

UNIVERSITY OF LJUBLJANA
INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
DEPARTMENT OF MATHEMATICS
JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

Preprint series, Vol. 37 (1999), 671

CLASSIFYING ARC-TRANSITIVE
CIRCULANTS OF SQUARE-FREE
ORDER

Caiheng Li Dragan Marušič
Joy Morris

ISSN 1318-4865

December 9, 1999

Ljubljana, December 9, 1999

CLASSIFYING ARC-TRANSITIVE CIRCULANTS OF SQUARE-FREE ORDER

CAIHENG LI¹

Department of Mathematics and Statistics
University of Western Australia
Nedlands, WA 6907
Australia

DRAGAN MARUŠIČ²

IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1000 Ljubljana
Slovenija

JOY MORRIS

Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B. C.
Canada, V5A 1S6

Abstract

A *circulant* is a Cayley graph of a cyclic group. Arc-transitive circulants of square-free order are classified. It is shown that an arc-transitive circulant Γ of square-free order n is one of the following: the lexicographic product $\Sigma[\overline{K}_b]$, or the deleted lexicographic $\Sigma[\overline{K}_b] - b\Sigma$, where $n = bm$ and Σ is an arc-transitive circulant, or Γ is a *normal* circulant, that is, $\text{Aut}\Gamma$ has a normal regular cyclic subgroup.

Keywords: circulant graph, arc-transitive graph, square-free order, cyclic group, primitive group, imprimitive group.

¹Li is grateful to the Institute of Mathematics, Physics and Mechanics at the University of Ljubljana for hospitality and financial support during his visit that led to the completion of this work.

²Supported in part by the “Ministrstvo za znanost in tehnologijo Republike Slovenije”, project no. J1-101-04965-99.

1 Introductory remarks

Throughout this paper, graphs are simple and undirected; the symbol \mathbb{Z}_n , where n is an integer, will be used to denote the ring of integers modulo n as well as its (additive) cyclic group of order n .

Let Γ be a graph and G a subgroup of its automorphism group $\text{Aut } \Gamma$. The graph Γ is said to be *G-arc-transitive* if G acts transitively on the set of arcs of Γ . In particular, Γ is said to be *arc-transitive* if Γ is $\text{Aut } \Gamma$ -arc-transitive. Note that an arc-transitive graph $G\alpha$ is necessarily vertex-transitive, that is, its automorphism group acts transitively on the vertex set $V\Gamma$ of Γ .

Given a group G and a symmetric subset $S = S^{-1}$ of $G \setminus \{id\}$, the *Cayley graph of G relative to S* , denoted by $\text{Cay}(G, S)$, has vertex set G and edges of the form $\{g, gs\}$, for all $g \in G$ and $s \in S$. By the definition, the group G acting by right multiplication is a subgroup of $\text{Aut } \Gamma$ and acts regularly on $V\Gamma = G$. The converse also holds (see [6]). A *circulant* is a Cayley graph of a cyclic group. Thus a graph Γ is a circulant of order n if and only if $\text{Aut } \Gamma$ contains a cyclic subgroup of order n which is regular on $V\Gamma$.

A classification of 2-arc-transitive circulants was given in [1]. It was proved that a connected, 2-arc-transitive circulant of order n , $n \geq 3$, is one of the following graphs: the cycle C_n , the complete graph K_n , the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$, $n \geq 6$, or $K_{\frac{n}{2}, \frac{n}{2}} - \frac{n}{2}K_2$ where $\frac{n}{2} \geq 5$ odd (the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$ minus a 1-factor).

In this paper we take the next step in our pursuit of a classification of all arc-transitive circulants, by classifying all such graphs of square-free order. To describe this classification, a few words on the notation are in order. For two graphs Γ and Σ , denote by $\Sigma[\Gamma]$ the *lexicographic product* of Γ by Σ . If in addition, Γ and Σ have the same vertex set then denote by $\Sigma - \Gamma$ the graph with vertex $V\Gamma$ and having two vertices adjacent if and only if they are adjacent in Σ but not adjacent in Γ . Furthermore, let $\bar{\Sigma}$ denote the complement of Σ , and for a positive integer m , denote by $m\Sigma$ the graph which consists of m disjoint copies of Σ . A circulant Γ is called a *normal circulant* if $\text{Aut } \Gamma$ contains a cyclic regular normal subgroup. The following is the main result of this paper.

