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1. Introduction

It is a classical result that each n-dimensional Stein manifold can be properly holomorphically embedded into $\mathbf{C}^N$ for $N \geq 2n+1$ ([GR], p. 226); this is analogous to the embedding theorem for real manifolds. Since it is known that an n-dimensional Stein manifold is homotopically equivalent to a real n-dimensional CW-complex, it seems natural to expect that any such manifold could be embedded into $\mathbf{C}^N$ for dimensions $N \geq 2n+1-[\frac{n}{2}]$. In 1971 Gromov and Eliashberg announced in [EG1] that this is possible for dimensions $N \geq [\frac{n}{2}]+n+2$; later (see [EG2]) they improved their bound to $N \geq [\frac{n+1}{2}]+n+1$ using the homotopy principle from [Gr]. Finally, Schürmann showed in [Sch1] that any Stein n-manifold can be embedded into $\mathbf{C}^N$ for $N = [\frac{n}{2}]+n+1$; this dimension cannot be lowered in general ([Fs], [Sch1]).

The main result of this paper is the following:

Theorem 1.1. *Let X be an n-dimensional Stein manifold, Y a discrete subset of X, $N = \max\{[\frac{n+1}{2}]+1, 3\}$, $N' = \max\{[\frac{n+1}{2}], 1\}$ and $\varphi : Y \longrightarrow \mathbf{C}^{n+q}$ a map for some $q \geq 0$. Then the following hold:*

(a) If $q \geq N$ and φ is proper and injective there exists a proper holomorphic embedding $\Phi : X \longrightarrow \mathbf{C}^{n+q}$ extending φ.

(b) If $q \geq N'$ and the map φ is proper then there exists a proper holomorphic immersion $\Phi : X \longrightarrow \mathbf{C}^{n+q}$ extending φ.

(c) If $q \geq 1$ and the map φ is proper then there exists a proper holomorphic map $\Phi : X \longrightarrow \mathbf{C}^{n+q}$ extending φ.

(d) If $q \geq 0$ then φ can be extended to an almost proper holomorphic map $\Phi : X \longrightarrow \mathbf{C}^{n+q}$.

The most important part of theorem 1.1 is statement (a), which is an analogue of the Eliashberg–Gromov–Schürmann embedding theorem with additional interpolation on discrete sets. Our embedding dimension N is optimal when n is even and could perhaps be lowered for one when n is odd; at this time we don't know whether the loss is due to our method. Part (b) is the analogous interpolation result for holomorphic immersions. If we assume that the image $\phi(Y)$ of our discrete set Y is a tame subset of $\mathbf{C}^{n+q}$ in the sense of [RR], theorem 1.1. can be improved to the following:

Theorem 1.2. *Let X be a Stein n-manifold, $Y \subset X$ a discrete set, $q \geq \max\{[\frac{n}{2}]+1, 2\}$ and $\varphi : X \longrightarrow \mathbf{C}^{n+q}$ a holomorphic map such that $\varphi|_Y : Y \longrightarrow \mathbf{C}^{n+q}$ is proper and injective and $\varphi(Y)$ is a tame subset of $\mathbf{C}^{n+q}$. For each $y \in Y$ let $m_y \in \mathbf{N}_0$ be a given integer. Then there exists a proper holomorphic embedding $\Phi : X \longrightarrow \mathbf{C}^{n+q}$ such that $j_{m_y}(\Phi)(y) = j_{m_y}(\phi)(y)$ for each $y \in Y$.*

Proof. The main result of [Sch1] provides an embedding $g : X \longrightarrow \mathbf{C}^{n+q}$ such that the image of any discrete set in X is a tame set in $\mathbf{C}^{n+q}$. Hence there is an automorphism G of $\mathbf{C}^{n+q}$ such that $j_{m_y}(G \circ g)(y) = j_{m_y}\varphi(y)$ for each $y \in Y$ ([BFo]). ♠

The following more general embedding theorem, with interpolation on any closed complex submanifold Y of X, was obtained by Aquistapace, Broglia and Tognolli in [ABT], based on ideas of Narasimhan ([Na]):

Theorem 1.3. *Let X be a Stein n-manifold and $Y \subset X$ a closed complex submanifold. Then any proper holomorphic embedding $f : Y \longrightarrow \mathbf{C}^m$ for $m \geq 2n+1$ can be extended to a proper holomorphic embedding $F : X \longrightarrow \mathbf{C}^m$.*

For $m = 2n$ the existence of such an extension is an open problem. However, for $m \leq 2n-1$ there are immediate counterexamples to theorem 1.3. which are constructed using the following:

Theorem 1.4. ([Fo]) *For each $k \in \{1, \ldots, l-1\}$ there exists a proper holomorphic embedding $f : \mathbf{C}^k \longrightarrow \mathbf{C}^l$ such that there is no nondegenerate holomorphic map $g : \mathbf{C}^{l-k} \longrightarrow \mathbf{C}^l \setminus f(\mathbf{C}^k)$.*

To see how this theorem implies the existence of counterexamples choose $m \leq 2n-1$, $m \geq n+1$ and set $l = m$, $k = m-n$. Let $f : \mathbf{C}^k \longrightarrow \mathbf{C}^m$ be a map from theorem 1.4. Assume that f can be extended to a proper holomorphic embedding $F : \mathbf{C}^n \longrightarrow \mathbf{C}^m$. This means that $F(\mathbf{C}^n \setminus \mathbf{C}^k)$ is a subset of $\mathbf{C}^m \setminus f(\mathbf{C}^k)$. Define $G : \mathbf{C}^n \longrightarrow (\mathbf{C}^n \setminus \mathbf{C}^k)$ by $G(z_1, \ldots, z_n) = (z_1, \ldots, z_k, \exp(z_{k+1}), \ldots, \exp(z_n))$. The map $F \circ G : \mathbf{C}^n \longrightarrow \mathbf{C}^m \setminus f(\mathbf{C}^k)$ is nondegenerate which contradicts theorem 1.4. In fact, f even does not have any injective holomorphic extension $F : \mathbf{C}^n \longrightarrow \mathbf{C}^m$. ♠

The structure of the paper is as follows. In the second section we deal with extension of almost proper and proper maps. In the third section some tools such as the h-principle are presented, and the last section contains the proof of the main theorem.

Some definitions and notations.
For $y \in \mathbf{C}^n$ let $|y| := \sup\{|y_i|, 1 \leq i \leq n\}$ denote the sup norm and $\|y\|$ the euclidean norm. With $\mathbf{N}_0$ we denote the set of nonnegative integers. Let $M^{k\times n}$ be the set of $k \times m$ matrices over $\mathbf{C}$. Let X be a complex manifold, $K \subset X$ a compact subset and $f : X \longrightarrow \mathbf{C}^n$ a continuous map. We will use the notations $|f|_K := \max\{|f(x)|,\ x \in K\}$ and $\|f\|_K := \max\{\|f(x)\|,\ x \in K\}$. By $B_n(r)$ we denote the ball in $\mathbf{C}^n$ with radius r and center 0. When $n = 1$ we will skip the subscript: $B(r) := B_1(r)$.

