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A Polynomial Algorithm for the Strong Helly Property

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Abstract

In this article we give a polynomial algorithm to recognize if any partial sub-hypergraph of a hypergraph has the Helly property.

1 Introduction

The Helly property is one of the most important concepts in hypergraph theory because a lot of classes of hypergraph have this property [4]. It was recently studied in several papers [2, 10, 11, 12], but there are few algorithm works about this property [1, 7, 5, 8]. The existence of an efficient polynomial algorithm for checking whether a hypergraph has the Helly property is still open. In this paper we study a related property which we call the strong Helly property.

The paper is organized as follows. In the next section we give formal definitions. In Section 3 we show that hypergraph has the strong Helly property if and only if its partial sub-hypergraphs have the Helly property. We also show that the strong Helly property is equivalent to the strong Helly property to order 3. Using this, we give a polynomial algorithm for testing the strong Helly property of a hypergraph in Section 4. We discuss correctness, time and space complexity of the algorithm. In the last appendix

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we give a characterization of hypergraph with the Helly property that may allow us to give a polynomial algorithm to recognize if a hypergraph has the Helly property.

2 The definitions

A *multigraph* G is a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set of elements called *vertices*, and $E(G)$ is a finite list of (not necessarily distinct) edges. An *edge* is a pair of distinct elements of $V(G)$. For brevity, an edge $\{u, v\}$ is shortly denoted by uv. Furthermore, we use V and E for $V(G)$ and $E(G)$, if it is clear which graph we consider.
A graph $G = (V, E)$ is bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and every edge joins a vertex of V_1 to a vertex V_2. We will denote a bipartite graph by $G(V_1, V_2)$.
A *walk* is a finite sequence of edges v_0v_1, v_1v_2, $\ldots$, $v_{k-1}v_k$. If $v_k = v_0$ and allother vertices are pairwise distinct, the walk is a *cycle*. Let $G = (V, E)$ be a graph, a cycle C_{2n}, $n > 2$ has a well chord if there exist an edge e belonging to G such that e divides C_{2n} in two cycles C_{n+1}.
A graph is *connected*, if there is a walk connecting every pair of vertices. The degree of vertex x of G is the number of edges e with $x \in e$. The maximal degree of a vertex of G is denoted by $\Delta(G)$. In this paper, we will consider only connected multigraphs. For notions concerning graphs and hypergraphs not defined here we refer to [3, 4, 13].

A *hypergraph* H on a finite set $\mathbf{S}$ is a family $(E_i)_{i\in I}$, $I = \{1, 2, \ldots, k\}$ $\quad k \in \mathbb{N}$ of non-empty subsets of $\mathbf{S}$ called hyperedges with:

$$\bigcup_{i\in I} E_i = \mathbf{S}$$

Let us denote : $H = (\mathbf{S}; (\mathbf{E_i})_{\mathbf{i\in I}})$. The degree of vertex x of H is the number of hyperedges E_i with $x \in E_i$.
The *rank* of H is the maximum cardinality of an edge. A hypergraph contains some *repeated hyperedges* if $E_i = E_j$ with $i \neq j$. In this article, without loss generality we will consider that all hyperedges are distinct.
Let $H = (\mathbf{S}; \mathbf{E} = (\mathbf{E_i})_{\mathbf{i\in I}})$ be a hypergraph, the hypergraph dual H^* s the hypergraph such that the set of vertices is the set of hyperedges, and the set of hyperedges is the set of stars of H (i. e. $H^* = (E, H(x)_{x\in S})$.
A *partial hypergraph* on $\mathbf{S}$ is a subfamily $(E_j)_{j\in J}$ of $(E_i)_{i\in I}$.
A *subhypergraph* of the hypergraph H is the hypergraph $H(Y) = (Y, (E_i \cap$

