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CYCLIC HAAR GRAPHS

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Abstract

For a given group Γ with a generating set A, a dipole with $|A|$ parallel directed edges labeled by elements of A gives rise to a voltage graph whose covering graph, denoted by $H(\Gamma, A)$ is a bipartite, regular graph, called a *bi-Cayley graph*. In the case when Γ is abelian we refer to $H(\Gamma, A)$ as a *Haar graph* of Γ with respect to the symbol A. In particular for Γ cyclic the above graph is referred to as a *cyclic Haar graph*. A basic theory of cyclic Haar graphs is presented.

1 Introduction

All graphs and groups in this paper are assumed to be finite. Recall that with a given group Γ and a subset A of $\Gamma \setminus \{1\}$ the *Cayley graph* $Cay(\Gamma, A)$ *of* G *with respect to* A is the graph with vertex set A and edges of the form $[g, ga]$, for all $g \in G$ and $a \in A$. One can view $Cay(\Gamma, A)$ as a covering graph over a bouquet of $|A|$ circles and semi-edges, the latter corresponding to involutions in A. In a similar way, taking a dipole D with $|A|$ parallel directed edges labeled by elements of A one obtains a voltage graph whose covering graph, denoted by $H(\Gamma, A)$ is a bipartite, regular graph, sometimes called a *bi-Cayley graph*. In the case when Γ is abelian we shall refer to $H(\Gamma, A)$ as a *Haar graph* of Γ with respect

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to the symbol A. In particular for Γ cyclic the above graph will be referred to as a *cyclic Haar graph.* The name Haar graph comes from the fact that the Schur norm of the corresponding adjacency matrix can be easily evaluated via the so called Haar integral on abelian Γ (see [12, 13]). Namely,

$$(1) \qquad ||H(\Gamma, A)|| := (1/|\Gamma|) \sum_{\gamma \in \Gamma} |\sum_{\delta \in A} \omega^{\gamma\delta}|,$$

where Γ is written in the standard form $\Gamma = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_s}$, with $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, $\delta = (\delta_1, \delta_2, \ldots, \delta_s)$, and $\omega^{\gamma\delta} = \omega_1^{\gamma_1\delta_1}\omega_2^{\gamma_2\delta_2} \ldots \omega_s^{\gamma_s\delta_s}$, for $\gamma_j, \delta_j \in \mathbb{Z}_{k_j}$, and each ω_j being the primitive root of unity: $\omega_j^{k_j} = 1$.

To each natural number n, a cyclic Haar graph $H(n)$, the so called *Haar graph of* n may be associated in the following way. Let $k = k(n) = 1 + \lfloor \log_2 n \rfloor$ denote the number of binary digits of an integer n and let $b(n) = (b_{k-1}, b_{k-2}, ..., b_1, b_0)$ be the binary vector consisting of the k binary digits of n with $b_{k-1} = 1$. The vector $b(n)$ can be viewed as a characteristic vector of the set $B = B(n) \subseteq \mathbb{Z}_k$ consisting of all i such that $b_i = 1$. The number $k = k(n)$ is called the *binary length* and the set $B(n)$ is called the *symbol* of n. Hence $H(n) = H(\mathbb{Z}_k, B)$ in the notation of the previous paragraph. Furthermore, the use of the same symbol $H(n)$ to denote its adjacency matrix should cause no confusion.

In the case of cyclic Haar graphs $H(n)$ the Schur norm formula (1) reduces to

$$(2) \qquad ||H(n)|| := (1/k) \sum_{j=0}^{k-1} |p(\omega^j)|,$$

where $p(\lambda) := \sum_{i=0}^{k-1} b_i \lambda^i$ and ω is a primitive root of unity: $\omega^k = 1$. Note that $p(2) = n$ (see [13] for details).

On the other hand, cyclic Haar graphs are interesting as they can be regarded as a generalization of bipartite circulant graphs.

The fact that the full information about the graph $H(n)$ is encoded in the natural number n makes the class of graphs worth investigating, in particular, as the connection between n and $H(n)$ is natural via binary representation of n.

We use standard graph-theoretic notation: say, C_n is the *cycle* on n vertices, K_n, the *complete graph* on n vertices, $K_{m,n}$ the *complete bipartite graph* with one part of size m and the other of size n, Π_n is the *graph of n-sided prism*, and M_n is the *Möbius ladder* graph on $2n$ vertices.

