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BEHZAD-VIZING CONJECTURE IS TRUE FOR CARTESIAN PRODUCT GRAPHS

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Behzad-Vizing conjecture is true for Cartesian product graphs

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Abstract

We prove the following theorem: if the Behzad-Vizing conjecture is true for graphs G and H, then is it true for the Cartesian product $G \Box H$.

Key words: Cartesian graph product, chromatic number, total chromatic number, Vizing conjecture.

1 Introduction

Graph products were first defined by Sabidussi [5] and Vizing [7]. A lot of work was done on various topics related to graph products, but on the other hand there are still many questions open. For a very recent survey, see [3].

Here we consider the total chromatic number on the Cartesian product graphs. It is well known and easy to see that the chromatic number of the Cartesian product is the maximum of the chromatic numbers of the factors. For the chromatic index, the Cartesian product is of type I exactly when at

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least one of the factors is of type I [4]. In this paper we give a corresponding result for the total chromatic number. Total chromatic numbers of Cartesian products of a path and a star, a cycle and a star, a path and a cycle and (in certain cases) of two cycles are given in [9] (see also [6]).

The *Cartesian product* $G\Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which the vertex (a, b) is adjacent to the vertex (c, d) whenever $a = c$ and b is adjacent to d, or $b = d$ and a is adjacent to c. The G and H *layers* are the induced subgraphs in $G\Box H$ on vertex sets $G_u = \{(x, u) \mid x \in V(G)\}$ and $H_v = \{(v, x) \mid x \in V(H)\}$, respectively. An element of a graph G is either a vertex or an edge of G. In a *proper total coloring*, any two elements that are either adjacent or incident are assigned different colors. The minimum number of colors needed for a proper total coloring is the *total chromatic number* of G, denoted by $\chi''(G)$. The maximum vertex degree in G is denoted by $\Delta(G)$.

Total coloring was introduced by Behzad [1, 2] and Vizing [8], both of whom conjectured that for any graph G, the following inequality holds:

$$\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2.$$

The lower bound is clearly best possible. Graphs that need only $\Delta(G)+1$ colors are called graph of *type I.* Examples include the cycles C_{3k}, paths and complete graphs with odd number of vertices K_{2k+1}. Graphs that need at least $\Delta(G)+2$ colors are called graphs of *type II.* Examples include the cycles C_{3k+1} and C_{3k+2}, complete graphs with even number of vertices K_{2k}, and the Cartesian product $K_2\Box C_5$. For a survey on total colorings see [10].

Our key lemma is the following

Lemma 1 *Let G and H be arbitrary graphs such that $\Delta(G) \leq \Delta(H)$. Then*

$$\chi''(G\Box H) \leq \Delta(G) + \chi''(H).$$

Clearly, if for H the Behzad-Vizing conjecture is true, then $\chi''(G\Box H) \leq \Delta(G) + \chi''(H) \leq \Delta(G) + \Delta(H) + 2$, hence the conjecture is true for $G\Box H$. In other words:

Theorem 1 *If the Behzad-Vizing conjecture is true for graphs G and H, then it is true for the Cartesian product $G\Box H$.*

Remark: From the lemma above it easily follows that:

1. If the factor with largest vertex degree is of type I, then the product is also of type I.

2. $\chi''(G\Box H) \leq \min\{\Delta(G) + \chi''(H), \Delta(H) + \chi''(G)\}$ provided $\Delta(G) = \Delta(H)$. In particular, it follows that the product of two cycles $C_{3k}\Box C_n$ is of type I.

3. Any graph product $P_m\Box S_n$ or $P_n\Box C_m$, $n > 2$ is of type I, where S_m is a star (graph with one universal vertex and m vertices of degree 1).

However, there are examples in which one factor is of type I and the other factor is of type II, and the Cartesian product is in some cases of type I and in other cases of type II. Furthermore, there are examples where the product of two type II factors is of type I. See section Examples.

2 Proof of the Lemma

Recall the statement of the lemma: Let G and H be arbitrary graphs such that $\Delta(G) \leq \Delta(H)$. Then we will prove that

$$\chi''(G\Box H) \leq \Delta(G) + \chi''(H).$$

We prove the assertion by constructing the total coloring as follows.

First color edges of G using colors -1,-2,...,-($\Delta(G)$ +1). (Color edges of all layers G_u in this way. Hence vertices and edges of layers H_v are left uncolored.) Clearly, there is at least one color at each vertex $v \in V(G)$ which is not used for any edge incident to v. Denote the color not used at v by $free(v)$.

Let I_1, I_2,...,$I_{\chi(G)}$ be independent vertex sets of G. Recall that $\chi(G) \leq \Delta(G) + 1 \leq \Delta(H) + 1 \leq \chi''(H)$.

Let T_1, T_2,...,$T_{\chi''(H)}$ be independent sets of H of some proper total coloring. (Hence $T_i \subseteq V(H) \cup V(G)$.)

Assign temporarily the colors to elements of $G\Box H$ as follows: Element of $I_i \times T_j$ receives color $((i - 1 + j - 1) \bmod \chi''(H)) + 1$. It is straightforward to see that colors 1,2,...,$\chi''(H)$ are used and that this is a proper coloring.

Finally replace color $\chi''(H)$ in layer H_v by the color $free(v)$. Observe that this does not introduce any violation of the proper coloring.

Summarizing we see that $\Delta(G) + 1 + \chi''(H)$ colors were used, but after recoloring we have a proper total coloring with $\Delta(G) + \chi''(H)$ colors as claimed.