$Y \neq \emptyset)_{i\in I})$, (with $Y \subseteq \mathbf{S}$).
A family of hyperedges is an *intersecting family* if every pair of hyperedges has a nonempty intersection.
A hypergraph has the *Helly property* if each intersecting family has a nonempty intersection.
Later we will define a related property which we call the *strong Helly property*. We will show that a hypergraph has the strong Helly property if every partial sub-hypergraph has the Helly property.
We say that the hypergraph $H = (V; E)$ has the *separation property* (briefly, SP) if for every pair of distinct vertices $x, y \in V$ there exists an hyperedge $E_i \in E$ such that either $x \in E_i$ and $y \notin E_i$ or $x \notin E_i$ and $y \in E_i$.

The *incidence graph* of a hypergraph $H = (S; E)$ is a bipartite graph with a vertex set $V = S \cup E$, where two vertices $x \in X$ and $e \in E$ are adjacent if and only if $x \in e$. We denote it $IG(H)$.

3 Theory

Given a fixed hypergraph H, we use the following notation:

- $k = k(H)$ is the number of hyperedges E_i in H.
- $r = \text{rank}(H)$ is the maximal size of a hyperedge of H.
- $\Delta = \Delta(H)$ is the *maximal degree* in H.

Construction of a multigraph $\mathcal{G}(H)$. Given a hypergraph H we construct an auxiliary edge labeled multigraph $\mathcal{G} = \mathcal{G}(H)$ as follows: $V(\mathcal{G}) = V(H)$ and vertices x and y are connected by an edge labeled with E_i when $\{x, y\} \subseteq E_i$. For edges in $\mathcal{G}$ we will use notation (xy, E), where xy is the edge, and E is its label.

Note that the number of edges in $\mathcal{G}$ is $\sum_{i=1}^{k}(|E_i|(|E_i| - 1)/2$, which is of order $O(kr^2)$. Furthermore, the maximal degree $\Delta(\mathcal{G}))$ of a vertex in $\mathcal{G}$ is clearly bounded by $r\Delta$.

We say that a hypergraph has the *Helly property to order k* if any intersecting family with k hyperedges at most has nonempty intersection. From the definition it is easy to observe:

Lemma 1 *H has the Helly property to order 3 if and only if for every triangle in the multigraph $\mathcal{G}(H)$ the following implication is true:*

if (xy, E_{xy}),(xz, E_{xz}),(yz, E_{yz}) *are edges of* $\mathcal{G}$
then $E_{xy} \cap E_{xz} \cap E_{yz} \neq \emptyset$.

Proof. First assume that the implication holds for any triangle in $\mathcal{G}$. Let $\{E_1, E_2, E_3\}$ be an intersecting family. Hence there are vertices $x \in E_1 \cap E_2$, $y \in E_1 \cap E_3$, $z \in E_2 \cap E_3$. By construction, there are the following three labeled edges in $\mathcal{G}$: (xy, E_1), (xz, E_2), (yz, E_3). But then $E_1 \cap E_2 \cap E_3 \neq \emptyset$, hence the Helly property to order 3 of H.

For the second part, let (xy, E_1), (xz, E_2), (yz, E_3) be any three edges forming a triangle in $\mathcal{G}$. Then, $x, y \in E_1$, $x, z \in E_2$ and $y, z \in E_3$. Consequently, $x \in E_1 \cap E_2$, $y \in E_1 \cap E_3$, $z \in E_2 \cap E_3$. Hence $\{E_1, E_2, E_3\}$ is an intersecting family. Helly property of H implies $E_1 \cap E_2 \cap E_3 \neq \emptyset$, concluding the proof. □

4 Strong Helly property and hypergraph

We now define the strong Helly property, but before we will give some definitions.
A family of hyperedges F covers an induced subgraph (denoted by GF) if any edge of GF is in an hyperedge of F.
A *triangle* in $\mathcal{G}(H)$ is an induced subgraph generated by three adjacent vertices of $\mathcal{G}(H)$.