2 Examples of cyclic Haar graphs and an Adam-like conjecture

Some well-known families of graphs contain Haar graphs. For instance, even prisms Π_{2n+2} can be obtained as $H(7 \cdot 2^{2n-1}) = H(2^{2n+1} + 3)$ and odd Möbius ladders M_{2n+3} are obtained as $H(7 \cdot 2^{2n}) = H(2^{2n+2}+3)$. This is the best that we can hope for as odd prisms and even Möbius ladders are not bipartite. Complete bipartite graphs $K_{n,n}$ are isomorphic to $H(2^n - 1)$ and even cycles C_{2n+4} are isomorphic to $H(3 \cdot 2^n) = H(2^{n+1} + 1)$.

It is obvious that the binary string defining a Haar graph can be shifted or even reversed (as long as we get 1 in the first position) and the corresponding graphs will remain isomorphic. This means that distinct natural numbers $n \neq m$ may give rise to isomorphic Haar graphs $H(m) = H(n)$.

To be more precise, one can define two equivalences. Two numbers m and n are called *Haar-equivalent* if and only if the corresponding Haar graphs $H(m)$ and $H(n)$ are isomorphic. On the other hand m and n are called *cyclically equivalent* if and only if $k(m) = k(n) = k$ and there exist $s \in \mathbb{Z}_k^*$ and $t \in \mathbb{Z}_k$ such that $B(n) = sB(m)+t$. It is obvious that cyclic equivalence implies Haar equivalence. The question whether the converse is true is a "bi-circulant" analog of the well-known Adam conjecture for circulant graphs. While it is known that Adam conjecture is false, our question remains open. For instance, the two circulant graphs $Cay(\mathbb{Z}_{16}, \{1, 2, 7, 9, 14, 15\})$ and $Cay(\mathbb{Z}_{16}, \{2, 3, 5, 11, 13, 14\})$ constitute a counter-example to the Adam conjecture, the corresponding Haar graphs

$$H(\mathbb{Z}_{16}, \{1, 2, 7, 9, 14, 15\}) = H(49798)$$

and

$$H(\mathbb{Z}_{16}, \{2, 3, 5, 11, 13, 14\}) = H(53336)$$

do not, since both 49798 and 53336 are cyclically equivalent to 33478. In fact, there are exactly 12 numbers cyclically equivalent to 33478, which is the smallest number in its equivalence class. Hence it makes sense to define for an arbitrary integer n the *canonical number*, that is, the smallest number cyclically equivalent to n.

Here are some further examples. It can be shown that $H(26) = H(19) = M_5$, the Möbius ladder on 10 vertices. (If only shifts and reversals are considered, 19 is not in the same class as 26.) It is perhaps of interest to note that there are exactly six numbers whose Haar graph is isomorphic to the Heawood graph (see Figure 2). These numbers are 69, 70, 81, 88, 98 and 104. The Möbius-Kantor graph [6, 15] or $G(8, 3)$ is $H(133)$; see Figure 2. Another interesting graph $\mathcal{T}_4 = H(137)$ depicted in Figure 2 and used in our Table 4 belongs to a general family $\mathcal{T}_n = H(2^{2n-1} + 2^{n-1} + 1)$.

3 Cyclic Haar Graphs as dihedrants and circulants

Two families of groups will be considered in this section: *cyclic* and *dihedral*. Recall that the cyclic group $\mathbb{Z}_k$ of order k has a standard presentation:

