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ABSTRACT. If G is a compact Lie group and X is a G-complex we consider the existence of a G-homotopy equivalent CW-complex Y with the property that the action map $G \times Y \to Y$ is a cellular map. Greenlees and May proved that for $G = S^1$ such a CW-complex Y exists for every S^1-complex X. In this paper we formulate a minimal condition on G which guarantees the existence of such a CW-complex for every G-complex and prove it is satisfied by the group $G = SU(2)$ and every abelian compact Lie group.

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1. Formulation of the results

In [1] Greenlees and May defined Tate cohomology of G-spaces for an arbitrary compact Lie group G. In case $G = S^1$ or $G = SU(2)$ and the G-space is a CW-complex with a cellular action of G they showed that the Tate cohomology of such a space is calculable in terms of the CW-decomposition. It is natural to ask the following

Question. Is every G-complex G-homotopy equivalent to a CW-complex with a cellular G-action?

For discrete groups G it is well known that every G-complex is also a CW-complex with a cellular action of G (compare for example [6], Propositon 1.16, p. 102). For general Lie groups, Illman [3] proved that for every G-complex and every compact subgroup H of G there exists an H-homotopy equivalent H-complex with similar fixed point sets and isotropy groups. If H is the trivial subgroup, it follows that a homotopy equivalent CW-complex always exists, but it does not follow that it has a cellular G-action.

The cases $G = S^1$ and $G = SU(2)$ are particularly interesting from the point of view of calculability of Tate cohomology. Greenlees and May [1], Lemma 14.1, proved that the question has a positive answer for $G = S^1$, but their proof uses the fact that S^1 is abelian, and that all quotients $S^1/H \cong S^1$ by closed subgroups have a very simple CW-decompositions, neither of which is true for $SU(2)$. In this note we prove that the answer to our question is affirmative for $G = SU(2)$ and for compact abelian Lie groups G.

Throughout this note G is a compact Lie group and X is a G-complex. Following the proof for $G = S^1$ in [1], we construct a G-homotopy equivalent nonequivariant CW-complex Y by induction on the G-skeleta of X. We begin by formulating minimal conditions which G must satisfy in order for this induction process to work.

Definition 1. A *representative family* $\mathcal{H}$ of subgroups of G is a family of closed subgroups, such that each conjugacy class of closed subgroups of G is represented by precisely one group $H \in \mathcal{H}$. A group G is admissible if there exists a representative family $\mathcal{H}$ of subgroups and a CW-decomposition of G such that the following two conditions are satisfied:

Condition 1. For each $H \in \mathcal{H}$ there is a CW-decomposition of G/H with respect to which the action $\mu\colon G \times G/H \to G/H$ is cellular.

Condition 2. In addition, for any given $K \in \mathcal{H}$ the fixed point set $(G/H)^K$ is a subcomplex of G/H.

Remark 1. Since we can describe the fixed point set of the action of K on G/H as

$$(G/H)^K = \{gH \mid g^{-1}Kg \subset H\} \tag{1}$$

it is obviously nontrivial only if K is conjugate to a subgroup of H. In order to show that Condition 2 is satisfied it therefore suffices to see that every fixed point set $(G/H)^K$, where $K \in \mathcal{H}$ is conjugate to a subgroup of H, is a subcomplex.

In Section 2 we prove that these two conditions are sufficient for the existence of an appropriate nonequivariant CW-decomposition in every G-homotopy class. More precisely, we prove

Theorem 1. *Assume that G is an admissible compact Lie group. Then there exists a CW-complex Y with a cellular action of G and a G-homotopy equivalence $h\colon X \to Y$.*

We also show that in the category of G-complexes with finitely many orbit types Condition 1 suffices.

In Section 3 we prove

Theorem 2. *The group $SU(2)$ is admissible.*

As an immediate consequence we obtain our main result:

Corollary 1. *For every $SU(2)$-complex X there exists a CW-complex Y with a cellular action of $SU(2)$ and a $SU(2)$-homotopy equivalence $h\colon X \to Y$.*

Finally, in Section 4 we prove that every compact abelian group is admissible, and obtain

Theorem 3. *If X is a G-complex where G is an abelian compact Lie group, then there exists a CW-complex Y with a cellular action of G and a G-homotopy equivalence $h : X \to Y$.*

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2. Proof of Theorem 1

Let $\mathcal{H}$ be a representative family of closed subgroups of G and assume that we have CW-decompositions of G and of G/H for each $H \in \mathcal{H}$ such that Conditions 1 and 2 are satisfied.