Theorem 1.1 *Let Γ be an arc-transitive circulant graph of square-free order n . Then one of the following holds:*

- (1) Γ is a complete graph;
- (2) Γ is a normal circulant graph;
- (3) $\Gamma = \Sigma[\overline{K}_b]$ or $\Gamma = \Sigma[\overline{K}_b] - b\Sigma$, where $n = mb$, and Σ is an arc-transitive circulant of order m .

Remark 1.2 Let Γ be a connected arc-transitive circulant. If $\Gamma = \Sigma[\overline{K}_b]$ or if $\Gamma = \Sigma[\overline{K}_b] - b\Sigma$, then the graph Γ may be easily reconstructed from a smaller arc-transitive circulant Σ . Thus the graphs in part (3) of Theorem 1.1 are well-characterized. As for arc-transitive normal circulants, the following observations are in order. For two groups G and H , denote by $G \rtimes H$ a semidirect product of G by H . Assume that $\Gamma = \text{Cay}(R, S)$ is normal. Let $\text{Aut}(R, S) = \{\sigma \in \text{Aut}(R) \mid S^\sigma = S\}$. Then by [4, Lemma 2.1], $\text{Aut}\Gamma = R \rtimes \text{Aut}(R, S)$, and since Γ is arc-transitive, $\text{Aut}(R, S)$ is transitive on S . Thus S may be written as $\{s^\sigma \mid \sigma \in \text{Aut}(R, S)\}$ where $s \in S$, that is, S is an $\text{Aut}(R, S)$ -orbit under the $\text{Aut}(R)$ -action. As R is cyclic, $\langle s \rangle = R$ if and only if $\langle S \rangle = R$. Hence, since Γ is connected, s generates R . This provides us with a general method for constructing connected arc-transitive normal circulants, that is, for any generating element g of R and a subgroup H of $\text{Aut}(R)$, $\text{Cay}(R, g^H)$ is a connected arc-transitive normal circulant. Note that, since R is cyclic, $\text{Aut}(R)$ is abelian.

2 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. We use a standard notation and terminology, see for example [3]. Let Γ be a finite graph, and assume that $G \leq \text{Aut}\Gamma$ is transitive on $V\Gamma$. Let $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ be a G -invariant partition of $V\Gamma$, that is, for each B_i and each $g \in G$, either $B_i^g \cap B_i = \emptyset$, or $B_i^g = B_i$. A partition \mathcal{B}' is called a *refined* partition of a partition \mathcal{B} if a block of \mathcal{B}' is a proper subset of a block of \mathcal{B} . For $B \in \mathcal{B}$, denote by G_B the subgroup of G which fixes B setwise, and by G_B^B the permutation group induced by G_B on B . The *kernel* N of G on \mathcal{B} is the subgroup of G in which every element fixes all $B \in \mathcal{B}$. Clearly, N is a normal subgroup of G . A partition \mathcal{B} is said to be *minimal* if \mathcal{B} has no refined partitions. It follows that if \mathcal{B} is a minimal partition of Ω , then G_B^B is primitive for each block $B \in \mathcal{B}$. For a G -invariant partition \mathcal{B} of $V\Gamma$, the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ induced on \mathcal{B} is the graph

with vertex set \mathcal{B} and B_i is adjacent in $\Gamma_{\mathcal{B}}$ to B_j if some $u \in B_i$ is adjacent in Γ to some $v \in B_j$. Two blocks $B, B' \in \mathcal{B}$ are said to be adjacent if they are adjacent in $\Gamma_{\mathcal{B}}$; denote by $\Gamma[B, B']$ the subgraph of Γ with vertex set $B \cup B'$ and with two vertices adjacent if and only they are adjacent in Γ .

As in Theorem 1.1, let n be a positive square-free integer, and let Γ be an arc-transitive circulant of order n . We will complete the proof of Theorem 1.1 by proving the following proposition, which is slightly stronger than Theorem 1.1.