Let $\mathcal{O}(X)$ denote the space of all holomorphic functions on a complex manifold X equipped with the standard topology of uniform convergence on compact sets. For an analytic set $Y \subset X$ let $\Gamma(X, \mathcal{J}(Y))$ denote the space of holomorphic functions on X which vanish on Y.

An open relatively compact set P in an n-dimensional complex manifold X is called *a special analytic polyhedron* if there exist holomorphic functions $f_1, \dots, f_n : X \longrightarrow \mathbf{C}$ such that P is the union of a finite number of connected components of the set $\{x \in X,\ |f_1(x)| < 1, \dots, |f_n(x)| < 1\}$. The functions $f_1, \dots, f_n$ are called *the defining functions for* P.

Let X be a complex manifold. With TX we denote the complex tangent bundle of X and with T_xX the complex tangent space of X at the point x. Let Y be another complex manifold an let $f : X \longrightarrow Y$ be a holomorphic map. Denote by $Df : TX \longrightarrow TY$ the derivative of the map f and let $D_xf : T_xX \longrightarrow T_{f(x)}Y$ be the derivative of f at x. By $j_m(f)(x)$ we denote the jet of order m of f at the point x.

A holomorphic map $f : X \longrightarrow Y$ is *almost proper* if for each compact set $K \subset Y$ the connected components of $f^{-1}(K)$ are compact. A *stratification of a complex manifold* X is a finite descending chain of analytic sets $A_m := X \supset A_{m-1} \dots \supset A_0$ such that $A_i \setminus A_{i-1}$ is a complex manifold ($i = 1, \dots, m$).

2. Almost proper and proper maps

In this section we are dealing with extensions of proper and almost proper maps. The statements (c) and (d) in theorem 1.1. are special cases of theorems in this section.

Proposition 2.1. *Let X be a n-dimensional Stein manifold, $Y \subset X$ a discrete subset and $F : X \longrightarrow \mathbf{C}^n$ a holomorphic map. Let $\{P_j\}_{j\in\mathbf{N}}$ be a sequence of special analytic polyhedra exhausting X, with $\overline{P_j} \subset P_{j+1}$ and $\partial P_j \cap Y = \emptyset$ for each $j \in \mathbf{N}$. For each $y \in Y$ let a number $m_y \in \mathbf{N}_0$ be given. Let $K \subset P_1$ be a compact set and $\varepsilon > 0$.*

Then for every sequence of positive real numbers $\{d_j\}_{j\in\mathbf{N}}$ there exists a holomorphic map $H : X \longrightarrow \mathbf{C}^n$ such that the following hold:

(*a*) $|H - F|_K < \varepsilon$,

(*b*) $\inf_{\partial P_j} |H| > d_j$ *for all* $j \in \mathbf{N}$ *and*

(*c*) $j_{m_y}(F)(y) = j_{m_y}(H)(y)$ *for each* $y \in Y$.

Remark. If the sequence $\{d_j\}_{j\in\mathbf{N}}$ goes to infinity then the map H is almost proper.

Proof. For each $m \in \mathbf{N}$ let $h_1^m, \dots, h_n^m$ be the defining functions for the polyhedron P_m. Choose a sequence of (positive) real numbers $\{\delta_m\}_{m\in\mathbf{N}}$ satisfying $\sum_m \delta_m < \min\{\varepsilon, 1\}$. The map H will be constructed as the limit of an appropriate sequence $\{H^m\}_{m\in\mathbf{N}}$ of holomorphic maps. The sequence $\{H^m\}_{m\in\mathbf{N}}$ will be constructed by induction on m.

Set $P_0 := K$ and define $\{y_1, \dots y_{n_0}\} := Y \cap K$ and $\{y_{n_{m-1}+1}, \dots, y_{n_m}\} := Y \cap (P_m \setminus P_{m-1})$ for each $m \in \mathbf{N}$. For the sake of simplicity set $m_k := m_{y_k}$ for each $k \in \mathbf{N}$. We proceed by induction on m.

$\underline{m = 1.}$ The construction of the map H^1 is componentwise. Write $F = (F_1, \dots, F_n)$. Since $|h_i^1|_K < 1$ and $|h_i^1(y_k)| < 1$ for $k = 1, \dots, n_1$ and $i = 1, \dots, n$, there exist constants $a_i > 1$ such that $|a_i h_i^1|_K < 1$, $|a_i h_i^1(y_k)| < 1$ for $k = 1, \dots, n_1$, $i = 1, \dots, n$ and $|a_i h_i^1|_{\partial P_1 \cap \{|h_i^1|=1\}} = a_i > 1$ $(i = 1, \dots, n)$. Hence we can choose a sufficiently large number $N \in \mathbf{N}$ such that

$$\prod_{k=1}^{n_1} |(a_i h_i^1)^N - (a_i h_i^1(y_k))^N|_K^{m_k+1} < \delta_1/2,$$

$$\inf_{\{|h_i^1|=1\}\cap\partial P_1} \prod_{k=1}^{n_1} |(a_i h_i^1)^N - (a_i h_i^1(y_k))^N|^{m_k+1} > d_1 + \sup_{\{|h_i^1|=1\}\cap\partial P_1} |F_i| + 1.$$

Define

$$\tilde{H}_i^1 := F_i + \prod_{k=1}^{n_1} \left[(a_i h_i^1)^N - (a_i h_i^1(y_k))^N\right]^{m_k+1}.$$

Note that $j_{m_k}(\tilde{H}_i^1)(y_k) = j_{m_k} F_i(y_k)$ for $k = 1, \dots, n_1$. By Cartan's theory there exist holomorphic functions $g_i : X \longrightarrow \mathbf{C}$, $i = 1, \dots, n$, satisfying

(1) $j_{m_k+1}(g_i)(y_k) = 0,\ k = 1, \dots, n_1$,

(2) $j_{m_k}(g_i)(y_k) = j_{m_k}(F_i - \tilde{H}_i^1)(y_k),\ k = n_1 + 1, \dots, n_2$, and

(3) $|g_i|_{\overline{P_1}} < \delta_1/2$.

