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IMPROVED LOWER BOUND ON SHANNON CAPACITY OF C_7

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Improved lower bound on the Shannon capacity of C_7

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Abstract

An independent set with 108 vertices in the strong product of four 7-cycles ($C_7 \boxtimes C_7 \boxtimes C_7 \boxtimes C_7$) is given. This improves the best known lower bound for the Shannon capacity of the graph C_7 which is the zero-error capacity of the corresponding noisy channel. The search was done by a computer program using the "simulated annealing" algorithm with a constant time temperature schedule.

Keywords: Shannon capacity; Combinatorial problems; Design of algorithms.

1 Introduction and definitions

The study of Shannon capacity was introduced by Shannon in [13], and motivated by the determination of the zero-error capacity of a noisy channel. Shannon formulated the problem in the form of graph theory and supplied some partial results. It turns out that the solution of the problem requires the determination of the independence number of product of graphs which contain odd cycles.

The Shannon capacity of the 7-cycle is one of the unsolved problems given in [6]. It may be interesting to note that the capacity of C_5 was studied already by Shannon [13] in 1956, and was determined only in 1979 by Lovász [10].

All graphs considered in this paper are connected finite undirected graphs without loops or multiple edges. If G is a graph, we shall write $V(G)$ or V for its vertex set and $E(G)$ or E for its edge set. $E(G)$ is a set of unordered pairs $xy = \{x, y\}$ of distinct vertices of G.

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The *strong product* of graphs G and H is the graph $G \boxtimes H$ with vertex set $G \times H$ and $(x_1, x_2)(y_1, y_2) \in E(G \boxtimes H)$ whenever $x_1y_1 \in E(G)$ and $x_2 = y_2$, or $x_2y_2 \in E(H)$ and $x_1 = y_1$, or $x_1y_1 \in E(G)$ and $x_2y_2 \in E(H)$. The strong product is commutative and associative in an obvious way, having the trivial graph as a unit.

A set $S \subseteq V(G)$ is *independent* if $xy \notin E(G)$ for any pair of vertices $x, y \in S$. Cardinality of a largest independent set S of G is the *independence* number of G, $\alpha(G)$.

The *Shannon capacity* $\Theta(G)$, of a graph G, is defined by:

$$\Theta(G) := \sup_n (\alpha(G^n))^{\frac{1}{n}}.$$

where $G^n = G \boxtimes G \boxtimes \cdots \boxtimes G$ (n times).

This paper is structured as follows. In the next section we recall an upper bound on the Shannon capacity and some known results on the independence number of C_7^n. In Section 3 we consider the allocation of potential independent vertices in C_7^4. In Section 4 we introduce the simulated "annealing" algorithm with a constant time temperature schedule and with heuristically restricted solutions. An independent vertex set with 108 vertices is obtained with the algorithm. This improves the best known lower bound for the Shannon capacity of C_7. We conclude with two questions presented in Section 5.

2 Preliminaries

Lovász [10] obtained general upper bounds on the Shannon capacity which yield the exact value of the capacity for many graphs, including the long-standing open case of the pentagon.

Theorem 1 [10] *For any d–regular graph G on n vertices*

$$\Theta(G) \leq \frac{-n \cdot \lambda_{min}}{d - \lambda_{min}},$$

where λ_{min} denotes the least eigenvalue of G.

However, the Shannon capacity of the 7-cycle is still an unsolved problem. In order to improve the lover bound, the independence number of C_7^4 is considered in this paper.

The independence numbers of the strong product of even cycles are well known. The problem is incomparably more difficult for more cycles with odd length. Hales [8], as well as Sonnemann and Krafft [15], discovered the explicit formula for the independence numbers of the strong product of two odd cycles.

Theorem 2 [8, 15] *For $j, k \in \mathbb{N}$, $j \geq k$*

$$\alpha(C_{2j+1} \boxtimes C_{2k+1}) = jk + \left\lfloor \frac{k}{2} \right\rfloor.$$

The independence numbers of products of more then two odd cycles are still unknown in most cases. Even for the product of three cycles there are several products with unknown independence numbers [16].

The general lower bound on the independence number follows from the well-known inequality

Lemma 3 *For all graphs G and H*

$$\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H).$$

The next lemma can be used to bound independent size growth in the strong products of cycles.