Proposition 2.1 *Let Γ be a G -arc-transitive circulant of square-free order, where $G \leq \text{Aut}\Gamma$ and let R be a cyclic regular subgroup of G . Then one of the following statements holds.*

- (1) G is 2-transitive on $V\Gamma$, and Γ is a complete graph; or
- (2) R is normal in G ; or
- (3) there exists a minimal G -invariant partition \mathcal{B} of $V\Gamma$ such that for the kernel N of the G -action on \mathcal{B} and for a block $B \in \mathcal{B}$, either
 - (i) N is unfaithful on B and $\Gamma = \Gamma_{\mathcal{B}}[\overline{K}_b]$, or
 - (ii) $K \cong K^B$ is 2-transitive on B and $\Gamma = \Gamma_{\mathcal{B}}[\overline{K}_b] - b\Gamma_{\mathcal{B}}$.

The proof of this proposition consists of a series of lemmas. As in the proposition, we denote by G a subgroup of $\text{Aut}\Gamma$ which is transitive on the set of arcs of Γ , and by R a cyclic subgroup of G . First, assume that G is primitive on $V\Gamma$. Then by Schur's Theorem (see [3, Theorem 3.5A, p.95]), either G is 2-transitive, or $|V\Gamma| = p$ and $\mathbb{Z}_p \leq G \leq \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ for some prime p . Thus we have the following lemma.

Lemma 2.2 *If G is primitive on $V\Gamma$, then either Γ is complete, or R is normal in G .*

Hence we assume that G is imprimitive on $V\Gamma$ in the rest of this section.

Lemma 2.3 *Let \mathcal{B} be a minimal G -invariant partition of $V\Gamma$, and let N be the kernel of the G -action on \mathcal{B} . Then either N^B is 2-transitive, or $\mathbb{Z}_p \leq N^B \leq \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, where $B \in \mathcal{B}$; in particular, in both cases N^B is primitive.*

PROOF. It is clear that G_B^B is primitive, $N^B \triangleleft G_B^B$, and N contains the subgroup of R of order $|B|$. Thus N^B and so G_B^B contains a cyclic regular subgroup on B . By Schur's theorem, either G_B^B is 2-transitive, or $\mathbb{Z}_p \leq G_B^B \leq \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$. By Burnside's theorem (see [3, Theorem 4.1B, p.107]), if G_B^B is 2-transitive then $\text{soc}(G_B^B)$ is non-abelian simple or elementary abelian. It then follows since n is square-free, that either $T \leq G_B^B \leq \text{Aut}(T)$ for some nonabelian simple group T , or $\mathbb{Z}_p \leq G_B^B \leq \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$. If $\mathbb{Z}_p \leq G_B^B \leq \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, then we have $\mathbb{Z}_p \leq N^B \triangleleft G_B^B \leq \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$. Assume that $T \leq G_B^B \leq \text{Aut}(T)$ with T nonabelian simple. Then T is transitive, and furthermore, N^B contains T . Suppose that N^B is imprimitive on B . Then there exists a N^B -invariant partition \mathcal{B}' of B such that the regular cyclic subgroup (on B) of N^B is transitive and unfaithful on \mathcal{B}' . Thus N^B has a normal subgroup which is intransitive on B , which is not possible since T is the unique minimal normal subgroup of G_B^B and transitive on B . Hence N^B is primitive, and so 2-transitive. ■

Next we deal with two different cases according to the actions of N on a block $B \in \mathcal{B}$.