Definition 1 *A hypergraph* H *has the strong Helly property to order* k *if and only if for every triangle* $T = (V(T), E(T))$ *of* $\mathcal{G}(H)$ *covered by an intersecting family* F *with at most* k *hyperedges , the following implication is true:*

if $(x_i x_j, E_{ij})$ *are edges of* T *in* $\mathcal{G}$
then $\exists \xi \in \{x \mid x \in V(T)\}$ *such that* $\xi \in \cap_{ij \in E(T)} E_{ij}\}$.

Finally we will say that H has the *strong Helly property* if it has the strong Helly property to order k for all k, $k \geq 3$.

For example consider a special case. The strong Helly property to order 3 holds for a hypergraph H if and only if for every triangle in the multigraph $\mathcal{G}(H)$ the following implication is true: if (xy, E_{xy}),(xz, E_{xz}),(yz, E_{yz}) are edges of $\mathcal{G}$ then $\exists \xi \in \{x, y, z\}$ such that $\xi \in E_{xy} \cap E_{xz} \cap E_{yz}$.

Lemma 2 *If hypergraph H has the strong Helly property to order k then it has the Helly property to order k.*

Proof. For $k = 3$ the statement directly follows from Lemma 1 and from the definition of the strong Helly property to order k. Suppose that the assertion is true for any p, $3 \le p \le k-1$. Let F be an intersecting family with $p+1$ hyperedges at most. By induction hypothesis we have:
$\exists x \in \cap_{i \in \{2,3,\ldots p+1\}} E_i$, $\exists y \in \cap_{i \in \{1,3,\ldots p+1\}} E_i$, and $\exists z \in \cap_{i \in \{1,2,4,\ldots p+1\}} E_i$.
This implies that there is a triangle in $\mathcal{G}(H)$ with weighted edges (xy, E_3) (xz, E_2) (yz, E_1). Since H has the strong Helly property to order 3, there is a vertex $\xi \in \{x, y, z\}$ which is in the intersection $E_1 \cap E_2 \cap E_3$. Consequently $\xi \in \cap_{i \in \{1,2,3,\ldots p+1\}} E_i$, Hence, $\xi \in \cap_i E_i$ and H has the Helly property to order $(p+1)$. □

To see that the two properties are not equivalent consider the following example: Let H be defined by a family of sets $\{E_1 = \{A, B, 3, 4\}, E_2 = \{A, C, 2, 4\}, E_3 = \{A, D, 2, 3\}, E_4 = \{B, C, 1, 4\}, E_5 = \{B, D, 1, 3\}, E_6 = \{C, D, 1, 2\}\}$. It is straightforward to verify that H has the Helly property to order 3 but it does not have the strong Helly property to order 3. (Hint: consider the triangle (A, B, E_1), (A, C, E_2) and (B, C, E_4).) From this example we conclude

Lemma 3 *The Helly property to order k does not imply the strong Helly property to order k.*

We are now going to link the strong Helly property to order k with partial subhypergraph.

Theorem 1 *A hypergraph has the strong Helly property to order k if and only if for any partial subhypergraph the Helly property to order k is true.*

Proof. Suppose that H has the strong Helly property to order k. Let H' be a partial subhypergraph of H and let $F' = (E'_i)_{i \in \{1,2,3,\ldots\}}$ be an intersecting family in H', with k hyperedges at most. There exists an intersecting family $F = (E_i)_{i \in \{1,2,3,\ldots\}}$ of H with k hyperedges at most, verifying the property of the above definition and such that for every $E'_i \in F'$ there is $E_i \in F$ with $E'_i \subseteq E_i$. Let $T' = (V(T'), E(T'))$ be a triangle of GF', GF' being an induced subgraph of GF, T' is a triangle of GF. Consequently:
if $(x_i x_j, E'_{ij} \subseteq E_{ij})$ are edges of T in $\mathcal{G}$
then $\exists \xi \in \{x \mid x \in V(T)\}$ such that $\xi \in \cap_{ij \in E(T)} E'_{ij} \subseteq E_{ij}$.