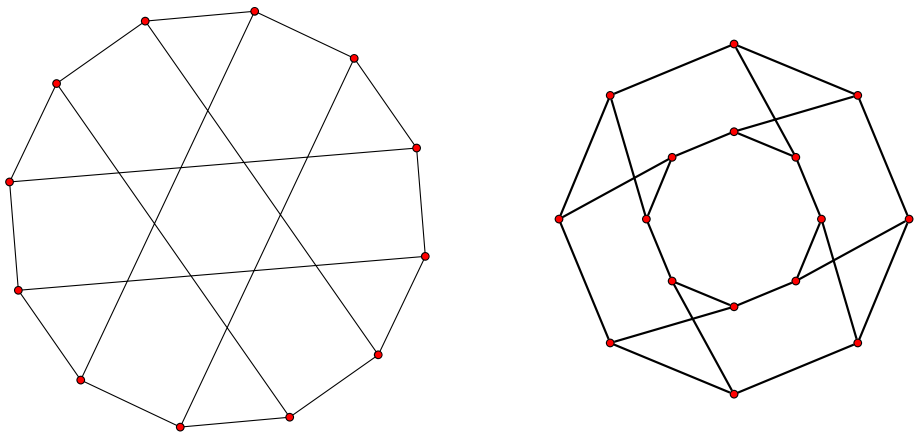

Figure 1: $H(37) = \mathcal{T}_3$ and $H(137) = \mathcal{T}_4$

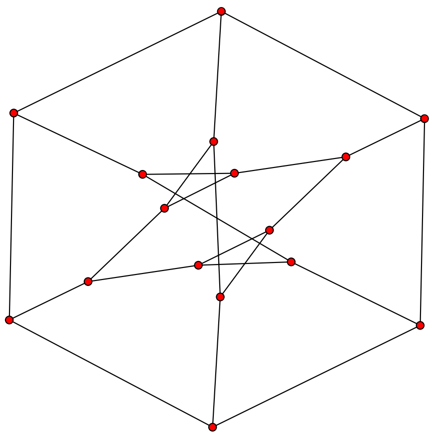

Figure 2: The Möbius-Kantor graph $G(8,3) = H(133)$

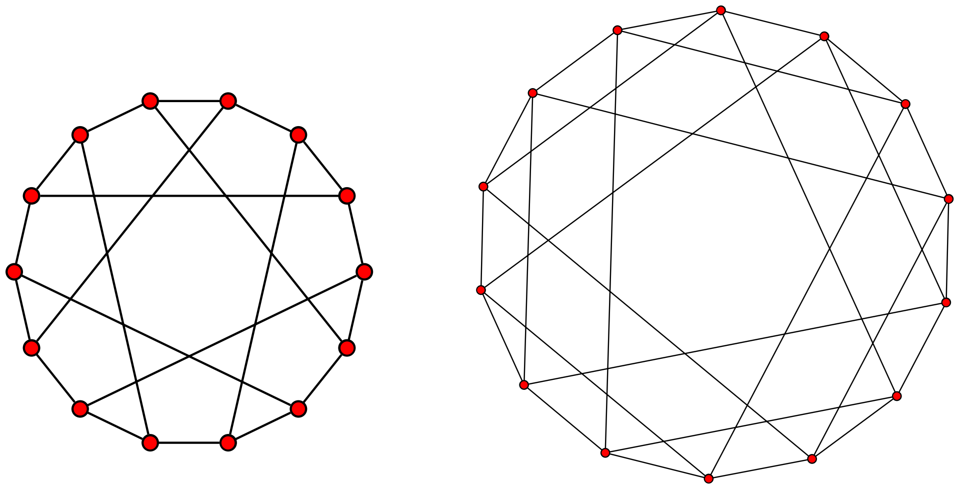

Figure 3: The Heawood graph and its bipartite complement $H(75)$

$$\mathbb{Z}_k = \langle \rho | \rho^k = 1 \rangle$$

and that the dihedral group $\mathbf{D}_k$ of order $2k$ has a standard presentation:

$$\mathbf{D}_k = \langle \sigma, \tau | \sigma^k = \tau^2 = 1, \sigma^{-1} = \tau\sigma\tau \rangle$$

See, for instance, the monograph by Coxeter and Moser [9].

Recall that a Cayley graph of the cyclic group $\mathbb{Z}_n$ of order n with respect to a subset S of $\mathbb{Z}_n$ is called a *circulant* and the set S its *symbol*. It has vertices $u_i, (i \in \mathbb{Z}_n)$ and edges of the form $u_i u_{i+s}$, for all $i \in \mathbb{Z}_n, s \in S$. The notation $Cir(n; S)$ will be used in this case.

(*** I think that this is not needed: Note that we assume that $S = -S, 0 \notin S$. In practice we omit the negatives since there is no danger of confusion.***)

For simplicity reasons a Cayley graph of the dihedral group D_n will be called a *dihedrant* with *symbol* (S, T) if it has vertices u_i, v_i, where $i \in \mathbb{Z}_n$ and edges of the form $u_i u_{i+s}$, $v_i v_{i+s}$ and $u_i v_{i+t}$ for all $i \in \mathbb{Z}_n, s \in S$, and $t \in T$. The notation $Dih(n; S, T)$ will be used in this case.

The theorem below gives a complete characterization of cyclic Haar graphs in terms of Cayley graphs of dihedral groups.