Lemma 2.4 *Assume that there exists a minimal G -invariant partition \mathcal{B} of $V\Gamma$ such that N is unfaithful on B , where N is the kernel of the G -action on \mathcal{B} , and $B \in \mathcal{B}$. Then $\Gamma = \Gamma_{\mathcal{B}}[\overline{K}_b]$, where $b = |B|$; as in part (3) (i).*

PROOF. Let M be the kernel of the N -action on B . Then $1 \neq M \triangleleft N$, and so $1 \neq M^{B'} \triangleleft N^{B'}$ for some $B' \in \mathcal{B}$. Since $N^{B'}$ is permutationally isomorphic to N^B and N^B is primitive (by Lemma 2.3), it follows that $M^{B'}$ is transitive on B' . As Γ is connected, there exists a sequence of blocks $B_0 = B, B_1, \dots, B_l = B'$ such that a vertex in B_j is adjacent in Γ to some vertices in B_{j+1} for each $0 \leq j \leq l-1$, and there exists $0 \leq i < l$ such that $M^{B_j} = 1$ for all $j \leq i$ and $M^{B_{i+1}} \neq 1$. Then for $u \in B_i$, $M^{B_i \cup B_{i+1}}$ is transitive on $\{\{u, v\} \mid v \in B_{i+1}\}$. Since $N^{B_i \cup B_{i+1}}$ is transitive on B_i and fixes B_{i+1} (setwise), each vertex in B_i is adjacent to all vertices in B_{i+1} . It follows that $\Gamma = \Gamma_{\mathcal{B}}[\overline{K}_b]$, where $b = |B|$. ■

Lemma 2.5 *Assume that there exists a minimal G -invariant partition \mathcal{B} of $V\Gamma$ such that $N \cong N^B$ is 2-transitive on B , where N is the kernel of G on \mathcal{B} , and $B \in \mathcal{B}$. Then $\Gamma = \Gamma_{\mathcal{B}}[\overline{K}_b] - b\Gamma_{\mathcal{B}}$, where $b = |B|$; as in part (3) (ii).*

PROOF. We note that, since Γ is a circulant, we may label the vertices of Γ by elements of \mathbb{Z}_n , in such a way that $\Gamma = \text{Cay}(R, S)$, where $S \subseteq \mathbb{Z}_n \setminus \{0\}$ satisfies $i \in S$ if and only if $n - i \in S$. The subset S will be called a *symbol* of Γ .

We are now going to distinguish two different cases, depending on whether the actions of the group N on the blocks in \mathcal{B} are permutationally equivalent or not. (Recall that by [3, Lemma 1.6B, p.21] two transitive actions of a permutation group on two sets are equivalent if and only if the point stabilizer of the action on the first set coincides with the stabilizer of a point in the action on the second set.)

CASE 1. The actions of N on the blocks in \mathcal{B} are equivalent.

It follows that for each block $B' \in \mathcal{B}$, there exists $v' \in B'$ such that $N_{v'} = N_v$, where $v \in B$. Let $\text{Equiv}(v)$ denote the collection of all such vertices v' , that is $\text{Equiv}(v) = \{v' \in V\Gamma \mid N_{v'} = N_v\}$. Then the 2-transitivity of the action of N on each of the blocks in \mathcal{B} implies that the stabilizer N_v has two orbits in B' , namely $\{v'\}$ and $B' \setminus \{v'\}$, or in other words, $B' \cap \text{Equiv}(v)$ and $B' \setminus \text{Equiv}(v)$. In particular, $|\text{Equiv}(v) \cap B'| = 1$ for each $B' \in \mathcal{B}$.

Assume first that $\Gamma(v) \cap \text{Equiv}(v) \neq \emptyset$, where $\Gamma(v)$ denotes the set of neighbors of v . Because of arc-transitivity we have that the bipartite graph induced by a pair of adjacent blocks is a perfect matching. Moreover, it may be seen that $\Gamma(v) \subseteq \text{Equiv}(v)$. But $\text{Equiv}(u) = \text{Equiv}(v)$ for any $u \in \text{Equiv}(v)$ and so the subgraph induced by the set $\text{Equiv}(v)$ is a connected component of Γ , isomorphic to $\Gamma_{\mathcal{B}}$, a contradiction for Γ is connected and $b \neq 1$.