Define $H_i^1 := \tilde{H}_i^1 + g_i,\ i = 1, \dots, n$. The map $H^1 = (H_1^1, \dots, H_n^1) : X \longrightarrow \mathbf{C}^n$ has the properties

(a_1) $|H^1 - F|_K < \delta_1$,

(b_1) $\inf_{\partial P_1} |H^1| > d_1 + 1 - \delta_1$, and

(c_1) $j_{m_k}(H^1)(y_k) = j_{m_k}(F)(y_k),\ k = 1, \ldots n_2.$

<u>$m \to m+1$.</u> Assume that we have already constructed a sequence of maps $H^1, \ldots, H^m : X \longrightarrow \mathbf{C}^n$ satisfying

(a_m) $|H^m - H^{m-1}|_{\overline{P_{m-1}}} < \delta_m$,

(b_m) $\inf_{\partial P_m} |H^m| > d_m + 1 - \sum_1^m \delta_i$, and

(c_m) $j_{m_k}(H^m)(y_k) = j_{m_k}(F)(y_k),\ k = 1, \ldots n_{m+1}.$

Note that H^1 satisfies these properties if we set $P_0 := K$ and $H^0 := F$. We construct the map H^{m+1} by the same procedure. For each $1 \leq i \leq n$ write $S_{i,m} = \{|h_i^{m+1}| = 1\} \cap \partial P_{m+1}$ and choose a constant $a_i > 1$ and a sufficiently large number $N \in \mathbf{N}$ such that

$$\prod_{k=1}^{n_{m+1}} |(a_i h_i^{m+1})^N - (a_i h_i^{m+1}(y_k))^N|_{\overline{P_m}}^{m_k+1} \quad < \quad \delta_{m+1}/2,$$
$$\inf_{S_{i,m}} \prod_{k=1}^{n_{m+1}} |(a_i h_i^{m+1})^N - (a_i h_i^{m+1}(y_k))^N|^{m_k+1} \quad > \quad d_{m+1} + \sup_{S_{i,m}} |H_i^m| + 1.$$

Define
$$\tilde{H}_i^{m+1} := H_i^m + \prod_{k=1}^{n_{m+1}} \Big((a_i h_i^{m+1})^N - (a_i h_i(y_k)^{m+1})^N\Big)^{m_k+1}.$$

Let $g_i : X \longrightarrow \mathbf{C}$, $i = 1, \ldots, n$, be holomorphic functions satisfying

(1) $j_{m_k+1}(g_i)(y_k) = 0,\ i = 1, \ldots, n_{m+1},$

(2) $j_{m_k}(g_i)(y_k) = j_{m_k}(H_i^m - \tilde{H}_i^{m+1})(y_k),\ k = n_{m+1}+1, \ldots, n_{m+2}$, and

(3) $|g_i|_{\overline{P_{m+1}}} < \delta_{m+1}/2$.

Define $H_i^{m+1} := \tilde{H}_i^{m+1} + g_i$ for $i = 1, \ldots, n$. The map $H^{m+1} = (H_1^{m+1}, \ldots, H_n^{m+1}) : X \longrightarrow \mathbf{C}^n$ has the properties

(a_{m+1}) $|H^{m+1} - H^m|_{\overline{P_m}} < \delta_{m+1}$,

(b_{m+1}) $\inf_{\partial P_{m+1}} |H^{m+1}| > d_{m+1} + 1 - \sum_1^{m+1} \delta_i$, and

(c_{m+1}) $j_{m_k}(H)^{m+1}(y_k) = j_{m_k}(F)(y_k),\ k = 1, \ldots n_{m+2}.$

This completes the induction step.

The property (a_m) shows that the sequence H^m converges uniformly on compact sets on X to a limit $H := \lim_{m\to\infty} H^m : X \longrightarrow \mathbf{C}^n$ and that $|H - f|_K < \varepsilon$. The property (c_m) insures that $j_{m_y} H(y) = j_{m_y}(F)(y)$ for each $y \in Y$, and (b_m) insures that $\inf_{\partial P_j} |H| > d_j + 1 - \sum_1^\infty \delta_i > d_j$ for each $j \in \mathbf{N}$. ♠

Proposition 2.2. *Let X be a Stein n-manifold, $Y \subset X$ a discrete set and $\varphi : X \longrightarrow \mathbf{C}^n$ a holomorphic map. For each $y \in Y$ let a number $m_y \in \mathbf{N}_0$ be given. Then the set*

$$A := \{F : X \longrightarrow \mathbf{C}^n, \;\; F \textit{ almost proper holomorphic, } j_{m_y}(F)(y) = j_{m_y}(\varphi)(y) \; \forall y \in Y\}$$

is residual in the set

$$B := \{F : X \longrightarrow \mathbf{C}^n, \; F \textit{ holomorphic, } j_{m_y}(F)(y) = j_{m_y}(\varphi)(y) \; \forall y \in Y\}.$$

Remark. Note that the statement (d) in theorem 1.1. is a special case of proposition 2.2.

Proof. It is sufficient to show that A is dense in B since it is known (see [Bi] and [Sch1]) that the set of all almost proper holomorphic maps $X \longrightarrow \mathbf{C}^n$ is residual in the Fréchet space $\mathcal{O}(X)^n$ and B is a closed affine subspace of $\mathcal{O}(X)^n$.

Choose $F \in B$, a compact set $K \subset X$ and $\varepsilon \in (0,1)$. Our goal is to find $H \in A$ satisfying $|H - F|_K < \varepsilon$. Let $\{K_j\}_{j \in \mathbf{N}}$ be a normal exhaustion of X by compact sets with $K_1 = K$. Choose an arbitrary almost proper holomorphic map $h : X \longrightarrow \mathbf{C}^n$. There exists a sequence of positive numbers $c_j \to \infty$ such that, if the polyhedron P_j ($j \in \mathbf{N}$) is the union of the (finitely many) connected components of the set $h^{-1}(B(c_j) \times \ldots \times B(c_j))$ intersecting K_j, then $\partial P_j \cap Y = \emptyset$. We may assume $K \subset P_1$. Choose a sequence of positive real numbers $\{d_j\}_{j \in \mathbf{N}}$ such that $d_j \to \infty$. Proposition 2.1. provides a map $H : X \longrightarrow \mathbf{C}^n$ satisfying

(a) $|H - F|_K < \varepsilon$,

(b) $\inf_{\partial P_j} |H| > d_j$ for all $j \in \mathbf{N}$, and

(c) $j_{m_y}(H)(y) = j_{m_y}(\varphi)(y)$ for each $y \in Y$.

As the sequence $\{d_j\}$ goes to infinity, the property (b) insures that H is almost proper and by (c) we have $H \in A$. ♠

Proposition 2.3. *Let X and Y be as in proposition 2.2., $m \in \mathbf{N}$ and $\varphi : X \longrightarrow \mathbf{C}^{n+m}$ a holomorphic map such that $\varphi|_Y : Y \longrightarrow \mathbf{C}^{n+m}$ is proper. For each $y \in Y$ let a number $m_y \in \mathbf{N}_0$ be given. Then the set*

$$A := \{F : X \longrightarrow \mathbf{C}^{n+m}, \;\; F \textit{ proper holomorphic, } j_{m_y}(F)(y) = j_{m_y}(\varphi)(y) \; \forall y \in Y\}$$

is everywhere dense in the set

$$B := \{F : X \longrightarrow \mathbf{C}^{n+m}, \; F \textit{ holomorphic, } j_{m_y}(F)(y) = j_{m_y}(\varphi)(y) \; \forall y \in Y\}.$$

Remark. The statement (c) in theorem 1.1. is a special case of proposition 2.3.