Theorem 1 *For the natural number n with the symbol $B(n)$ and the binary length $k = k(n)$ its Haar graph $H(n)$ is isomorphic to the bipartite Cayley graph $Cay(\mathbf{D}_k, X(n))$, where $X(n) = \{\sigma^i \tau | i \in B(n)\}$. In particular $H(n)$ is bipartite and vertex transitive. Conversely, any bipartite Cayley graph $Cay(\mathbf{D}_k, A)$ of a dihedral group $\mathbf{D}_k$ whose symbol A consists of reflections is isomorphic to a cyclic Haar graph.*

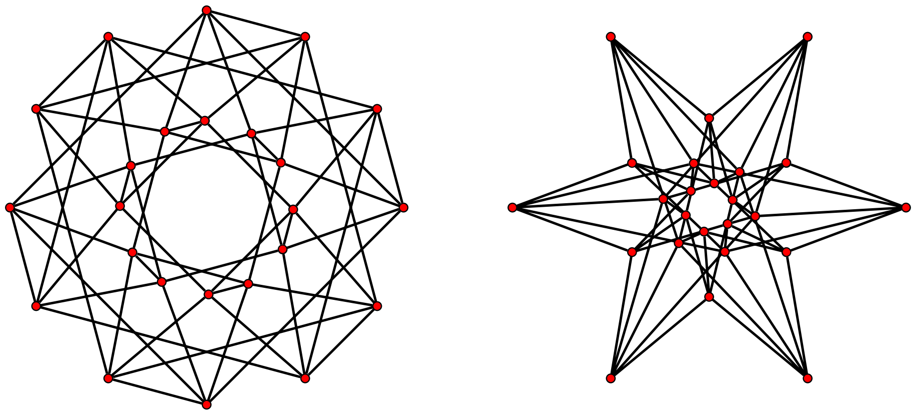

Figure 4: Two drawings of a bipartite non-Haar Cayley graph of a dihedral group $\mathbf{D}_{12}$.

PROOF.From the construction of $H(n)$ we conclude that the graph is bipartite on $2k$ vertices. Using the vertex labeling from the definition of $H(n)$ we denote by i the i-th white vertex and by j' the j-th black vertex of $H(n)$. Furthermore, the vertices i and j' are adjacent if and only if $j - i \in B(n)$. Permutations $s = (012\dots k-1)(0'1'2'\dots(k-1)')$ and $t = (0(k-1)')(1(k-2)')\dots((k-1)0')$ induce automorphisms of $H(n)$. Namely, for any edge $e = ij'$ we have $s(e) = (i+1)(j+1)'$ ane $t(e) = (k-j-1)(k-i-1)'$. Note that the group generated by s and t is isomorphic to $\mathbf{D}_k$ and clearly acts transitively on the vertex set of $H(n)$. Furthermore, the isomorphism from $H(n)$ to $Cay(\mathbf{D}_k, X(n)$ is obtained by mapping each vertex i to σ^i and each vertex j' to $\sigma^j\tau$. As all generators from $X(n)$ are involutions the valence of $H(n)$ is indeed equal to $|B(n)|$, the number of digits 1 in $b(n)$.

($\Leftarrow$) The converse is clear. ■

For instance, for $n = 26$ we get $H(26) = Cay(\mathbf{D}_5, \{\tau, \sigma\tau, \sigma^3\tau\})$. Here $B(n) = T = \{0, 1, 3\}, S = \emptyset$. Unfortunately, it is not true that each bipartite Cayley graph of a dihedral group is a cyclic Haar graph. For instance, the Cayley graph $Cay(\mathbf{D}_{12}, \{\sigma, \tau, \sigma^2\tau, \sigma^6\tau\}$ depicted in Figure 4 is 5-valent bipartite graph that is not isomorphic to any 5-valent Haar graph on 24 vertices. It has $S = \{-1, 1\}, T = \{0, 2, 6\}$. This was also checked by computer, using VEGA and *nauty*. Even more, there are Cayley graphs of dihedral group generated by involutions that are not cyclic Haar graphs. Namely, if k is even, the group $\mathbf{D}_k$ contains in addition to the reflections of the form $\sigma^i\tau$ an extra involution $\sigma^{k/2}$ which may cause problems. In particular, $Cay(\mathbf{D}_8, \{\sigma^2\tau, \sigma^4\tau, \sigma^6\tau\,\sigma^4\})$ is isomorphic to the four-cube $Q_4 = K_2 \times K_2 \times K_2 \times K_2$ which is not a cyclic Haar graph.

If $X = Dih(n; S, T)$ is bipartite then we may assume with no loss of generality that $S \subseteq 2\mathbb{Z}_n + 1$ and that T is either a subset of $2\mathbb{Z}_n$ or a subset of $2\mathbb{Z}_n + 1$, where n is even of course. Note that $S = -S$, but T need not be symmetric.