To see the converse, asssume that any partial subypergraph has the Helly property to order k. Let $(x_{i,j}, E_{i,j})$ Be edges of an arbitrary triangle T in H. Since every partial subhypergraph has the Helly property to order k $I = \bigcap_{ij \in E(T)} E_{ij} \neq \emptyset$. We claim that $V(T) \cap I \neq \emptyset$. This is true because if $V(T) \cap I = \emptyset$, then the partial subhypergraph induced on T would not have the Helly property. □

The idea of the algorithm given in section 5 is based on the following:

Theorem 2 *Hypergraph H has the strong Helly property if and only if H has the strong Helly property to order 3.*

Proof. Clearly the strong Helly property implies the strong Helly property to order 3.
We will prove the reversed implication by induction. Assume that the i-Helly property holds for $i = 3, 4, \ldots \ell$.
Let $E_1, E_2, \ldots, E_{\ell+1}$ be arbitrary intersecting family of hyperedges of H.
By induction, $\exists x \in \cap_{i \neq 1} E_i$, $\exists y \in \cap_{i \neq 2} E_i$, $\exists z \in \cap_{i \neq 3} E_i$.
This implies that there is a triangle in $\mathcal{G}(H)$ with weighted edges (xy, E_3), (xz, E_2) and (yz, E_1). Since H has the strong Helly property to order 3, there is a vertex $\xi \in \{x, y, z\}$ which is in the intersection $E_1 \cap E_2 \cap E_3$. Hence, $\xi \in \cap_i E_i$ and H has the strong Helly property to order $(\ell + 1)$. □

From this theorem we have the following result:

Corollary 1 *A hypergraph H with the separation property SP has the strong Helly property if and only if its dual H^* has the strong Helly property.*

Proof. It is easy to see that the incidence graph of a hypergraph with the SP property is isomorphic to the incidence graph of its dual H^*. Moreover, from theorem above, H has the strong 3-Helly property if and only if every C_6 of $IG(H)$ is well chorded. The corollary is proved □

5 The algorithm

We assume that the hypergraph H is given as a set (family) of sets (hyperedges). Recall that the number of hyperedges is denoted by k and that the maximal cardinality of a hyperedges, rank(H) is denoted by r.
First we give some remarks on the data structures used.
The input hypergraph H can be naturaly organized as an array (of size k)

of sorted lists of vertices (of maximal size r).
Remark: if the lists of vertices are not sorted, k lists of maximal size r can be sorted in total time of order $k \times r \times \log r$.
In the algorithm, we will also need the following data structure representing the multigraph $\mathcal{G}$ by neighborhood lists. More precisely, this is an array (of size $n = |V(\mathcal{G})|$) of lists of maximal size $\Delta(\mathcal{G})$. An element of the list is a pair (v, A), where $v \in V(\mathcal{G})$ is a vertex and $A \in E(H)$ is the edge label.

Lemma 4 *The neighborhood lists of the multigraph $\mathcal{G}(H)$ can be build in time $O(kr^2)$.*

Proof. For every pair of vertices x, y in every hyperedge E_i we have to add (y, E_i) to the neighborhood list of x and (x, E_i) to the neighborhood list of y. □

Observation: The neighborhood lists in $\mathcal{G}$ are at most Δr long.

For sorting the neighborhood lists of $\mathcal{G}$ we need time proportional to $n\Delta r \log(\Delta r)$ or $kr\Delta r \log(\Delta r)$. (We use the fact that n, the number of vertices of $\mathcal{G}$, is at most kr.) Hence the following

Lemma 5 *For preparing the sorted neighborhood lists of the auxiliary multigraph $\mathcal{G}(H)$ we need at most $O(kr\Delta r \log(\Delta r)) = O(kr^2\Delta \log(\Delta r))$ time.*

For checking all sets of three hyperedges forming a triangle in H, we have to do the following computation on the multigraph $\mathcal{G}$.