Let X be a G-complex. We will construct a CW-complex Y and a G-homotopy equivalence $h\colon X \to Y$ by induction on the G-skeleta $X^{(i)}$ of X.

The 0-skeleton $X^{(0)}$ is a disjoint union of orbits G/H_i, where $H_i \in \mathcal{H}$. Let Y_0 be $X^{(0)}$ with the CW-decomposition which is given on every G-cell G/H_i. Condition 1 implies that the action $\mu\colon G \times Y_0 \to Y_0$ is cellular, and Condition 2 implies that

$$(Y_0)^K = \coprod (G/H_i)^K$$

is a subcomplex of Y_0 for every $K \in \mathcal{H}$. We define the G-homotopy equivalence on the 0-skeleton by $h_0 = \mathrm{id}\colon X^{(0)} \to Y_0$.

By induction we assume that there exists a CW-complex Y_{n-1} with a cellular action of G, such that for every $K \in \mathcal{H}$ the fixed point set $(Y_{n-1})^K$ is a subcomplex, and a G-homotopy equivalence

$$h_{n-1} : X^{(n-1)} \to Y_{n-1}.$$

For any G-cell $e^n_\nu \in X^{(n)}$, the attaching G-map $G/H_\nu \times S^{n-1} \to X^{(n-1)}$ is determined by its restriction

$$\varphi_\nu : S^{n-1} \to (X^{(n-1)})^{H_\nu}.$$

Let ψ_ν be a nonequivariant cellular approximation of the composition

$$h_{n-1} \circ \varphi_\nu : S^{n-1} \to (Y_{n-1})^{H_\nu}.$$

Since the action of G on Y_{n-1} is cellular, the G-extension

$$\tilde{\psi}_\nu : G/H_\nu \times S^{n-1} \to Y_{n-1}$$

of ψ_ν is also cellular, and the space

$$Y_n = \coprod_{e_\nu^n \in X^{(n)}} (G/H_\nu \times D^n) \cup_{\coprod \tilde{\psi}_\nu} Y_{n-1}$$

is a CW-complex with a cellular action of G. For each $K \in \mathcal{H}$ the fixed point set $(Y_n)^K$ is obtained by gluing the subcomplexes $(G/H_\nu)^K$, corresponding to the n-cells, and the subcomplex $(Y_{n-1})^K$ along a cellular map, so it is a subcomplex of Y_n. It is easy to see that the G-homotopy h_n is obtained so that h_{n-1} is extended G-cell by G-cell over the whole space Y_n. In the direct limit we obtain the desired CW-complex Y and G-homotopy equivalence h. □

Remark 2. The above proof shows that even if G is not admissible, but X is a G-complex such that there exists a family of closed subgroups representing the conjugacy classes of isotropy subgroups and satifying Conditions 1 and 2, we still obtain a G-homotopy equivalent CW-complex Y with the given action of G cellular.

In some situations, for example in the category of complexes with finitely many orbit types, Condition 1 suffices. This follows from the following Proposition.

Proposition 1. *If $\mathcal{K}$ is a family of closed subgroups such that for every $H \in \mathcal{K}$ the set of fixed point sets $(G/H)^K, K \in \mathcal{K}$, is finite, then every orbit G/H has a CW-decomposition with respect to which every fixed point set $(G/H)^K$, $K \in \mathcal{K}$, is a subcomplex.*

Proof. For every pair $H, K \in \mathcal{K}$ the orbit (G/H) is a smooth K-manifold, and the fixed point set $(G/H)^K$ is a submanifold (compare [6], p. 42) which is, by (1) nontrivial only if K is conjugate to a subgroup of H. For a given H, the family $\{(G/H)^K, K \in \mathcal{K}\}$ is a finite family of smooth submanifolds of G/H which, by the differentiable slice theorem (compare for example [2], Theorem I.5), intersect transversally. By [4], 10.11, 10.14, this implies that there exists a triangulation of G/H such that each $(G/H)^K$, $K \in \mathcal{K}$, is a subcomplex. □

