bullet v$ denote the vertex obtained from $v$ by replacing $a_i$ with $a_{i+1}$, for each $i \in \mathbb{Z}_n$, in the canonical expression for $v$. Note that the $\bullet$-operator is an automorphism of $T_n$ fixing id and rotating its neighbors. To each element $f \in F_n$ we associate a well defined permutation $\phi_f$ of the vertex set of $Y_n$ according to the rule

$$[v_i^\varepsilon] \cdot \phi_f = [(fv)_i^\varepsilon], \quad i \in \mathbb{Z}_n, \ \varepsilon \in \{+,-\}.$$

Further, let $\rho$ denote a well defined permutation on $V(Y_n)$ mapping according to the formula

$$[v_i^\varepsilon] \cdot \rho = [(\bullet v)_{i+1}^\varepsilon], \quad i \in \mathbb{Z}_n, \ \varepsilon \in \{+,-\}.$$

Obviously, $\bar{F}_n = \{\phi_f \mid f \in F_n\}$ is a group isomorphic to $F_n$. We now identify $Y_n$ as a Cayley graph.

**Lemma 3.1** *The group $H_n = \langle \bar{F}_n, \rho \rangle$ is a subgroup of $\mathrm{Aut}\, Y_n$ and acts regularly on the vertex set of $Y_n$.*

PROOF. To prove that $\phi_f$, $f \in F_n$, is an automorphism of $Y_n$, take an edge $e$ of $Y_n$. Then either $e = [v_i^+][v_{i-1}^-]$ or $e = [v_i^+][v_{i+1}^-]$. Since

$$[v_i^+] \cdot \phi_f = [(fv)_i^+], \quad [v_{i-1}^-] \cdot \phi_f = [(fv)_{i-1}^-], \quad [v_{i+1}^-] \cdot \phi_f = [(fv)_{i+1}^-],$$

the image $e\phi_f$ is indeed an edge in $Y_n$. Therefore $\phi_f \in \mathrm{Aut}\, Y_n$. The proof that $\rho$ is an automorphism of $Y_n$ is done in a similar manner and is left to the reader.

To prove regularity of $H_n$, denote the orbits of $\bar{F}_n$ by $B_i = \{[v_i^+] \mid v \in V(T_n)\} = \{[v_i^-] \mid v \in V(T_n)\}$, $i \in \mathbb{Z}_n$. Clearly, $\bar{F}_n$ acts regularly on each of its orbits. Since $\bar{F}_n$ is normal in $H_n$ and $\rho$ is a permutation of order $n$ cyclically permuting the orbits $B_0, \ldots, B_{n-1}$, it follows that $H_n$ acts regularly on $V(Y_n)$. ∎

In view of Lemma 3.1, $Y_n$ is a Cayley digraph of the group $H_n$. In fact, the following result holds.

**Proposition 3.2** *There is a group isomorphism between $G_n$ and $H_n$ giving rise to a graph isomorphism $X_n \cong Y_n$.*

Proof. Let $\phi_i = \phi_{a_i}$, $i \in \mathbb{Z}_n$. Recalling that $n$ is odd, set $k = (n-1)/2$ and define

$$r = \phi_0 \rho^{k+1}, \quad l = \phi_0 \rho^k.$$

Since $\rho = l^{-1} r$ and $\phi_i = \rho^{-1} \phi_{i-1} \rho$, we have $\langle r, l \rangle = H_n$.

Choosing the vertex $[\mathrm{id}_0^+]$ in $Y_n$, where $\mathrm{id} \in F_n$, we have that $r$ and $l$, respectively, map it to its two successors $[\mathrm{id}_1^-]$ and $[\mathrm{id}_{2k}^-]$ and so $Y_n = \mathrm{Cay}(\langle r, l \rangle \mid \{r, l\})$.

A short computation shows that $(rl^{-1})^n = 1$ and $((rl^{-1})^{(n-1)/2} r)^2 = 1$. One infers that $H_n$ is a homomorphic image of $G_n$. However, the two relations in $r$ and $l$ which hold in $H_n$ are actually the defining relations for $H_n$, thereby implying that $H_n$ is in fact isomorphic to $G_n$. To this end we have to see that any relation in $H_n$ is a consequence of either the two given relations or the group axioms. Assume that some relation $S$ is a counterexample of minimal length. Clearly, $S$ gives rise to a reduced closed walk $W$ in $Y_n$. Consider the sequence $\mathcal{C} = C_0 C_1 \ldots C_{m-1}$ of alternating cycles in $Y_n$ in which the maximal alternating subwalks of $W$ are contained (in this order). If $m = 1$ then $S$ is a multiple of one basic relation, and therefore not a counterexample. Hence $m > 1$. We may assume that any two consecutive cycles $C_i, C_{i+1}$, $i \in \mathbb{Z}_m$, are distinct (but all cycles are not necessarily pairwise distinct). The sequence $\mathcal{C}$ determines a walk in the tree $T_n$. So there exists some $j \in \mathbb{Z}_m$ such that $C_{j-1} = C_{j+1}$. Let $w_1$ and $w_2$ be the two antipodal vertices where $C_{j-1}$ and $C_j$ are glued together. Since the relation $S$ is reduced, we can assume that $W$ enters $C_j$ at $w_1$ and leaves it at $w_2$. The walk $W$ leaves $C_{j+1}$ at some vertex $v \neq w_2$. Consider the part of $W$ which contains the alternating path from $w_1$ to $w_2$ in $C_j$ (this path has length at least $n$) and the path in $C_{j+1}$ from $w_2$ to $v$. Substituting this part by the shorter of the two paths in $C_{j+1}$ which connect $w_1$ to $v$, we obtain a closed walk $W'$ in $Y_n$ shorter than $W$. Also, $W'$ gives rise to a relation $S'$ in $H_n$ which is not a consequence of either the two given relations or of the group axioms, contradicting the minimality of $S$.