3 Examples

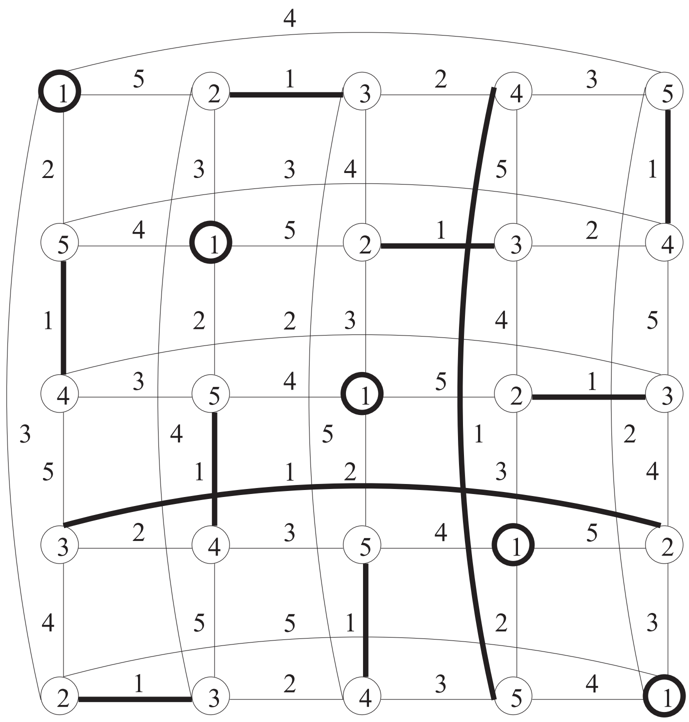

Figure 1: Type I Cartesian graph product of type II graphs.

In this section we consider examples of Cartesian product graphs with $\Delta(G) = \Delta(H)$ (Examples 1. and 2.) and examples of Cartesian product graphs with $\Delta(G) < \Delta(H)$ (Examples 3. and 4.).

Example 1. $\chi''(K_2 \Box K_2) = \chi''(C_4) = 4 = \Delta(C_4) + 2$
Recall that K_2 is of type II and therefore C_4 is an example of type II Cartesian graph product of type II graphs.

Example 2. $\chi''(C_5 \Box C_5) = 5 = \Delta(C_5 \Box C_5) + 1$
Since C_5 is type II graph, this is an example of type I Cartesian graph product of type II graphs. For a proper 5-total coloring see Fig. 1.

Examples 1. and 2. confirm that our result is the best possible.

Example 3. Analog to Examples 1. and 2., $K_2 \Box C_4$ and $K_2 \Box C_5$ are examples of type I and type II Cartesian graph products of type II graphs, provided $\Delta(G) < \Delta(H)$. For proper total colorings see Fig. 2 and 3.

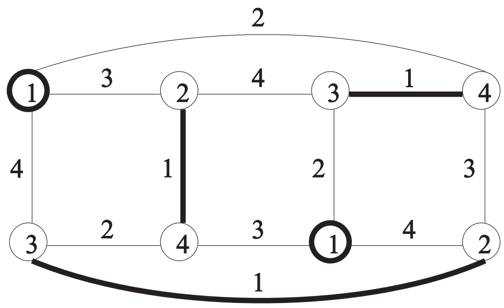

Figure 2: Type I Cartesian graph product of type II graphs.

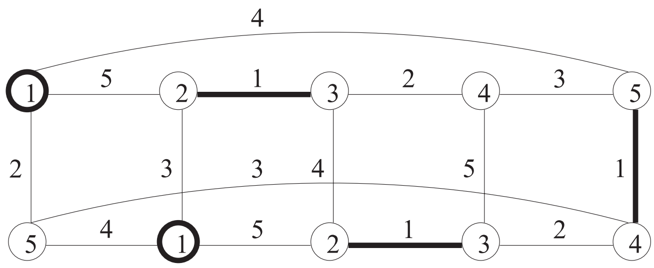

Figure 3: Type II Cartesian graph product of type II graphs.

If $\Delta(G) < \Delta(H)$, type of graph H is significant in the hand provided by the Lemmma. However, the type of the product may again be I, even if H id of type II.

Example 4. $\chi''(K_3 \Box K_4) = 6 = \Delta(K_3 \Box K_4) + 1$
Recall that K_3 is of type I and K_4 is of type II and $\Delta(K_3) < \Delta(K_4)$. In Fig. 4 we give proper 6-coloring of graph $K_3 \Box K_4$.

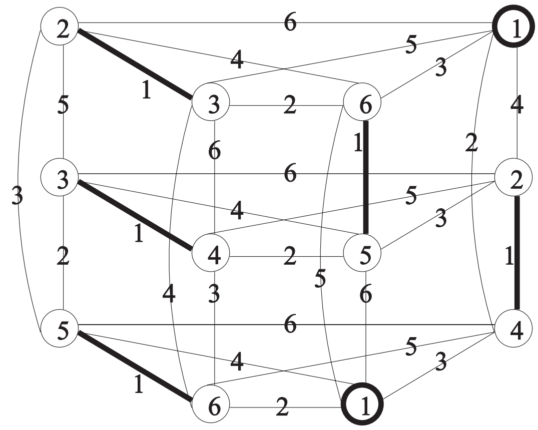

Figure 4: Type I Cartesian graph product of type I and type II graphs.

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