Proof. Choose $F \in B$, a compact set $K \subset X$ and $\varepsilon > 0$. We are going to construct a holomorphic map $(H,G) : X \longrightarrow \mathbf{C}^{n+m}$ which belongs to A and satisfies $|F - (H,G)|_K < \varepsilon$. Write $\varphi' := (\varphi_1, \ldots, \varphi_n)$, $\varphi'' = (\varphi_{n+1}, \ldots, \varphi_{n+m})$ and $F' := (F_1, \ldots, F_n)$, $F'' = (F_{n+1}, \ldots, F_{n+m})$. Using a strictly plurisubharmonic exhaustion function on X we can choose a normal exhaustion of X by holomorphically convex compact sets $\{C_j\}_{j\in\mathbf{N}}$ such that $K \subset C_1$, for each $j \geq 2$ the set $(C_j \setminus C_{j-1}) \cap$Y contains exactly one point which we denote by y_j, and the set $\partial C_j \cap Y$ is empty for each $j \in \mathbf{N}$. We simplify the notation and set $m_j := m_{y_j}$ for each $j \in \mathbf{N}$. For technical reasons we assume that $0 \notin \varphi''(Y)$ (since Y is discrete this is always possible by translating the coordinate system). For each y_j there are an open neighbourhood $U_j \subset C_j$ of y_j and a holomorphic map $f_j : U_j \longrightarrow \mathbf{C}^m$ such that $j_{m_j}(f_j)(y_j) = j_{m_j}(\varphi'')(y_j)$ and $|f_j|_{U_j} > |f_j(y_j)|/2$. Let V_{j-1} be an open set containing C_{j-1}. We may assume that $U_j \cap V_{j-1} = \emptyset$. Since the sets $C_j \cup \{y_{j+1}\}$ are holomorphically convex, we can apply Bishop's results ([Bi], Theorem 2) to approximate the sets $C_j \cup \{y_{j+1}\}$ by special analytic polyhedra P_j such that for each $j \in \mathbf{N}$ the following hold:

- $(C_j \cup \{y_{j+1}\}) \subset P_j \subset (V_j \cup U_{j+1})$ and $K \subset P_1 \cap V_1$;
- $\partial P_j \cap Y = \emptyset$, $(P'_j \setminus P'_{j-1}) \cap Y = \{y_j\}$ where $P'_j := P_j \cap V_j$;
- the set $P''_j := P_j \cap U_{j+1}$ is connected and contains y_{j+1}.

Obviously $\{P_j\}$ is a normal exhaustion of X. Set $n_k := |\varphi(y_k)|$ for each $k \in \mathbf{N}$ and choose an increasing sequence of positive numbers $\{d_j\}_{j\in\mathbf{N}}$ such that $d_j \neq n_k$ for each $j, k \in \mathbf{N}$ and $d_j > \max\{n_1, \ldots, n_{j+1}\} + 1$. Since φ is proper on Y, the sequence n_k goes to infinity and so does the sequence $\{d_j\}_{j\in\mathbf{N}}$. It follows from proposition 2.1. that there is an almost proper map $H : X \longrightarrow \mathbf{C}^n$ satisfying $|F' - H|_K < \varepsilon$, $j_{m_y}(H)(y) = j_{m_y}(\varphi')(y)$ for each $y \in Y$, and $\inf_{\partial P_j} |H| > d_j, j \in \mathbf{N}$.

For each $j \in \mathbf{N}$ let the analytic polyhedron Q'_j be the union of the (finitely many) connected components of the set $H^{-1}(B(d_j) \times \ldots \times B(d_j))$ contained in P'_j and let Q''_j be a connected component of the set $H^{-1}(B(d_j) \times \ldots \times B(d_j))$ which contains y_{j+1}. Note that by the choice of d_j, the set Q''_j is a subset of P''_j and therefore a subset of U_{j+1}. Set $Q_j := Q'_j \cup Q''_j$. It is easy to check that $\{Q_j\}_{j\in\mathbf{N}}$ is a normal exhaustion of X with $K \subset Q'_1$, $\partial Q_j \cap Y = \emptyset$ and $(Q_j \setminus Q_{j-1}) \cap Y = \{y_{j+1}\}$.

Now we are going to construct a holomorphic map $G : X \longrightarrow \mathbf{C}^m$ such that $j_{m_y}(G)(y) = j_{m_y}(\varphi'')(y)$ for each $y \in Y$ and $(H,G) : X \longrightarrow \mathbf{C}^{n+m}$ is proper, which means that $|G|$ must be big enough on the sets where $|H|$ is too small. We define the sets where $|H|$ is too small by

$$L_1 := Q_1 \text{ and } L_j := \{z \in Q_j \setminus \overline{Q_{j-1}},\ |H(z)| < n_{j+1} - 1/2\},\ j \geq 2.$$

The choice of d_j insures that the set $\overline{L_j}$ is a compact subset of $(Q_j \setminus \overline{Q_{j-1}})$ and $L := \cup_1^\infty L_j$ is Runge in X. Write $L_j = L'_j \cup L''_j$ where $L'_j \subset Q'_j$ and $L''_j \subset Q''_j$ is the connected component of L_j containing y_{j+1}. If $y_{j+1} \notin L_j$ we take $L''_j = \emptyset$. Note that $y_{j+1} \in L_j$ immediately implies

$$n_{j+1} = |\varphi''(y_{j+1})| = |f_{j+1}(y_{j+1})|. \tag{1}$$

We define the map $g : L \longrightarrow \mathbf{C}^m$ by setting $g|_{L_1} = F''|_{L_1}$ and $g|_{L'_j} := n_{j+1}$, $g|_{L''_j} := f_{j+1}$ for $j \geq 2$.

Note that by equation (1) we have $|g(x)| > n_{j+1}/2$ for each $x \in L''_j$. Put $K_1 := K$ and $K_j := \{z \in L_j,\ |H(z)| \leq n_{j+1} - 1\}$ for $j \geq 2$. There is a map $G : X \longrightarrow \mathbf{C}^m$ such that $j_{m_y}(G)(y) = j_{m_y}(\varphi'')(y)$ for each $y \in Y$ and $|G - g|_{K_j} < \varepsilon$ for $j \in \mathbf{N}$. Since

$$|(G, H)|_{Q_j \setminus Q_{j-1}} \geq \min\{n_{j+1} - 1 - \varepsilon, n_{j+1}/2 - \varepsilon\} \geq n_{j+1}/2 - \varepsilon - 1 \text{ for each } j \geq 2$$

and $\lim_{j\to\infty} n_j = \infty$, the map $(H, G) : X \longrightarrow \mathbf{C}^{n+m}$ is proper. By construction $j_{m_y}(H, G)(y) = j_{m_y}(\varphi)(y)$ for each $y \in Y$ and $|(H, G) - F|_K < \varepsilon$. ♠