Proposition 2 *Let n be even and $X = Dih(b; S, T)$, with $S \neq \emptyset, S \subseteq 2\mathbb{Z}_n + 1, T \subseteq 2\mathbb{Z}_n$ and $T = -T$. Then X is isomorphic to a cyclic Haar graph $Dih(n; \emptyset, S \cup T)$.*

PROOF. We first observe that for any dihedrant of order $2n$ with symbol $(R, Q), Q = -Q$ the permutation $(u_0v_1u_2v_3 \ldots u_{2i}v_{2i+1} \ldots)(v_0u_1v_2u_3 \ldots v_{2i}u_{2i+1} \ldots)$ is an automorphisms. Applying this fact to X we see that the sets $W = \{u_0, v_1, u_2, \ldots\}$ and $W' = \{v_0, u_1, v_2, \ldots\}$ are orbits of an automorphism of order n and since by assumption $S \subseteq 2\mathbb{Z}_n + 1, T \subseteq 2\mathbb{Z}_n$ we have that $\{W, W'\}$ is a bipartition for X. It follows that X is a cyclic Haar graph. ■

Corollary 3 *Let n be even and let $X = Dih(n; S, T)$ with $S \neq \emptyset$ be a bipartite graph. If $S \subseteq 2\mathbb{Z}_n + 1, T \subseteq 2\mathbb{Z}_n$ and $T' = -T'$ for some even translate $T' = T + 2i$ of T then X is a cyclic Haar graph.*

The above result suggests that in order to construct bipartite dihedrants which are not cyclic Haar graphs one has to avoid those with "symmetric" symbol (S, T), those which are also Cayley graphs of abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$. A construction of an infinite family of such bipartite dihedrants which are not Haar graphs is given below.

In fact we construct an infinite family of bipartite dihedrants which are graphical regular representations of dihedral groups with respect to a generating set of the form $\{\sigma, \sigma^{-1}, \sigma^t\tau; t \in T\}$ where σ and τ are the usual generators of dihedral group

$$\mathbf{D}_{2n} =< \sigma, \tau | \sigma^n = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} >$$

To construct such graphs we choose the symbol (S, T) in such a way that $S = \{1, -1\}$, whereas T must posses certain properties limitation, the number of 4-cycles in the dihedrant with symbol $(\emptyset, T)$. This can be done in many ways. Here is one of the possibilities.

Lemma 4 *Let $0 \in T \subseteq \mathbb{Z}_n$ and $|T| \geq 3$ and no translate of T is symmetric be such that the girth of $Dih(n; \emptyset, T)$ is 6. Then the graph $Dih(n; \{1, -1\}, T)$ is a bipartite GRR for $\mathbf{D}_{2n}$ and is not a cyclic Haar graph.*

PROOF. Since by assumption $Dih(n; \emptyset, T)$ has girth 6 we have that an edge of the form u_iv_{i+t} is contained in a 4-cycle only in two possible ways:

(i) $u_iv_{i+t}v_{i+t+1}u_{i+1}$ for all t

(ii) $u_i v_{i+t} u_{i+t-t'} u_{i+1}$ for $t, t' \in T$ such that $t - t' = 2$

Clearly this means that all edges of the form $u_i u_{i+1}, v_i v_{i+1}$ are contained on more 4-cycles than the edges of the form $u_i v_{i+t}, t \in T$. This implies that the orbits $\{u_i | i \in \mathbb{Z}_n\}$ and $\{v_i | i \in \mathbb{Z}_n\}$ of σ are blocks of imprimitivity of Aut X. (There might be a few special cases to consider.) But this forces the automorphism group of $Dih(n; \{1, -1\}, T)$ to be $< \sigma, \tau >$ and we have a GRR. Clearly this cannot be a cyclic Haar graph, for the latter would force "another" dihedral groups inside Aut $Dih(n; \{1, -1\}, T)$. ■

Corollary 5 *For each $i \geq 2$ let $T_i = \{0, 2, 6, \ldots, i(i+1)\}$. Let $n = 2k \geq i(i+3)$. The dihedrant $X(n, i) = Dih(n; \{1, -1\}, T_i)$ is a bipartite GRR and is not a cyclic Haar graph.*

Proof. The smallest case $n = 12, i = 2$ is done separately and corresponds to our $Dih(12; \{1, -1\}, \{0, 2, 6\})$. (There are a few more 4-cycles on the edges $u_i v_i$ here.) But in general we have a family of graphs satisfying the assumptions of Lemma 3.4, giving us the desired result. ■

Note that a graph G has an adjacency matrix that is symmetric right-shift matrix of its first row if and only if it is isomorphic to a circulant graph.