Lemma 4 *For any graph G and $k \geq 2$*

$$\alpha(G \boxtimes C_{2k+1}) \leq k\alpha(G) + \left\lfloor \frac{\alpha(G)}{2} \right\rfloor.$$

The independence numbers of C_7^n are known only for $n \leq 3$. For $n = 2$ the result is obtained using Theorem 2 and for $n = 3$ the result was obtained by a computer search [2, 16].

For $n > 3$ this numbers are bounded by Lemma 4 and Lemma 3. The situation is showed in Table 1 for $n \leq 5$. Note that there is an exception: the lower bound for $\alpha(C_7^5)$ is obtained by a construction of a set of 343 independent vertices in [2].

n	1	2	3	4	5
$\alpha(C_7^n)$	3	10	33	100 - 115	343 - 402

Table 1: Independence numbers of C_7^n, $n \leq 5$

One can obtain the following lower bound and the upper bound on the Shannon capacity of C_7 applying the lower bound for $\alpha(C_7^5)$ and Theorem 1, respectively.

Proposition 5 $\sqrt[5]{343} = 3.21409 \leq \Theta(C_7) \leq 3.3176.$

In order to improve the lower bound the computer search for larger independent sets in C_7^4 has been applied. We used a randomized heuristics based on the simulated annealing, one of the most popular heuristics.

3 The structure of a solution

At the beginning of our computer search, the allocation of the independent vertices in the initial solution as well as in the intermediary solutions was not restricted. Since the search was not particularly successful (only independent sets with at most 100 vertices were found),

we constrained the distribution of the independent vertices in the solutions as follows.

It is a straightforward exercise to obtain the following well-known result.

Lemma 6 *For a graph G and $n \in \mathbb{N}$*

$$\alpha(G \boxtimes K_n) = \alpha(G).$$

The following obvious corollary will help us to establish the structure of a possible solution.

Corollary 7 *(i)* $\alpha(C_7^2 \boxtimes K_4) = \alpha(C_7^2) = 10.$
(ii) $\alpha(C_7^3 \boxtimes K_2) = \alpha(C_7^3) = 33.$

This result suggests that vertices in a large independent set should be allocated quite evenly in the graph.

The C_7^4 can be visualized as a a square array of 49 cells where a cell corresponds to a copy of $C_7 \boxtimes C_7$. For $i \in \{0, 1, \ldots, 48\}$, a copy of $C_7 \boxtimes C_7$ will be denoted $(C_7 \boxtimes C_7)_i$. Consider Table 2, where the number in a cell denotes the number of potentially independent vertices in the corresponding product of two 7-cycles $(C_7 \boxtimes C_7)_i$. From Corollary 7 it follows that (i) four mutually adjacent cells can have at most ten independent vertices together and (ii) two adjacent rows (columns) can have at most 33 independent vertices together. One can easily verify that both conditions are fulfilled in the represented distribution.

3	2	3	2	2	2	2
2	2	2	2	3	2	2
2	3	2	2	2	2	3
2	2	2	3	2	2	2
3	2	2	2	2	3	2
2	2	3	2	2	2	2
2	2	2	2	2	3	2

Table 2: A distribution of 108 potentially independent vertices

There are 10 cells with three independent vertices in the table. Without the numbers in the cells, Table 2 can also be seen as a representation of the graph

$C_7 \boxtimes C_7$, where a cell corresponds to a simple vertex. Note that the positions of the cells with three independent vertices correspond to the positions of the vertices in the largest independent vertex set of $C_7 \boxtimes C_7$. ($\alpha(C_7 \boxtimes C_7 = 10$ by Theorem 2.) Thus, the allocation of 108 potentially independent vertices seems to be very well balanced. It is also obvious that larger independent sets cannot be balanced in this way.

4 Simulated "annealing" with a constant temperature schedule

Simulated annealing (see, for example [9]) can be understood as a random relaxation of the iterative improvement algorithm [1]. The moves to higher cost solutions are accepted only with a certain probability. This acceptance probability depends on the parameter, which is usually called the temperature. Sometimes very good solutions are found by SA, but it is also well known that SA can be extremely time consuming. Furthermore, the best known convergence results hold only if the so-called cooling schedule is very slow [11, 7]. On the other hand, it is known for some time that at least asymptotically and with a usual implementation, where rejected moves are explicitly counted, the probability of success of simulated annealing is worse than as simple a procedure as multi-starts of the iterative improvement algorithm [5]. Recently, a detailed analysis of various temperature schedules was given [4].