Proposition 2.4. *Let X be a Stein n-manifold, $Y \subset X$ a discrete set, $\varphi : X \longrightarrow \mathbf{C}^n$ a holomorphic map and $q' = [\frac{n+1}{2}]$. For each $y \in Y$ let a number $m_y \in \mathbf{N}$ be given. The set of all almost proper holomorphic maps $F : X \longrightarrow \mathbf{C}^n$ satisfying*

(1) $j_{m_y}(F)(y) = j_{m_y}(\varphi)(y)$ *for each* $y \in Y$ *and*

(2) $dim\{x \in X \setminus Y,\ rank_x F \leq n - i\} < 2(q' - i + 1),\ i = 1, \ldots, n$

is residual in the set $\mathcal{G}$ of all holomorphic maps G satisfying $j_{m_y}(G) = j_{m_y}(\varphi)(y)$ for each $y \in Y$. If Y is empty or $m_y = 0$ for each $y \in Y$, then we can replace (2) *by*

(2′)$dim\{x \in X,\ rank_x F \leq n - i\} < 2(q' - i + 1),\ i = 1, \ldots, n$

Remark. If A is an analytic set with $dim\, A < 0$, then A is empty by definition.

Proof. According to proposition 2.2. the set $\mathcal{G}_1$ of all almost proper holomorphic maps satisfying (1) is residual in $\mathcal{G}$. It remains to show that the set $\mathcal{G}_2$ of all $F \in \mathcal{G}$ satisfying (2) is residual in $\mathcal{G}$.

Let $T := TX$, $S := X \times \mathbf{C}^n$ and $V = Hom_{\mathbf{C}}(T, S)$, i.e., $V = \cup_{x \in X}\{L,\ L : T_x \longrightarrow S_x,\ \text{ L is } \mathbf{C}\text{-linear}\}$. Denote by $pr_X : V \longrightarrow X$ the vector bundle projection. For each $p = 0, \ldots, n-1$ define $V^p = \cup_{x \in X}\{L \in V_x,\ rankL = p\}$. It is known that V^p is a complex submanifold of V with $codim_V V^p = (n-p)^2$. For each set $A \subset X$ define $V^p|_A := \cup_{x \in A}\{L \in V_x,\ rankL = p\}$. Define the map $\psi(f) : (X \setminus Y) \longrightarrow V^p|_{(X \setminus Y)}$ by

$$\psi(f)(x) = D_x(f),\ x \in X \setminus Y.$$

Let $\mathcal{H}^p$ be the set of all holomorphic maps f such that $\psi(f)$ is transverse to $V^p|_{X \setminus Y}$ and set $\mathcal{H} := \cap_{p=0}^{n-1}\mathcal{H}^p$. We claim that each $\mathcal{H}^p$ and hence $\mathcal{H}$ is a residual subset of $\mathcal{G}$. Assuming this for a moment, we can complete the proof as follows. Transversality of $\psi(f)$ to V^p implies that

$$dim\{x \in X \setminus Y,\ \psi(f)(x) \in V^p\} = dim[\psi(f)(X \setminus Y)] - dimV + dimV^p.$$

After rearrangement we get for every $p = 0, \ldots, n-1$

$$dim\{x \in X \setminus Y,\ \psi(f)(x) \in V^p\} = n - (n-p)^2,$$

or, writing $i = n - p$,

$$dim\{x \in X \setminus Y,\ rank_x f \leq n - i\} = n - i^2$$

for $i = 1, \ldots, n$. It is easy to check that $n - i^2 < 2(q' - i + 1)$ for $i = 1, \ldots, n$. This means that each $f \in \mathcal{H}$ satisfies (2) in proposition 2.4. and hence $\mathcal{H}$ is a subset of $\mathcal{G}_2$. Since $\mathcal{H}$ is residual in $\mathcal{G}$, so is $\mathcal{G}_2$.

To prove that $\mathcal{H}^p$ is residual in $\mathcal{G}$ it suffices to show that for each compact set $C \subset V^p|_{(X \setminus Y)}$ the set $\mathcal{H}^p_C$ of all holomorphic maps $f \in \mathcal{G}$, such that $\psi(f)$ is transverse to V^p on C, is open and dense in $\mathcal{G}$ for each $p = 0, \ldots, n-1$.

Fix such a compact set C. Since transversality is an open condition on compact sets, the set $\mathcal{H}^p_C$ is an open subset of $\mathcal{G}$. To prove that it is dense we choose an arbitrary $F \in \mathcal{G}$, a compact set $K \subset (X \setminus Y)$ and $\varepsilon > 0$. We may assume that K contains $pr_X(C)$ in its interior. Now we choose holomorphic functions $g_1, \ldots, g_k : X \longrightarrow \mathbf{C}$ such that the map $g = (g_1, \ldots, g_k)$ has maximal rank at each point of K and $j_{m_y}(g_i) = 0$ for each $y \in Y$, $i = 1, \ldots, k$. Note that in case $m_y = 0$ for each $y \in Y$ (or $Y = \emptyset$) such functions exist for each compact set $K \subset X$. Define the map $\Psi : M^{n \times k} \times K \longrightarrow V|_K$ by

$$\Psi(A, y) := D_y(F + Ag),\ A \in M^{n \times k}, y \in K.$$

The map Ψ is an (open) surjective submersion over some neighbourhood of K and is therefore transversal to every set $V^p|_K$.

By Thom's transversality theorem the set

$$M_C := \{A \in M^{n\times k},\ \Psi(A,\cdot) \text{ is transverse to } V^p \text{ on } C\}$$

is dense in $M^{n\times k}$. If we choose $A \in M_K$ sufficiently close to the zero matrix then the map $G := F + Ag$ is close to F on K and $\psi(F + Ag)$ is transverse to V^p on C, i.e., $F + Ag$ belongs to $\mathcal{H}^p_C$.