We have thus shown that the mapping $R \mapsto r$, $L \mapsto l$ defines an isomorphism from $G_n$ to $H_n$. Moreover, this mapping extends to a graph isomorphism of the respective Cayley graphs $X_n$ and $Y_n$. ■

**Theorem 3.3** *For each odd $n \geq 3$ the Cayley digraph $X_n \cong Y_n$ is highly arc-transitive and does not have property Z.*

PROOF. The fact that $Y_n$ does not possess property $Z$ has already been discussed in Section 2. It remains to show that $Y_n$ is $s$-arc-transitive for any $s \geq 0$. We adopt an inductive argument. By Lemma 3.1, $Y_n$ is 0-arc-transitive. Assume it is $s$-arc-transitive and prove that it is $(s+1)$-arc-transitive. It suffices to prove that there exists an automorphism of $Y_n$, which fixes an $s$-arc with terminal vertex $[(\mathrm{id})_0^+] = [(a_0)_0^-]$ and exchanges the successors $[(\mathrm{id})_1^-]$ and $[(\mathrm{id})_{n-1}^-]$.

Given an arbitrary vertex $v$ of the tree $T_n$, let $a_{i_1} a_{i_2} \cdots a_{i_t}$ be its canonical expression. We define

$$\mathrm{beg}(v) = a_{i_1}$$

and

$$\updownarrow v = a_{-i_1} a_{-i_2} \cdots a_{-i_t}.$$

Let $\alpha$ be the permutation of $V(Y_n)$ mapping according to the rule

$$[v_i^\varepsilon]\alpha = \begin{cases} [v_i^\varepsilon], & \text{if } \mathrm{beg}(v) = a_0; \\ [(\updownarrow v)_i^\varepsilon], & \text{if } \mathrm{beg}(v) \neq a_0, \end{cases} \qquad \text{for each } i \in \mathbb{Z}_n.$$

That $\alpha$ is an automorphism of $Y_n$ is an immediate consequence of the fact that the intersection graph of the alternating cycles of $Y_n$ is a tree. ■

If $D$ is a digraph, then the *line digraph* $L(D)$ has vertex set $E(D)$ and (directed) edges of the form $ee'$, where the latter gives rise to a 2-arc in $D$. It was proved in [1, Lemma 4.1(a)] that $L(D)$ is highly arc-transitive if $D$ is highly arc-transitive. Considering $X_n$ (or $Y_n$) we immediately see that $L(X_n)$ has alternating cycles of length 4. Also, it was shown in [3, Proposition 1] that $L(D)$ has property $Z$ if and only if $D$ has the same property. This gives the following result.

**Corollary 3.4** *For each $n \geq 3$ the digraphs $L(Y_n)$ are highly arc-transitive digraphs with alternating cycles of length 4 and without property $Z$.*