Corollary 2. *If G is such that every finite family of conjugacy classes of subgroups can be represented by a finite family $\mathcal{K}$ of subgroups satisfying Condition 1, then for every G-complex with finitely many orbit types X, there exists a CW-complex Y with a cellular action of G and a G-homotopy equivalence $h: X \to Y$.*

Proof. Every cell e_ν^n is of type G/H, where $H \in \mathcal{K}$, and $\mathcal{K}$ is finite. For every H, also the family of fixed point sets $(G/H)^K$, $K \in \mathcal{K}$ is finite, and so, by Proposition 1 G/H has a decomposition, such that every fixed point set $(G/H)^K$ is a subcomplex. At every step of the induction in the proof of Theorem 1 $(Y_n)^K$ is a subcomplex, and Y can be constructed in the same way as above. □

3. A PROOF FOR $G = SU(2)$

Throughout this section G will denote the group $SU(2)$ of complex matrices of the form

$$\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}, \quad |z_1|^2 + |z_2|^2 = 1,$$

with matrix multiplication as operation. It is isomorphic to the group $Sp(1)$ of unit quaternions which is topologically the sphere S^3. The projection $\mathrm{pr}\colon Sp(1) \to Sp(1)/(\pm 1)$ is a twofold covering over the special orthogonal group $SO(3)$ viewed as the subgroup of rotations of $Sp(1) \approx S^3$ leaving the real axis fixed, which are given by conjugation by unit quaternions.

The proof of Theorem 2 will follow from two propositions. In the first we will prove that G with respect to the obvious CW-decomposition satisfies Condition 1 for any representative family of closed subgroups and in the second we will show that G satisfies Condition 2 for a sensible choice of $\mathcal{H}$.

Proposition 2. *If $G = SU(2)$ is given the standard CW-decomposition into one 0-cell and one 3-cell then, for any closed subgroup H, the orbit G/H can be given a CW-structure such that the action $\mu\colon G \times G/H \to G/H$ is cellular.*

Proof. Choose $e^0 = I \in SU(2)$. For any closed subgroup H the quotient G/H is a connected manifold of dimension 2 or 3. For any CW-decomposition of G/H the 0, 1 and 2 skeleta of $G \times G/H$ consist of cells of the form $e^0 \times f^j_\nu$, where f^j_ν is a j-cell of G/H, and $j = 0, 1$ or 2. Since multiplication by $e^0 = I$ is the identity,

$$\mu(e^0 \times f^j_\nu) = f^j_\nu \subset (G \times G/H)^{(j)}.$$

For $j \geq 3$ the j-skeleton of $G \times G/H$ is mapped to $G/H = (G/H)^{(3)}$. $\square$

The conjugacy classes of closed subgroups of $SU(2)$ are known. Since there are no noncommutative 2-dimensional Lie groups (compare for example [2]), the dimension of a proper closed subgroup is at most 1. The 0 and 1-dimensional subgroups can be reconstructed from the conjugacy classes of closed subgroups of $SO(3)$ (compare [7], p. 155). The only 1-dimensional subgroups are the maximal torus $T \cong S^1$ and its normalizer NT. The finite subgroups of G consist of two infinite families and three additional groups (compare also [8], Appendix I, and [5], p. 404).

1. The cyclic subgroups $\mathbb{Z}/n$.
2. The generalized quaternionic groups
$$G_{2n} = \langle x, y \mid x^n = y^2, y^{-1}xy = x^{-1} \rangle$$
which are lifts of the dihedral subgroups
$$D_n = \langle x, y \mid x^2 = y^2 = (xy)^n = 1 \rangle \subset SO(3).$$
3. A copy of the special linear group $SL_2(\mathbb{F}_3)$ of 2×2 matrices over a field with 3 elements (compare [5] p. 83), which is a lift of the tetrahedral subgroup of $SO(3)$.
4. A copy of the special linear group $SL_2(\mathbb{F}_5)$ which is of order 120 and is the lift of the icosahedral subgroup of $SO(3)$.