Among various temperature schedules, a constant temperature schedule for simulated "annealing" was found competitive or at least worth consideration [4]. Similar experience with a SA-like algorithm for graph coloring [12], namely very fast convergence of the algorithm on an interval of "good temperatures" [14], motivated us to try constant time schedule.

While all attempts with standard cooling schedules failed, we succeeded using a constant time temperature schedule.

Based on the discussion described above we present the following procedure. The cost function $C(S)$ is given by $C(S) = |S| - m_S$, where m_S is the number of edges in the subgraph induced by the set of vertices S.

Procedure CONSTANT ANNEALING(T);

1. Randomly select a subset of vertices S such that the number of vertices in $(C_7 \boxtimes C_7)_i$ equals the number in the corresponding cell of Table 2;
3. **for** $j := 1$ **to** *iteration length* **do begin**
   4. Randomly select i in $\{0, 1, \ldots, 48\}$.
   5. Randomly select vertices v and u in $(C_7 \boxtimes C_7)_i$, such that $v \in S$ and $u \notin S$;
   6. $S_j := S \cup \{u\} \setminus \{v\}$;
   7. **If** $C(S) \geq C(S_j)$ **then** $S := S_j$

8. **else if** $e^{\frac{C(S)-C(S_j)}{T}}$ > rnd[0,1) **then** $S := S_j$;
end;
end.

The algorithm was implemented in Delphi (Object Pascal) and tested on the personal computer with Pentium II processor. Experiments with CONSTANT ANNEALING for various ranges of the parameter T have been performed. Table 3 shows the results produced after 20 runs of the procedure with *iteration length* $= 10^9$ and with T ranging from 0.05 to 0.14. $\bar{S}$ and S_{max} are the average size and the largest size of the independent set found, respectively.

T	0.05	0.06	0.07	0.08	0.09
S_{max}	104	105	104	105	106
$\bar{S}$	102.70	103.15	103.60	104.40	104.20
T	0.10	0.11	0.12	0.13	0.14
S_{max}	108	106	105	103	104
$\bar{S}$	104.35	103.85	103.30	102.60	102.70

Table 3: Influence of the parameter T on the performance of CONSTANT ANNEALING

It appears that the procedure yields best results for T between 0.08 and 0.11. However, we cannot deduce that the optimal value of the parameter T is achieved at $T = 0.1$ from the Table 3. Indeed, the independent set with 108 vertices depicted in Figure 1 was found in one of the preliminary runs of the procedure at $T = 0.11$.

This improves the best known lower bound for the Shannon capacity of the graph C_7. The new lower bound together with the upper bound is stated in the next theorem.

Theorem 8 $\sqrt[4]{108} \leq \Theta(C_7) \leq \dfrac{7(2\sqrt{7}\cos(\frac{2\arctan(\frac{\sqrt{3}}{9})+\pi}{6})+1)}{7+2\sqrt{7}\cos(\frac{2\arctan(\frac{\sqrt{3}}{9})+\pi}{6})}$.

Corollary 9 $3.2237 \leq \Theta(C_7) \leq 3.3176$.

5 Conclusions

In this paper we have reported on searching for independent vertices in the strong product of four 7-cycles $C_7 \boxtimes C_7 \boxtimes C_7 \boxtimes C_7$. For computer search we used simulated annealing, one of the most popular randomized heuristics.