Assume now that $\Gamma(v) \cap \text{Equiv}(v) = \emptyset$. Then for a block B' adjacent to B we must have that $\Gamma(v) \cap B' = B' \setminus \text{Equiv}(v) = B' \setminus \{v'\}$. Let Γ' denote the graph obtained from Γ by joining two non-adjacent vertices of Γ if and only if they belong to two adjacent blocks in $\Gamma_{\mathcal{B}}$. In view of the comments of the previous paragraph $\Gamma' \cong b\Gamma_{\mathcal{B}}$ and so $\Gamma = \Gamma_{\mathcal{B}}[\overline{K}_b] - b\Gamma_{\mathcal{B}}$.

CASE 2. The actions of N on the blocks in \mathcal{B} are not (all) equivalent.

Using the classification of 2-transitive groups (see [3, Section 7.7]) we deduce that a 2-transitive group can have at most two inequivalent actions (of the same degree). Hence the set \mathcal{B} decomposes into subsets \mathcal{B}_0 and \mathcal{B}_1 such that the actions of N on B and $B' \in \mathcal{B}$ are equivalent when $B' \in \mathcal{B}_0$ and inequivalent when $B' \in \mathcal{B}_1$. Moreover,

in view of the fact that Γ is arc-transitive and thus the bipartite graphs induced by pairs of adjacent blocks are all isomorphic, it follows that $\{\mathcal{B}_0, \mathcal{B}_1\}$ is a bipartition of $V\Gamma_{\mathcal{B}}$ with $|\mathcal{B}_0| = |\mathcal{B}_1|$. In particular, $|\mathcal{B}| = m$ is an even number. Let ρ be a generator of the cyclic regular group R of G . Letting $B_i = B\rho^i$, we have that \mathcal{B}_0 consists of all the blocks B_i with $i \in \mathbb{Z}_m$ even and \mathcal{B}_1 consists of all the blocks B_i with $i \in \mathbb{Z}_m$ odd. Let $v_i^j = \rho^{i+mj}$, for all $i \in \mathbb{Z}_m$ and all $j \in \mathbb{Z}_b$.

Now the quotient graph $\Gamma_{\mathcal{B}}$ is a circulant. Assume that $2i+1$ belongs to the symbol of $\Gamma_{\mathcal{B}}$. (Note that the symbol of $\Gamma_{\mathcal{B}}$ contains only odd numbers.) Let $\sigma = \rho^{2i+1}$ and consider the blocks B_0, B_{2i+1} and B_{4i+2} . Let T be the subset of \mathbb{Z}_b consisting of all those t such that $v = v_0^0$ is adjacent to v_{2i+1}^t . Then $v_{2i+1}^0 = v^\sigma$ is adjacent to $(v_{2i+1}^t)^\sigma = v^\sigma \rho^{2i+1+mt} = v^{\rho^{4i+2+mt}} = v_{4i+2}^t$, where $t \in T$. Therefore

$$v_{2i+1}^j \sim v_{4i+2}^l \iff l - j \in T. \quad (1)$$

Let $a \in \mathbb{Z}_b$ be such that $N_v = N_u$, where $u = v_{4i+2}^a$. Recall that the bipartite graphs induced by pairs of adjacent blocks are isomorphic, and moreover by the classification of 2-transitive groups [3, Section 7.7], N_v has two orbits of different cardinalities on B_{2i+1} . Hence u and v must have the same neighbors in B_{2i+1} and so $\Gamma(u) \cap B_{2i+1} = \{v_{2i+1}^t \mid t \in T\}$. Combining this together with (1) we have that $a - t \in T$ for each $t \in T$ and so

$$a - T = T. \quad (2)$$

Now because of the 2-transitivity of the action of N on each block, it follows that $|\Gamma(v_0^0) \cap \Gamma(v_0^j) \cap B_{2i+1}|$ is constant for all $j \in \mathbb{Z}_b \setminus \{0\}$. This implies the existence of a positive integer λ such that $|T \cap (T + j)| = \lambda$, for all $j \in \mathbb{Z}_b \setminus \{0\}$. Hence, in view of (2),

$$|T \cap (-T + a + j)| = \begin{cases} \lambda & \text{if } j \neq -a, \\ |T| & \text{if } j = -a. \end{cases} \quad (3)$$