5. A lift of the octahedral subgroup of $SO(3)$ which is an extension of the symmetric group S_4.

Proposition 3. *There exists a representative family $\mathcal{H}$ of closed subgroups of $G = SU(2)$ such that for each given $H \in \mathcal{H}$ the family of fixed point sets $(G/H)^K, K \in \mathcal{H}$, is finite.*

Proof. We will choose the representative family $\mathcal{H}$ as follows. As a representative of the conjugacy class of maximal tori we choose the group of real rotations

$$T = \left\{ a_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad t \in \mathbb{R} \right\}.$$

A representative of the conjugacy class of the normalizers is NT which is the product $T \cdot \langle u \rangle$, where

$$u = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{2}$$

is of order 4. The cyclic groups $\mathbb{Z}/n$ are represented by subgroups of T generated by rotations through $2\pi/n$, and the generalized quaternionic groups G_{2n} by subgroups of NT, where the generator x is rotation by π/n and $y = u$. For every $n \neq 2$ the normalizer of G_{2n} is G_{4n}. The normalizer of G_4 is a representative for the special group $SL_2(\mathbb{F}_3)$. For the remaining two conjugacy classes we choose any representative.

For a finite group $H \in \mathcal{H}$, there are only finitely many closed subgroups of G conjugate to a subgroup of H, so the family of fixed point sets $(G/H)^K$ is finite.

The closed subgroups of $H = T$ are the cyclic groups $\mathbb{Z}/n$. A computation shows that for every choice of $n \neq 2$,

$$(G/T)^{\mathbb{Z}/n} = NT/T = \mathbb{Z}/2.$$

If $n = 2$, then $\mathbb{Z}/2$ is the center of G, and $(G/T)^{\mathbb{Z}/2} = G/T$. The family $(G/T)^K, K \in \mathcal{K}$, therefore has two members.

If $H = NT$, then it contains the cyclic groups and the generalized quaternionic groups. A computation shows that all elements of the non-identity component of NT are of the form

$$u(t) = \begin{bmatrix} i\cos t & i\sin t \\ i\sin t & -i\cos t \end{bmatrix}, \tag{3}$$

and are of order 4. This implies that for all $n \neq 2, 4$ every subgroup of NT, conjugate to $\mathbb{Z}/n$ must be contained in T, so it is $\mathbb{Z}/n$. Any conjugation $c_g : G \to G$ which maps $\mathbb{Z}/n$ into NT must therefore map the generator of $\mathbb{Z}/n$ to an element of $\mathbb{Z}/n$, so by (1)

$$(G/NT)^{\mathbb{Z}/n} = (G/NT)^T = NT/NT$$

is a point. Similarly, the only subgroup of NT, conjugate to G_{2n} is G_{2n}, and any conjugation $c_g : G \to G$ which maps G_{2n} into NT must preserve the subgroup $\mathbb{Z}/n$, and it must map u into some element of the form (3). A simple computation shows that this is true for every $g \in NT$. On the other hand no element $g \notin NT$ has this property since no such element preserves rotations. So, $(G/NT)^{G_{2n}} = NT/NT$ is a point for all n. For $n = 4$ there is an infinite family of embeddings of $\mathbb{Z}/4$ into NT, and

$$(G/NT)^{\mathbb{Z}/4} = \{gNT \mid g^{-1}a_{\pi/2}g = u(t) \text{ for some } t\}.$$

The remaining finite subgroups of $SU(2)$ do not embed in NT. For $n = 2$, $(G/NT)^{\mathbb{Z}/2} = G/NT$. The family $(G/NT)^K$ therefore has three members. □

Proof of theorem 2. By Proposition 2 any representative family of subgroups of $SU(2)$ satisfies Condition 1. The representative family $\mathcal{H}$ of Proposition 3 satisfies the assumptions of Proposition 1, and therefore Condition 2. □

4. Abelian compact Lie groups

For abelian groups, the only representative family is the family of all closed subgroups of G.