In case when $Y = \emptyset$ or $m_y = 0$ for each $y \in Y$ the proof is the same except that we choose compact sets C as subsets of V^p (instead of $V^p|_{X\setminus Y}$). ♠

Corollary 2.1. *Let X be a Stein n-manifold, $Y = \{y_j\}_{j\in\mathbf{N}} \subset X$ a discrete set, $\varphi : Y \longrightarrow \mathbf{C}^n$ a map, $\{d_j\}_{j\in\mathbf{N}}$ a sequence of positive numbers, and $\{P_j\}_{j\in\mathbf{N}}$ a normal exhaustion of X by special analytic polyhedra such that $\partial P_j \cap Y = \emptyset$ and (after a rearrangement of the y_j-s) $(P_{j+1} \setminus P_j) \cap Y = \{y_j\}$ for each $j \in \mathbf{N}$. Then there exists an almost proper map $H : X \longrightarrow \mathbf{C}^n$ satisfying*

(1) $H(y_j) = \varphi(y_j)$ *for each* $j \in \mathbf{N}$,

(2) $|H|_{\partial P_j} > d_j$ *for each* $j \in \mathbf{N}$, *and*

(3) $dim\{x \in X,\ rank_x H \le n - i\} < 2([\frac{n+1}{2}] - i + 1),\ i = 1, \dots n.$

Proof. Let V and V^p, $p = 0, \dots, n-1$, be as in the proof of proposition 2.4. There we have proved that the set $\mathcal{H}$ of all holomorphic maps $F : X \longrightarrow \mathbf{C}^n$ such that the map $x \mapsto D_xF$ is transverse to each V^p is residual in the set $\mathcal{G}$ of all extensions of φ. Note that in the present case $m_y = 0$ for each $y \in Y$. The transversality theorem implies that for each $F \in \mathcal{G}$, each compact set $K \subset X$ and each relatively compact open neighbourhood U of K there exists an $\varepsilon > 0$ such that the following holds:

$(*)$if $G : X \longrightarrow \mathbf{C}^n$ satisfies $|G - F|_U < \varepsilon$ then the map $x \mapsto D_xG$ is transverse to V^p, $p = 0, \dots, n-1$, at any point $x \in K$.

For each $j \in \mathbf{N}$ let $U_j \subset P_{j+1}$ be an open neighbourhood of $\overline{P}_j$. By induction we will construct a sequence of holomorphic maps $H^j : X \longrightarrow \mathbf{C}^n$ and a decreasing sequence $\{\varepsilon_j\}$, such that $\varepsilon_j \in [0, 1)$ for each $j \in \mathbf{N}$ and

(a_j) $|H^j|_{\partial P_i} > d_i + 1 - \sum_{k=i+1}^{j} \varepsilon_k/2^{k+1}$ for $i = 1, \dots, j$

(b_j) H^j belongs to $\mathcal{H}$ and $|H^j - H^{j-1}|_{U_{j-1}} < \varepsilon_{j-1}/2^j$ for $j \ge 2$,

(c_j) ε_j satisfies $(*)$ for $F := H^j$, $U := U_j$ and $K := \overline{P}_j$.

$\underline{j=1.}$ Let $h^1 : X \longrightarrow \mathbf{C}^n$ satisfy (1). Since $\mathcal{H}$ is residual in $\mathcal{G}$, there exists $H^1 \in \mathcal{H}$ such that $|H^1|_{\partial P_1} > d_1 + 1$. By the transversality theorem there exist $\varepsilon_1 \in [0, 1)$ such that (c) holds.
$\underline{j \to j+1.}$ Assume that H^j has already been constructed. There exists $h^{j+1} : X \longrightarrow \mathbf{C}^n$ such that $|H^j - h^{j+1}|_{U_{j+1}} < \varepsilon_j/2^{j+2}$ and $|h^{j+1}|_{\partial P_{j+1}} > d_{j+1} + 1$. Since $\mathcal{H}$ is residual in $\mathcal{G}$, there exists $H^{j+1} \in \mathcal{H}$ such that $|h^{j+1} - H^{j+1}|_{U_{j+1}} < \varepsilon_j/2^{j+2}$. Now H^{j+1} satisfies (a_{j+1}) and (b_{j+1}). By the same argument as above there exists $\varepsilon_{j+1} \in [0, 1)$ such that (c_{j+1}) holds as well.

Because of (b_j) the sequence $\{H^j\}$ converges uniformly on compact sets in X and therefore has a limit $H : X \longrightarrow \mathbf{C}^n$. It follows from (a_j) that H satisfies (1). Since $|H - H^j|_{U_j} < \sum_j^\infty \varepsilon_i/2^{i+1} < \varepsilon_j$ for each $j \in \mathbf{N}$, the map $x \mapsto D_x H$ is transverse to all V^p, $p = 0, \dots, n-1$, which means that H satisfies (2). ♠

3. Tools

Before proceeding to the proof of the statements (a) and (b) of the main theorem we need some additional tools. As before let X be a n-dimensional Stein manifold and $Y \subset X$ a discrete set. Recall that we have already defined

$$N := \max\{[\tfrac{n+1}{2}] + 1, 3\} \text{ and } N' := \max\{[\tfrac{n+1}{2}], 1\}.$$

Let $\varphi = (\varphi', \varphi'') : Y \longrightarrow \mathbf{C}^{n+q}$ be a proper map for some $q \in \mathbf{N}$ and let $H : X \longrightarrow \mathbf{C}^n$ be an almost proper holomorphic extension of $\varphi' : X \longrightarrow \mathbf{C}^n$ provided by corollary 2.1. Throughout this section we keep q and the maps H and φ fixed.

For $R > 0$ let X^R be an arbitrary union of finitely many connected components of the set $H^{-1}(B_n(R)) \subset X$ and let $Z^R = H(X^R) = B_n(R)$.

Lemma 3.1. *There are stratifications $X_n := X^R \supset X_{n-1} \ldots \supset X_0 \supset X_{-1} = \emptyset$ and $Z_n := Z^R \supset Z_{n-1} \ldots \supset Z_0 \supset Z_{-1} = \emptyset$, with $X_0, Z_0 \neq \emptyset$, satisfying*

(1) *$X_0 \supset X^R \cap Y$ and $Z_0 \supset H(Y \cap X^R)$,*

(2) *$X_j = H^{-1}(Z_j) \cap X^R$,*

(3) *the sets X_j and Z_j have dimension at most j and the sets $X_j^* := X_j \setminus X_{j-1}$, $Z_j^* = Z_j \setminus Z_{j-1}$ are complex j-dimensional manifolds (or empty),*

(4) *if X_j^* is not empty, the map $H : X_j^* \longrightarrow Z_j^*$ is an immersion for $j \in \{0, \dots, n\}$,*

(5) *the rank of H is constant on each connected component of the set X_j^* for each $j \in \{0, \dots, n\}$.*

Proof. Lemma 6.1 from [Sch1] provides stratifications $\{X'_j\}$ and $\{Z'_j\}$ satisfying all the conditions except condition (1). Define:

$$\begin{aligned} X_n &:= X'_n, \text{ and } Z_n := Z'_n, \\ X_j &:= X'_j \cup [X^R \cap H^{-1}(H(Y \cap X^R))], \text{ and } Z_j := Z'_j \cup H(Y \cap X^R),\ 0 \le j \le n-1, \\ X_{-1} &:= X'_{-1} = Z_{-1} := Z'_{-1} = \emptyset. \end{aligned}$$

The new stratifications have all the desired properties. ♠

We quote two more results from [Sch1] (the almost proper map H and proper map φ are fixed).