We now make the following observation about the intersection $T \cap (-T + l)$. (See also [1, Lemma 2.1].) Whenever $x \in T \cap (-T + l)$ there must exist some $y \in T$ such that $x = -y + l$. Clearly, we get that $y \in T \cap (-T + l)$ by reversing the roles of x and y . So the elements in the intersection $T \cap (-T + l)$ are paired off with one exception occurring when $l \in 2T$. Then the equality $l = 2x$ ($x \in T$) gives rise to a

unique element in the intersection $T \cap (-T + l)$. Therefore the parity of $|T \cap (-T + l)|$ depends solely on whether l belongs to $2T$ or not. More precisely, $|T \cap (-T + l)|$ is an odd number if $l \in 2T$ and an even number if $l \notin 2T$. Combining this fact with (3) we see that, in particular, $\mathbb{Z}_b \setminus \{-a\}$ is either a subset of $2T$ or of $\mathbb{Z}_b \setminus 2T$. But then in the first case $|T| = |2T| = b - 1$ and in the second case $|T| = |2T| = 1$. In both case, a contradiction is derived with the assumption that the actions of N on B_0 and B_{2i+1} are inequivalent, completing the proof of Lemma 2.5. ■

Remark 2.6 Note that the bipartite graph arising from a pair of inequivalent 2-transitive group actions is the incidence graph of a symmetric block design with a 2-transitive automorphism group. (See [5] for the classification of all such designs.) Therefore the last part of the proof of Lemma 2.5 thus amounted to proving that in an arc-transitive circulant of square-free order, the bipartite graph induced by two adjacent blocks cannot be the incidence graph of a symmetric design with the automorphism group acting 2-transitively on the sets of points and blocks of the design, respectively.

In view of the above lemmas, to complete the proof of Proposition 2.1, we may assume that

for each minimal G -invariant partition \mathcal{X} of $V\Gamma$, letting F be the kernel of G on \mathcal{X} and $X \in \mathcal{X}$, $F \cong F^X$ is not 2-transitive on X .

Now let \mathcal{B} be a minimal G -invariant partition of $V\Gamma$, and let N be the kernel of the G -action on \mathcal{B} . Take a block $B \in \mathcal{B}$. Then by Lemma 2.3,

$$\mathbb{Z}_p \leq N \cong N^B < \mathbb{Z}_p \times \mathbb{Z}_{p-1},$$

where p is a prime. Let $M = \text{soc}(N)$, which is isomorphic to \mathbb{Z}_p . Then $M \triangleleft G$.

Lemma 2.7 *There is a subgroup H of \mathbb{Z}_{p-1} and a group C such that $G = (M \times C).H$ and $M \leq R \leq M \times C$.*

PROOF. Take $v \in V\Gamma$, and denote by G_v the stabilizer of v in G . Let P be a Sylow p -subgroup of G_v . Since n is square-free, $p|P|$ exactly divides $|G|$, and so $\langle M, P \rangle = M \times P$ is a Sylow p -subgroup of G , that is, a Sylow p -subgroup of G is a

split extension of M by P . By [7, Theorem 8.6, p. 232], G is a split extension of M by a subgroup L of G , where $L \cong G/M$, that is, $G = M \rtimes L$. Let $C = \mathbf{C}_L(M)$. Then $M \cap C = 1$, $C \triangleleft G$, and $G/(MC)$ is isomorphic to a subgroup of $\text{Aut}(M)$ which is isomorphic to \mathbb{Z}_{p-1} . Thus $G = (M \times C).H$, where $H \leq \mathbb{Z}_{p-1}$. Since R is abelian and $M < R$, we have that $R < \mathbf{C}_G(M) = M \times C$. ■