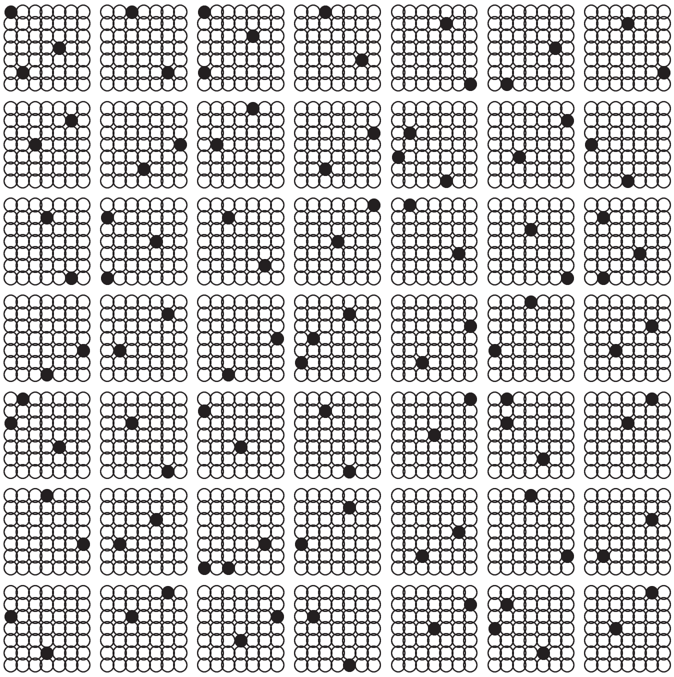

Figure 1: 108 independent vertices of $C_7 \boxtimes C_7 \boxtimes C_7 \boxtimes C_7$

In order to build an effective algorithm we also heuristically restricted the structure and the cardinality of independent vertex sets. The algorithm was efficient enough to find an independent set with 108 vertices in the graph. This improves the best known lower bound for the Shannon capacity of the graph C_7.

We think that the following two questions regarding the perspective of computer search in order to improve the Shannon capacity of C_7 are considerable.

Question 1. Is $\alpha(C_7^4) > 108$?

Although many experiments were done on searching an independent set with more than 108 vertices, all attempts failed. Therefore we conjecture that $\alpha(C_7^4) = 108$. Furthermore, we showed in the paper that the independent set with more than 108 vertices cannot be well balanced in a way we did it for the smaller one. This in some sort supports the conjecture.

Question 2. Is $\alpha(C_7^5) > 343$?

The graph C_7^5 is a huge one. It has 16807 vertices and it seems that the problem exceeds the capability of todays personal computers. Perhaps the search could be performed on a supercomputer or as the alternative one could consider a parallel implementation.

References

[1] E.H.L.Aarts, J.K.Lenstra, *Local search algorithms*, John Wiley & Sons, Chichester 1997.

[2] L.D. Baumert, R.J. McEliece, E. Rodemich, H.C. Rumsey, R. Stanley, H. Taylor, A combinatorial packing problem, in: *Computers in Algebra and Number Theory* (proc. SIAM-AMS Sympos. Appl. Math.), 1971, Amer. Math. Soc., Providence, R.I., 97-108.

[3] K.D.Boese, D.E.Franklin, A.B.King, Training minimal artificial network architectures for subsoil object detection, Manuscript.

[4] H.Cohn, M.Fielding, Simulated annealing: searching for an optimal temperature schedule, *SIAM J. Optim.* 9 (1999) 779-802.

[5] A.Ferreira, J.Žerovnik, Bounding the probability of success of stochastic methods for global optimization, *Comput. Math. Appl.* 25 (1993) 1-8.

[6] C.D. Godsil, Problems in algebraic combinatorics, *Electron. J. Combin.* 2 (1995) 20 pp.

[7] B.Hajek, G.Sasaki, Simulated annealing - to cool or not, *System Control Lett.* 12 (1989) 443-447.

[8] R.S. Hales, Numerical invariants and the strong product of graphs, *J. Combin. Theory Ser. B* 15 (1973) 146-155.

[9] P.J.M.Laarhoven, E.H.L.Aarts, *Simulated Annealing, Theory and Applications*, D.Reidel Publishing Company, Dordrecht 1987.

[10] L.Lovasz, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory* 25 (1979) 1-7.

[11] D.Mitra, F.Romeo, A. Sangiovanni-Vincentelli, Convergence and Finite Time Behavior of Simulated Annealing, *Adv. Appl. Prob.* 18 (1986) 747-771 .

[12] A.Petford, D.Welsh, A Randomised 3-colouring Algorithm, *Discrete Math.* 74 (1989) 253-261.

[13] C.E.Shannon, The zero-error capacity of a noisy channel, *IRE Trans. Inform. Theory* 2 (1956) 8-19.

[14] J.Shawe-Taylor, J.Žerovnik, Adapting temperature for some randomized local search algorithms, *Preprint series department of mathematics* 36 (1998) no. 614.

[15] E. Sonnemann and O. Krafft, Independence Numbers of Product Graphs, *J. Combin. Theory Ser. B* 17 (1974) 133-142.

[16] A.Vesel, The independence number of the strong product of cycles, *Comput. Math. Appl.* 36 (1998) 9-21.