Theorem 3.1. *Let $R > 0$ and let X^R be the union of a finite number of connected components of the set $H^{-1}(B_n(R))$. For $r \in (0, R)$ let $X^r := X^R \cap H^{-1}(B_n(r))$. If $q \ge N$ and $\varphi : Y \longrightarrow \mathbf{C}^{n+q}$ is injective, there exists a holomorphic map $G : X^r \longrightarrow \mathbf{C}^q$ satisfying the conditions*

$\alpha(r)$ the map $(H, G) : X^r \longrightarrow \mathbf{C}^{n+q}$ is an injective immersion,

$\beta(r)$ the map $(H, G_1, \ldots, G_{N'}) : X^r \longrightarrow \mathbf{C}^{n+N'}$ is an immersion,

$\gamma(r)$ $(H, G)|_{Y \cap X^r} = \varphi|_{Y \cap X^r}$, and

$\delta(r)$ $((H, G)(X^r \setminus Y)) \cap (\varphi(Y \setminus X^r)) = \emptyset$.

If $q \ge N'$ and φ is not necessarily injective, there exists a map $G : X^r \longrightarrow \mathbf{C}^q$ satisfying only the conditions $\beta(r)$ and $\gamma(r)$.

Proof. We follow the proof of theorem 3.3 in [Sch1] with a slight adjustment. Choose $r' \in (r, R)$ and set $X^{r'} := X^R \cap H^{-1}(B_n(r'))$.

First assume that $q \ge N$ and the map φ is injective. On the zero-dimensional stratum X_0 we define $g' : X_0 \longrightarrow \mathbf{C}^q$ such that $g'|_{X_0 \cap Y} = \varphi''|_{X_0 \cap Y}$ and its value at each point of $X_0 \setminus Y$ belongs to $\mathbf{C}^q \setminus ((\varphi_{n+1}, \ldots, \varphi_{n+q})(Y))$. Theorem 3.3 in [Sch1] gives us a map $G' : X^{r'} \longrightarrow \mathbf{C}^q$ which coincides with g' over X_0 and has the properties $\alpha(r'), \beta(r')$ and $\gamma(r')$. By a perturbation which is small enough over X^r we obtain a map $G : X^r \longrightarrow \mathbf{C}^q$ satisfying $\alpha(r), \beta(r), \gamma(r)$ and $\delta(r)$.

If $q \ge N'$ and the map φ is not assumed to be injective, we define $g' : X_0 \longrightarrow \mathbf{C}^q$ with $g'|_{X_0 \cap Y} = \varphi''|_{X_0 \cap Y}$ and $g'|_{X_0 \setminus Y} = 0$. Again by theorem 3.3 in [Sch1] there is a map $G : X^r \longrightarrow \mathbf{C}^q$ which satisfies $\beta(r)$ and $\gamma(r)$. ♠

Theorem 3.2. *Let $R, r > 0$, X^R and X^r be as in theorem 3.1. Choose $r' \in (r, R)$ and set $X^{r'} := X^R \cap H^{-1}(B_n(r'))$.*

If a holomorphic map $G : X^r \longrightarrow \mathbf{C}^q$ with $q \geq N'$ satisfies $\beta(r)$ and $\gamma(r)$ from theorem 3.1., it can be approximated arbitrarily well on the set X^r by a map $G' : X^{r'} \longrightarrow \mathbf{C}^q$ satisfying $\beta(r')$ and $\gamma(r')$ from theorem 3.1.

Assume that $q \geq N$ and $\varphi : Y \longrightarrow \mathbf{C}^{n+q}$ is injective. Let $G : X^r \longrightarrow \mathbf{C}^q$ be a map satisfying the conditions $\alpha(r), \beta(r), \gamma(r)$ and $\delta(r)$ in theorem 3.1. *Then the map G can be approximated as well as we wish on the set X^r by a map $G' : X^{r'} \longrightarrow \mathbf{C}^q$ satisfying $\alpha(r'), \beta(r'), \gamma(r')$ and $\delta(r')$ from theorem* 3.1.

Proof. We follow the proof of theorem 3.4 in [Sch1].

If $q \geq N'$ and the map φ is not assumed to be injective, we define $g' : X_0 \cup X^r \longrightarrow \mathbf{C}^q$ with $g'|_{X_0 \cap Y} = \varphi''|_{X_0 \cap Y}$, $g'|_{X^r} := G$, and $g'|_{X_0 \setminus (Y \cup X^r)} = 0$. By theorem 3.4 in [Sch1] there is a map $G' : X^{r'} \longrightarrow \mathbf{C}^q$ which satisfies $\beta(r')$ and $\gamma(r')$ and which approximates G.

If $q \geq N$ and φ is injective, define $g' : X_0 \cup X_r \longrightarrow \mathbf{C}^q$ such that $g'|_{X^r} = G$, $g'|_{X_0 \cap Y} = \varphi''|_{X_0 \cap Y}$, $g'(x) \in \mathbf{C}^q \setminus ((\varphi_{n+1}, \ldots, \varphi_{n+q})(Y))$ for each $x \in X_0 \setminus (Y \cup X_r)$, and g' is injective. Choose $r'' \in (r', R)$. Theorem 3.4 in [Sch1] gives us a map $G'' : X^{r''} \longrightarrow \mathbf{C}^q$ with the properties $\alpha(r''), \beta(r'')$ and $\gamma(r'')$ which approximates G on the set X^r. By a perturbation which is small enough over $X^{r'}$ we obtain a map G satisfying the conditions $\alpha(r'), \beta(r'), \gamma(r')$, and $\delta(r')$. ♠

Lemma 3.2. *Let $X \subset \mathbf{C}^N$ be an n-dimensional analytic set, $X_0 \subset X$ an analytic subset of dimension at most $n-1$ such that $X \setminus X_0$ is a manifold, and let $k \in \mathbf{N}$. Let Σ be a closed subset of $X \times \mathbf{C}^k$ such that $pr_X : V := (X \times \mathbf{C}^k) \setminus \Sigma \longrightarrow X$ is a topological bundle over $X \setminus X_0$ with $(n-1)$-connected fibres.*

Choose $d \in \mathbf{R}$ and set $K := \{x \in X,\ \|x\| \leq d\}$ (K may be empty). Every continuous section $c' : X_0 \cup K \longrightarrow V|_{X_0 \cup K}$ can be extended to a continuous section $c : X \longrightarrow V$. In particular, every continuous section $c' : X_0 \longrightarrow V|_{X_0}$ can be extended to a continuous section $c : X \longrightarrow V$ (we take $d < 0$).

Remark. The above lemma could be proved by using the fact that any n-dimensional Stein space is homotopically equivalent to a real n-dimensional CW-complex ([Ha1],[Ha2]). Since in our case we don't need such precise information about the homotopy type of analytic sets, we provide a more elementary proof following the method of Henkin and Leiterer ([HL]).