We are now ready to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. By Lemma 2.7, $G = (M_0 \times C_0).H_0$ such that $M_0 \leq R \leq M_0 \times C_0$ and $H_0 \leq \mathbb{Z}_{p_0-1}$, where p_0 is a prime. In particular, C_0 is normal in G and intransitive on $V\Gamma$. If $C_0 = 1$, then $R = M_0$ is normal in G , as required. Assume that $C_0 \neq 1$. Let \mathcal{C}_1 be the set of the C_0 -orbits in $V\Gamma$. Then \mathcal{C}_1 is a G -invariant partition of $V\Gamma$. Let $\mathcal{B}^{(1)}$ be a minimal G -invariant partition of $V\Gamma$ which is a refined partition of \mathcal{C} . Take a block $B^{(1)} \in \mathcal{B}^{(1)}$. Let N_1 be the kernel of G on $B^{(1)}$, and let $M_1 = \text{soc}(N_1)$. By our assumption, N is faithful and is not 2-transitive on $B^{(1)}$. Then by Lemma 2.3, $M_1 \cong \mathbb{Z}_{p_1}$ for some prime p_1 . By Lemma 2.7, $G = (M_1 \times C_1).H_1$ such that $M_1 \leq R \leq M_1 \times C_1$. Now $M_0 \times M_1 \leq R \leq (M_0 \times C_0) \cap (M_1 \times C_1)$. It follows that $R \leq (M_0 \times C_0) \cap (M_1 \times C_1) = M_0 \times M_1 \times C'_1$, and $G = (M_0 \times M_1 \times C'_1).H'_1$. If $C'_1 = 1$, then $R = M_0 \times M_1$ is normal in G , as required. Assume that $C'_1 \neq 1$, and assume inductively that $G = (M_0 \times M_1 \times \dots \times M_i \times C'_i).H'_i$ such that $i \geq 1$, $\mathbb{Z}_{p_j} \cong M_j \leq R$ for each j , and $R \leq M_0 \times M_1 \times \dots \times M_i \times C'_i$. Now C'_i is normal in G and intransitive on $V\Gamma$, and hence we may repeat our arguments with C'_i in place of C_0 so that we have $G = (M_{i+1} \times C_{i+1}).H_{i+1}$ such that $M_{i+1} \cong \mathbb{Z}_{p_{i+1}}$ for some prime p_{i+1} , and $M_{i+1} \leq R \leq M_{i+1} \times C_{i+1}$. Since $M_0, M_1, \dots, M_{i+1} \leq R \leq (M_0 \times M_1 \times \dots \times M_i \times C'_i) \cap (M_{i+1} \times C_{i+1})$. It follows that $R \leq (M_0 \times M_1 \times \dots \times M_i \times C'_i) \cap (M_{i+1} \times C_{i+1}) = (M_0 \times M_1 \times \dots \times M_{i+1} \times C'_{i+1})$ such that $G = (M_0 \times M_1 \times \dots \times M_i \times M_{i+1} \times C'_{i+1}).H'_{i+1}$. Therefore, repeating this argument, we finally obtain $G = (M_0 \times M_1 \times \dots \times M_k).H$ such that $R = M_0 \times M_1 \times \dots \times M_k$, which is normal in G , as required. ■

In view of the comments in the paragraph preceding the statement of Proposition 2.1, this completes the proof of Theorem 1.1.

References

- [1] B. Alspach, M. D. E. Conder, D. Marušič, M.Y. Xu, A classification of 2-arc-transitive circulants, *J. Alg. Combin.*, **5** (1996), 83–86.
- [2] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, London, New York, 1992.
- [3] J. D. Dixon and B. Mortimer, *Permutation Groups*, Springer-Verlag, New York, 1996.
- [4] C. Godsil, On the full automorphism group of a graph, *Combinatorica* **1** (1981), 243-256.
- [5] W. Kantor, Classification of 2-transitive symmetric designs, *Graphs Combin.* **1** (1985), 165–166.
- [6] G. O. Sabidussi, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964).
- [7] M. Suzuki, *Group Theory I*, Springer-Verlag, Berlin, New York, 1982.

CIRCULANTS OF SQUARE-FREE ORDER

Keywords: circulant graph, arc-transitive graph, square-free order, cyclic group, primitive group, imprimitive group.

CIRCULANTS OF SQUARE-FREE ORDER

Contact address:

DRAGAN MARUŠIČ
IMFM, Oddelek za matematiko
Univerza v Ljubljani
Jadranska 19, 1000 Ljubljana
Slovenija
tel. 386-61-1892320
fax 386-61-347997
email `dragan.marusic@uni-lj.si`