Some cyclic Haar graphs are circulants, and some are not. For instance the cube $Q_3 = H(11)$ is not a circulant graph. None of the even prisms Π_{2n} is circulant.

If n has an odd number of ones in the binary expansion and $H(n)$ is connected circulant, then $k(n)$ must be necessarily odd. Note that $42 = 101010_2$ has disconnected Haar graph that is circulant. There are cases with even $k(n)$ that give rise to circulant Haar graphs. For instance $H(45) = C(\mathbb{Z}_{12}, \{1, 6\})$. Note, that 45 is periodic (see definition below).

The following criterion, restated in our own terminology, for n that has circulant $H(n)$ was first obtained by Alspach

Theorem 6 (Alspach) *A connected Haar graph $H(n)$ is a circulant if and only if $b = b(n)$ can be shifted cyclically to a palindrome $b' = b'^R$. A disconnected Haar graph $H(n)$ is a circulant if and only if any of its connected components is circulant.*

One can then say that n is a *circulant* number if $H(n)$ is a circulant graph. It seems that among the first n integers the fraction of circulants is about $\frac{\log n}{\sqrt{n}}$.

Proposition 7 *A Cayley graph of a cyclic group $Cir(m, S)$ is a cyclic Haar graph if and only if it is bipartite.*

Hence m is even $m = 2k$ and $S = -S$ consists of odd numbers only: $S = 2T + 1$. As an exercise the reader can figure out how to obtain n from a given $Cir(m, S)$.

In the next section we study connectivity of Haar graphs.

4 Connected, periodic, and primitive numbers

In general a cyclic Haar graph may be disconnected. Let us call a natural number *connected* if its Haar graph is connected. The smallest disconnected natural number other than power of 2 is 10. Its Haar graph is $2C_4$.

Note that $H(2^k) = (k+1)K_2$. It is composed of $(k+1)$ copies of the graph K_2. The graphs $H(34)$ and $H(40)$ are isomorphic. They are both disconnected and equal to $2C_6$.

It is possible to classify the natural numbers that are disconnected. Let b be a binary vector of length k. Let $n(b,m)$ be the number corresponding to b in base m:

$$n(b,m) = \sum_{i=0}^{k-1} b_i m^i$$

Recall that for a given n we denote by $b(n)$ the binary vector consisting of $k(n)$ digits of the binary expansion of n and by $B(n)$ is the symbol of n (which is the set with the characteristic vector $b(n)$.)

Let $b^R(n)$ be the reverse of $b(n)$ and $B^R(n) = \{i \in \{0,1,...,k(n)-1\} | b_i^R(n) = b_{k(n)-i}(n) = 1\}$.

Proposition 8 *Let $d(n) = \gcd\{k(n), B^R(n)\}$. Then there exists a unique binary vector b of the length $k(n)/d(n)$ such that*

$$n = 2^{d(n)-1} n(b, 2^{d(n)})$$

and the Haar graph $H(n)$ consists of $d(n)$ disjoint copies of the graph $H(n(b,2))$ Hence $H(n)$ is connected if and only if $d(n) = 1$.

PROOF.

One only has to check that vertex i is reachable from vertex j from the same bipartition class if and only if j is of the form $i + pd(n)$ for some p. ■

For instance, if $n = 40$, then $k = k(40) = 6, B^R(40) = \{0,2\}$, and $d(40) = \gcd\{6,0,2\} = 2$.

A list of all disconnected integers < 100 with their Haar graphs is given in Table 1.

Corollary 9 *If $k(n)$, the number of binary digits of n, is prime and $n \neq 2^{k(n)-1}$ then $H(n)$ is connected.*

PROOF. Let $k(n)$ be prime. Then $d(n)$ from of Proposition 4.1 equals 1 unless the graph is of valence 1. ■

			$n = 2^{d(n)-1} n(b, 2^d(n))$	
n	$b(n)$	$H(n)$	b	$d(n)$
2	10	$2K_{1,1}$	1	2
4	100	$3K_{1,1}$	1	3
8	1000	$4K_{1,1}$	1	4
10	1010	$2K_{2,2}$	11	2
16	10000	$5K_{1,1}$	1	5
32	100000	$6K_{1,1}$	1	6
34	100010	$2C_6$	101	2
36	100100	$3K_{2,2}$	11	3
40	101000	$2C_6$	110	2
42	101010	$2K_{3,3}$	111	2
64	1000000	$7K_{1,1}$	1	7

Table 1: Small disconnected natural numbers and their Haar Graphs. For definition of b, see Proposition 4.1.