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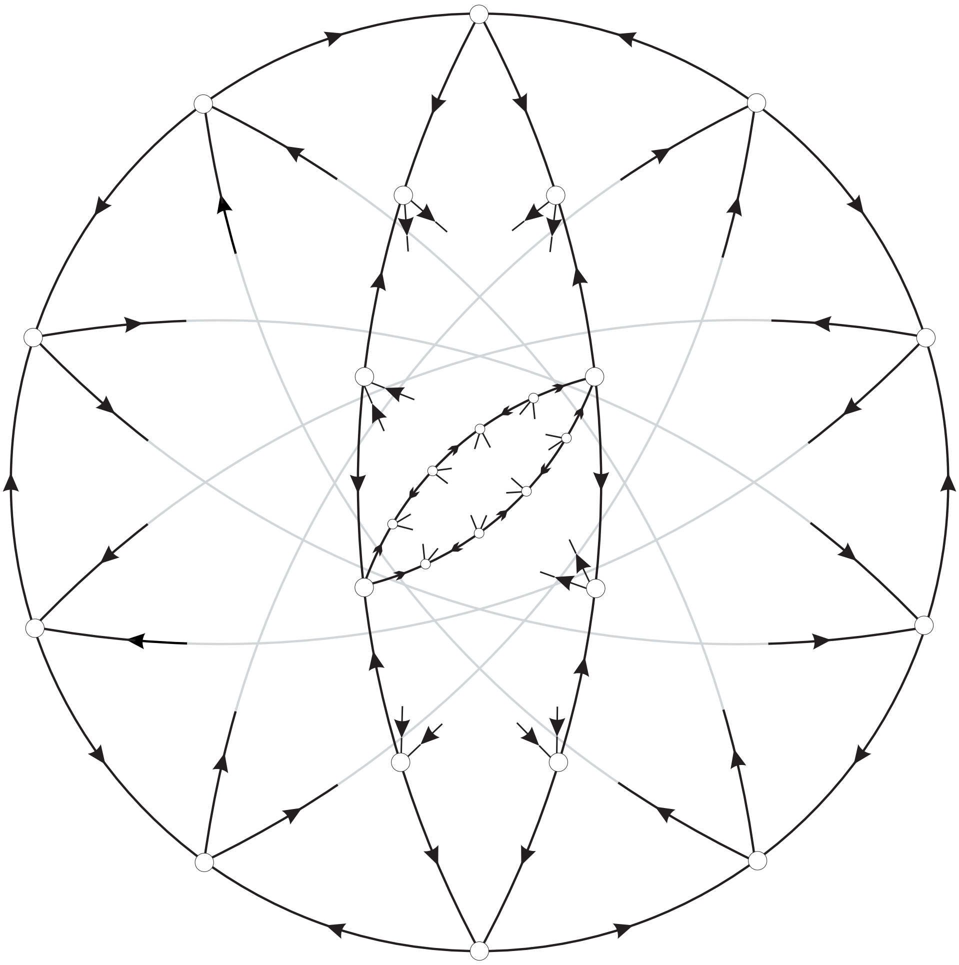

Figure 1: The construction of a highly arc-transitive digraph without property $Z$ and having alternating cycles of length $2n$, for $n = 5$.

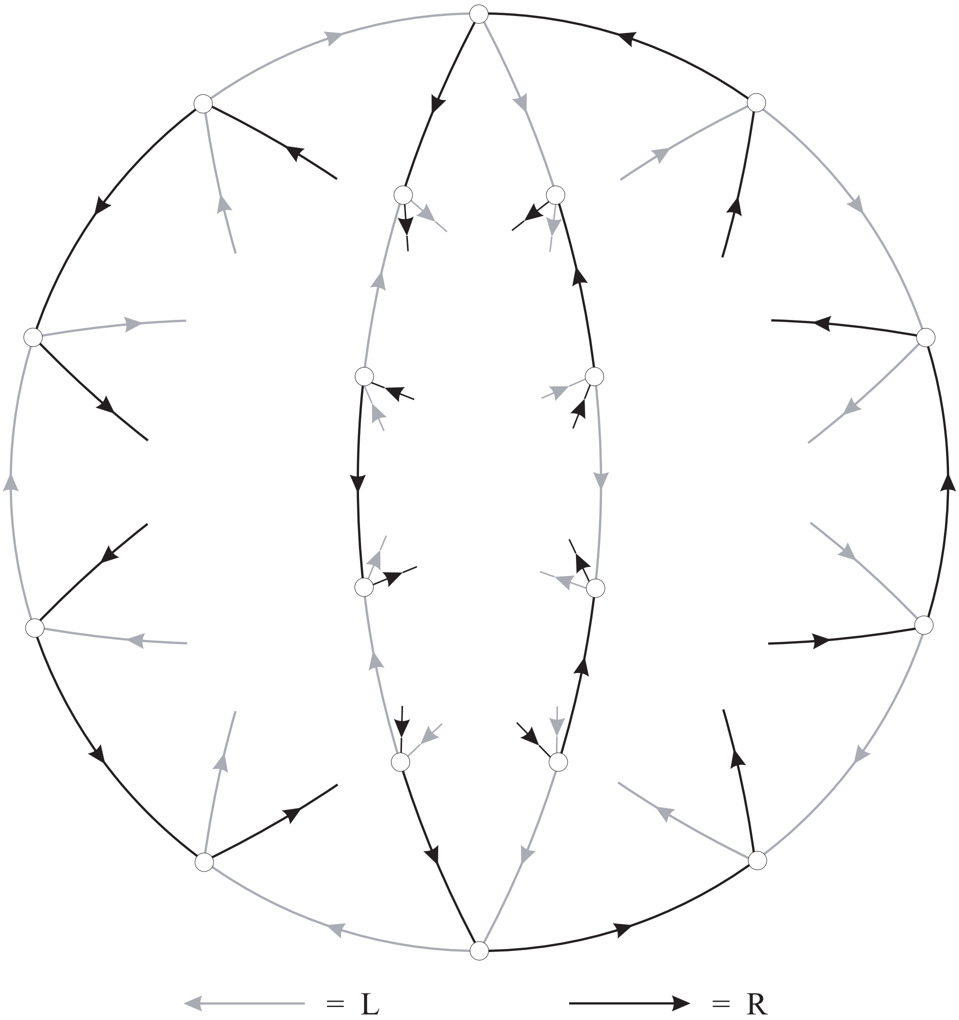


Figure 2: The representation of the digraph in Figure 1 as a Cayley digraph $X_5$.