ALGORITHM A: check all triangles in $\mathcal{G}$

```
for every edge (x, y, A_1) of G do
    for every pair of edges (x, z, A_2), (y, z, A_3) do
        if x ∉ A_1 ∩ A_2 ∩ A_3 and
            y ∉ A_1 ∩ A_2 ∩ A_3 and
            z ∉ A_1 ∩ A_2 ∩ A_3
        then begin
            output(the STRONG HELLY PROPERTY DOES NOT HOLD)
            stop.
        end
end.
```

Lemma 6 *The algorithm* **A** *is correct.*

Proof. Correctness of **A** is a straightforward implication from the definition of the strong 3-Helly property and Theorem 2. □

To estimate the time complexity of **A** recall that: the number of edges in $\mathcal{G}$ is of order $O(kr^2)$, the length of the neighborhood lists in $\mathcal{G}$ is at most Δr and the maximal size of a hyperedge is r. The total time needed is therefore $O(kr^2\Delta rr) = O(k\Delta r^4)$

Lemma 7 *The algorithm* **A** *runs in* $O(k\Delta r^4)$ *time.*

Finally, recall that the total space complexity is $O(kr^2)$. The total time complexity is the maximum of $O(kr\Delta r\log(\Delta r))$ needed for preparing the sorted neighborhood lists and $O(kr^2\Delta rr) = O(k\Delta r^4)$ for algorithm **A**. Summarizing we can write:

Theorem 3 *The strong Helly property of a hypergraph H with k hyperedges, maximal degree Δ and rank r can be tested in $O(k\Delta r^4)$ time and $O(kr^2)$ space.*

6 Appendix: Another characterization of Helly property

In this section we will, without loss of generality, consider that any vertex is adjacent to itself.
We prove the following result:

Theorem 4 *Let H be a hypergraph, H has the Helly property if and only if for every triangle $T = (V(T), E(T))$ covered by an intersecting family F of H the following property is true:*

(a) If (x_ix_j, E_{ij}) are edges of T in $\mathcal{G}$ then $\exists z \in V$, z adjacent to all vertices of T such that $z \in \cap_{i,j\in E(T)} E_{ij}$.

Proof. For any triangle covered by an intersecting family clearly the Helly property involves the condition (a).
We are going to show the converse by induction on the maximum cardinality of intersecting families.
Let $F = (E_1, E_2, E_3)$ be an intersecting family with three hyperedges. Let (x, y, z) be a triangle covered by F. Suppose that (xy, E_1); (yz, E_2); (xz, E_3) are edges of T in $\mathcal{G}$. From the condition (a) F is a star, so H has the Helly

property.
Suppose now that any intersecting family F with at most k hyperedges is a star.
Let $F = (E_1, E_2, \dots E_{k+1})$ be an intersecting family with $k+1$ hyperedges. By induction hypothesis we have: $\exists x \in \bigcap_{i\in\{2,3,\dots k+1\}}$, $\exists y \in \bigcap_{i\in\{1,3,\dots k+1\}}$, $\exists z \in \bigcap_{i\in\{1,2,4,\dots k+1\}}$. It is easy to see that x, y, z generates a triangle T. This triangle is covered by the family F. Hence there exists u adjacent to x, y and z such that $u \in \cap_{i,j\in E(T)} E_{ij}\}$. But the edge (xy, E_1) is contained in $(E_{i\in\{2,3,\dots k+1\}}$, (yz, E_2) is contained in $(E_{i\in\{1,4,\dots k+1\}}$ and (zx, E_3) is contained in $(E_{i\in\{1,2,4\dots k+1\}}$. Consequently $u \in \bigcap_{i\in\{1,2,3,4,\dots k+1\}}$ and F is a star. □

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