Corollary 10 *If n is odd then $H(n)$ is hamiltonian.*

Proof. If n has two consecutive ones in the binary expansion they give rise to a Hamilton cycle. In particular, if n is odd two consecutive ones are obtained by a suitable cyclic shift. Note that it follows that for n odd $H(n)$ is connected. ■

At first glance it seems that the problem of hamiltonicity is not difficult for Haar graphs. We decided to check small numbers in order to get the feeling for the problem. In order to simplify the computer search we introduce the notion of a *periodic and aperiodic numbers.*

Number n with its binary vector $b(n)$ is called *periodic* if $b(n)$ (or any of its shifts) can be written as a concatenation of $p > 1$ equal binary strings c of length r Hence $b(n) = (c, c, ..., c)$. The shortest c is called *period*, r is the *length* and p *the exponent of the period.* If $b(n)$ is equal to its period n is called *aperiodic.*

Here is a characterization of periodic numbers:

Proposition 11 *A natural number n is periodic with length r and exponent p if and only if it can be written in the form $n = n_0(2^{pr} - 1)/(2^r - 1)$ for some $r > 0, p > 1$, and $1 \leq n_0 < 2^r$.*

Proof. The proof is straightforward and is an application of elementary mathematics. Note that $c = b(n_0)$. ■

Now we can reduce any natural number to a so-called *primitive number.* Natural number n is *primitive* if it is connected, aperiodic and canonical. To each number n we may associate a unique primitive number $\pi(n)$ so that we first select n_1 a number corresponding to a connected component of n, then n_2, the period of n_1 and finally, $\pi(n)$ the canonical representative of the equivalence class to which n_2 belongs. In order to show that all connected n are hamiltonian it suffices to show that all primitive n are hamiltonian. The smallest possible counter-example must be primitive. Another reason for considering primitive numbers (and hence primitive cyclic Haar graphs) is the fact that the Schur norms of $H(n)$ and $H(\pi(n))$ are the same;[13]. The reduction is indeed substantial. For instance, there are only 85 primitive integers < 1024 and each of them is cyclically equivalent to an odd number. They are analyzed in Table 4.

We checked many more primitive numbers by computer. Since each of them turned out to be equivalent to an odd number, we expected an easy proof for hamiltonicity of Haar graphs. One has to be careful, since there exist even primitive numbers. The smallest example is 534 with gives rise to a four-valent cyclic Haar graph $H(534)$ on 20 vertices that can be described as a Cartesian graph bundle over the five-cycle C_5 with the fiber C_4; see for instance [16] for the definition of graph bundles. However, 534 has two consecutive ones in its binary expansion and is cyclically equivalent to an odd number, say 537. This is a technical problem that could have been avoided if we define canonical numbers in a different way.

Alas, it was Mark Watkins, who found back in 1974 the first zero-symmetric Cayley graph of a dihedral group that is generated by irredundant generating set of three involutions; see [8]. The corresponding primitive number 536870930 has the property that it has no two consecutive ones in its binary expansion and is not cyclically equivalent to any odd number. It seems it is the smallest positive integer with this property. Its cubic Haar graph is depicted in Figure 5. Note that zero-symmetric graphs are cubic graphs belonging to a GRR. Characterization of zero-symmetric Haar graphs was given by Foster and Powers; see [8]. Our Corollary 4.3 does not ensure the existence of Hamilton cycle in these graphs.

5 Cycles in cyclic Haar graphs

Alspach and Zhang [1] proved that every cubic Cayley graph of a dihedral group is hamiltonian. Compare also [2]. This result covers also cubic Haar graphs.

Proposition 12 (Alspach and Zhang) *Cubic connected cyclic Haar graphs are Hamiltonian.*

Unfortunately for the valence greater than 3 the status of the hamiltonicity problem remains the same as for the general Cayley graphs of dihedral groups.