Proposition 4. *Every abelian compact Lie group G satisfies Condition 1.*

Proof. We represent G as a product of a torus and a discrete torus:

$$\begin{aligned} G &= (S^1)^\nu \times \mathbb{Z}/n_1 \times \ldots \times \mathbb{Z}/n_\lambda \\ &= \{(e^{it_1}, \ldots, e^{it_\nu}, e^{i2k_1\pi/2n_1}, \ldots, e^{i2k_\lambda\pi/n_\lambda}) \mid t_i \in \mathbb{R}, k_j \in \mathbb{Z}\} \subset \mathbb{C}^{\nu+\lambda}. \end{aligned}$$

The standard (product) CW-decomposition of the unit component $G_1 = (S^1)^\nu$ consists of one 0-cell $e^0 = (1, \ldots, 1)$, of ν one-cells e^1_i, $i = 1, \ldots, \nu$, with characteristic maps which are inclusions of S^1 into the i-th copy of $\mathbb{C}$ in $\mathbb{C}^\nu$,

$$\chi^1_i(t) = (1, \ldots, e^{it}, \ldots, 1), \quad t \in [0, 2\pi].$$

For $j \geq 1$, every j-cell e^j_I is determined by a subset $I = \{i_1, \ldots, i_j\}$ of $\{1, \ldots, \nu\}$, and its characteristic map is a product

$$\chi^j_I = \chi^1_{i_1}(t_1) \cdots \chi^1_{i_j}(t_j), \quad t_1, \ldots, t_j \in [0, 2\pi].$$

It follows that each closed j-cell of G_1 is a j-dimensional closed subgroup of G_1.

For a given closed subgroup $H \leq G_1$, the quotient map $q\colon G_1 \to G_1/H$ is a homomorphism of groups, therefore the image $q(e^j_I)$ is a closed subgroup of G_1/H of dimension $\leq j$. To obtain a CW-decomposition of G_1/H we first subdivide the nontrivial 1-dimensional subgroups $K^1_i = q(e^1_i)$. Two such subgroups either coincide or intersect in a finite subgroup of G_1/H. Let the 0-skeleton of G_1/H consist of all points of all finite subgroups $K^0_{i_1,i_2} = K^1_{i_1} \cap K^1_{i_2}$. Let the 1-skeleton of G_1/H consist of all arcs on all K^1_i between these points. For every $j > 1$, the image $q(e^j_I) = q(e^j_{i_1}) \cdots q(e^j_{i_j})$ is given the subdivision induced by the product structure.

Since G/G_1 is discrete, G as well as G/H is a disjoint union of cosets γG_1 and $\gamma(G_1/H)$ respectively, where $\gamma \in \mathbb{Z}/n_1 \times \ldots \times \mathbb{Z}/n_\lambda$, and we give each coset γG_1 and $\gamma(G_1/H)$ the CW-decomposition induced by the diffeomorphism $x \to \gamma x$.

It remains to show the the G-action on G/H is a cellular map with respect to these CW-decompositions of G and G/H, but this follows immediately from the coset structure of the cells. The action of G on G/H maps an element $(\gamma x, gq(y))$ of a j-cell, $j = k + m$,

$$\gamma(e^k_I) \times q(e^{m'}_{I'}), \quad m' \geq m$$

to the product $(\gamma x) \cdot (q(y)) = \gamma q(xy)$ which is an element of

$$\gamma q(e_I^k \cdot e_{I'}^{m'}) = \gamma q(e_I^k) \cdot q(e_{I'}^{m'}).$$

Since $q(e_I^k)$ is a union of cells of dimension $k' \leq k$, and $q(e_{I'}^{m'})$ is a union of cells of dimension m, the product is contained in the $k' + m$-skeleton $(k' + m \leq j)$ of G/H. □

Proof of Theorem 3. Since G is abelian, the fixed point set $(G/H_i)^{H_j}$ is the whole space G/H_i when $H_j \leq H_i$. In all other cases, $(G/H_i)^{H_j}$ is empty. This implies that for any G-complex X, the fixed point set X^{H_j} is the union of all G-cells of type G/H_i, where $H_j \leq H_i$, so it is a CW-subcomplex. □

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