It looks like it make sense to introduce the notion of the *irredundant* and *redundant* numbers. Natural n is *redundant* if the generating set for the Cayley

	n	order	girth	valence	$H(n)$
1	1	2	–	1	K_2
2	5	6	6	2	C_6
3	9	8	8	2	C_8
4	11	8	4	3	Q_3
5	17	10	10	2	C_{10}
6	19	10	4	3	M_5
7	23	10	4	4	$Cir(10; 1, 3)$
8	33	12	12	2	C_{12}
9	35	12	4	3	$K_2 \times C_6$
10	37	12	4	3	T_3
11	39	12	4	4	$Cir(12; 1, 3)$
12	43	12	4	4	$K_2 \times K_{3,3}$
13	47	12	4	5	$K_{6,6} - 6K_2$
14	65	14	14	2	C_{14}
15	67	14	4	3	M_7
16	69	14	6	3	Heawood
17	71	14	4	4	$Cir(14; 1, 3)$
18	75	14	4	4	$H(75)$
19	79	14	4	5	$Cir(14; 1, 3, 7)$
20	95	14	4	6	$K_{7,7} - 7K_2$
21	129	16	16	2	C_{16}
22	131	16	4	3	$K_2 \times C_8$
23	133	16	6	3	$G(8, 3)$
24	135	16	4	4	$Cir(16; 1, 3)$
25	137	16	4	3	T_4

Table 2: The first 25 primitive numbers n and their Haar graphs $H(n)$.

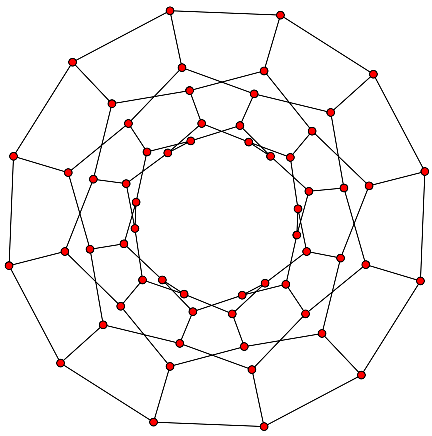

Figure 5: The Watkins zero-symmetric graph $H(536870930)$.

graph $H(n)$ is redundant. Let n' be the number obtained from n by replacing a single "1" in the binary representation of n by a "0". Clearly n is redundant if some n' is connected.

It seems that (connected) irredundant Haar graphs are quite rare. There are none for valence 1. All even cycles are irredundant graphs of valence 2. For valence 3 we get the series of Powers, [8], started by the Watkins graph.

Cubic connected Haar graphs are of special interest. They are certainly the first non-trivial case of Haar graphs. But they also have some very nice properties.

Proposition 13 ([17, 1]) *Each cubic cyclic Haar graph admits an embedding into torus with hexagonal faces in such a way that the resulting map is regular of type* $\{6, 3\}$.

In [15] the intersection of the classes of Haar graphs and generalized Petersen graphs is determined. Recall the definition of the generalized Petersen graph. For a positive integer $n \geq 3$ and $1 \leq r < n/2$, *the generalized Petersen graph* $G(n, r)$ has vertex set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edges of the form $u_i v_i, u_i, u_{i+1}, v_i v_{i+r}, i \in \{0, 1, \ldots, n-1\}$ with the arithmetic modulo n.

Proposition 14 ([15]) $G(8, 3)$ *is the only generalized Petersen graph except for the trivial examples* $G(n, 1), n \geq 3$, *that is a Cayley graph of a dihedral group. In particular, the only Haar graphs that are generalized Petersen graphs are even prisms and the Möbius-Kantor graph:* $G(2m, 1) = H(2^{2m-1} + 3)$ *and* $G(8, 3) = H(133)$.

We end this paper with the following result which classifies connected cyclic Haar graphs in terms of their girth. The proof is straightforward and is omitted.

Proposition 15 *Let $X = H(n)$ be a connected cyclic Haar graph. Then one of the following is true.*

(i) $n = 1$ *and* $X \cong K_2$ *has infinite girth;*

(ii) $n = 2^{k-1} + 1$ *and* $X \cong C_{2k}$ *has girth* $2k$*;*

(iii) X *has valence greater than 2 and girth* 4 *which occurs if and only if there are* $a, b, c, d \in B(n)$ *such that* $\{a, b\} \cap \{c, d\} = \emptyset$) *and* $a + b = c + d$.

(iv) X *has valence greater than 2 and girth* 6 *which occurs if and only if whenever we have* $a, b, c, d \in B(n)$ *such that* $a + b = c + d$ *then* $\{a, b\} = \{c, d\}$.

This result has interpretation in terms of configurations. For the terms used in the following corollaries, see for instance [3, 7, 11].

Corollary 16 *The cyclic Haar graphs of girth 6 correspond precisely to the so-called Levi graphs of cyclic configurations .*

Corollary 17 *Each cyclic configuration is symmetric and point- and line-transitive.*

Corollary 18 *There are no triangle-free cyclic configurations.*

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