Outline of the proof. Write $a_0 := d$, $c_0 := c'$ and choose an increasing sequence of positive real numbers $\{a_i\}_{i \in \mathbf{N}}$ which goes to infinity, with $a_0 < a_1$, and set $K_i := \{x \in X,\ \|x\| \leq a_i\}$ for each $i \geq 0$ (note that $K = K_0$.). We will prove the lemma by induction on i, i.e., we are going to prove that every continuous section $c_i : X_0 \cup K_i \longrightarrow V|_{X_0 \cup K_i}$ can be extended to a continuous section $c_{i+1} : X_0 \cup K_{i+1} \longrightarrow V_{X_0 \cup K_{i+1}}$. In the limit we obtain a global section $c := \lim_{i \to \infty} c_i : X \longrightarrow V$ which extends $c_0 = c'$.

The initial step is trivial since the section $c_0 : X_0 \cup K_0 \longrightarrow V|_{X_0 \cup K_0}$ is already defined.

For the induction step assume that the section $c_i : X_0 \cup K_i \longrightarrow V|_{X_0 \cup K_i}$ has already been constructed. Our goal is to extend the section c_i to a section $c_{i+1} : X_0 \cup K_{i+1} \longrightarrow V$. Since $V = X \times \mathbf{C}^k \setminus \Sigma$, the section c_i has the form $c_i(x) = (x, \gamma_i(x))$, where $\gamma_i : X_0 \cup K_i \longrightarrow \mathbf{C}^k$ is a continuous map. By Tietze's theorem there is a continuous extension $\tilde{\gamma}_i : X \longrightarrow \mathbf{C}^k$ of the map γ_i. Since the set Σ is closed and c_i avoids Σ, there exists an open neighbourhood $U \subset X$ of $X_0 \cup K_i$ such that the section $\tilde{c}_i := (id_X, \tilde{\gamma}_i)$ avoids Σ over U. It remains to extend the section $\tilde{c}_i$ from U to a neighbourhood of $X_0 \cup K_{i+1}$. Let L be a compact set which contains K_{i+1} in its interior. There exists a smooth function $\rho : X \longrightarrow \mathbf{R}$ with 0 as a regular value and such that $X_0 \cup K_i \subset \{\rho < 0\} \subset U$ and ρ is strictly plurisubharmonic in a neighbourhood of $L \setminus (X_0 \cup K_i)$. By an arbitrarily small perturbation of ρ over $L \setminus \{\rho < 0\}$ we can deform ρ to a function which is still strictly plurisubharmonic in a neighbourhood of $L \setminus \{\rho < 0\}$ and has only nondegenerate critical points on the set $L \setminus \{\rho < 0\}$. By [HL] there exists a finite sequence of functions $\rho_0 := \rho, \rho_1 \dots, \rho_m$, which are strictly plurisubharmonic in a neighbourhood of $L \setminus \{\rho < 0\}$, $X_0 \cup K_{i+1} \subset \{\rho_m < 0\}$ and such that for each $i = 1, \dots, n$ we have $supp(\rho_i - \rho_{i-1}) \subset (L \setminus X_0)$, and the sublevel set $\{\rho_i < 0\}$ is homotopically equivalent to $\{\rho_{i_1} < 0\}$ with an attached l-cell. This means that in a finite number of steps we deform the initial sublevel set $\{\rho < 0\}$ to a sublevel set $\{\rho_m < 0\}$ such that each deformation is equivalent to attaching an l-cell (pseudoconvex bumps in [HL]). Note that the functions ρ_i are strictly plurisubharmonic in a neighbourhood of $L \setminus \{\rho < 0\}$ which implies $l \leq n$. Since we have attached cells outside X_0 and $V|_{X \setminus X_0}$ is a topological bundle with $(n-1)$-connected fibres, we can at each step extend our section over the attached cell to a continuous section of V. After the last step we obtain a continuous section $\tilde{c}_{i+1} : \{\rho_m < 0\} \longrightarrow V$ which extends c_0. The section $c_{i+1} := \tilde{c}_{i+1}|_{X_0 \cup K_{i+1}}$ is the desired section. ♠

Definition 3.1. *Let Z and X be complex manifolds, $h : Z \longrightarrow X$ a surjective submersion, and let $U \subset X$ be an open set. The submersion h admits a spray over U if for some $m \in \mathbf{N}$ there exists a holomorphic map $s \colon h^{-1}(U) \times \mathbf{C}^m \longrightarrow h^{-1}(U)$ such that*

$s(z, 0) = z$ *for each* $z \in h^{-1}(U)$,

$s(z, \mathbf{C}^m) \subset h^{-1}(h(z))$ *for each* $z \in h^{-1}(U)$, *and*

$\frac{\partial}{\partial t} s(z,t)|_{t=0} : \mathbf{C}^m \longrightarrow ker D_z h$ *is surjective.*

The following result can be found in [Gr], [FP2, theorems 1.2 and 1.5]:

Theorem 3.3. (The h-principle)
Let X be a Stein manifold, Z a complex manifold and $h : Z \longrightarrow X$ a holomorphic submersion onto X. Assume that each $x \in X$ has a neighbourhood $U \subset X$ such that h admits a spray over U. Let d a metric on Z compatible with the manifold topology. Then the following hold:

(a) *Each continuous section $f_0 : X \longrightarrow Z$ can be deformed to a holomorphic section $f_1 : X \longrightarrow Z$ through a continuous one-parameter family of continuous sections (a homotopy) $f_t : X \longrightarrow Z$, $t \in [0,1]$.*

(b) *If $K \subset X$ is a compact holomorphically convex set and the initial section f_0 is holomorphic in a neighbourhood of K, then for each $\varepsilon > 0$ there exists a homotopy $f_t : X \longrightarrow Z$, $t \in [0,1]$, such that $d(f_t(x), f_0(x)) < \varepsilon$ for each $x \in K$ and $t \in [0,1]$, each f_t is holomorphic in a neighbourhood of K and f_1 is holomorphic on X. In this case it suffices to assume that the submersion $h : Z \longrightarrow X$ has a spray over small open subsets of $X \setminus K$.*

Proposition 3.1. (Existence of sprays; lemma 7.1 in [FP2]).
Let $U \subset \mathbf{C}^n$ be an open set ($n \geq 1$) and let $\Sigma \subset U \times \mathbf{C}^q$ for $q \geq 2$ be a closed analytic subset such that each fiber $\Sigma_x = \{w \in \mathbf{C}^q \colon (x,w) \in \Sigma\}$ has complex codimension at least two in $\mathbf{C}^q$ (it may be empty). Assume that there exists a nonempty open set $\Omega \subset \mathbf{CP}^{q-1}$ such that for each $[v] \in \Omega$ the linear projection $\tilde{\pi}_v \colon U \times \mathbf{C}^q \to U \times \mathbf{C}^{q-1}$, defined by

$$\tilde{\pi}_v(x,w) = (x, \pi_v(w)) \quad (x \in U,\ w \in \mathbf{C}^q),$$

is proper when restricted to Σ. Then the projection $h \colon (U \times \mathbf{C}^q) \setminus \Sigma \to U$, given by $h(x,w) = x$, admits a spray.

Lemma 3.3. *Let d be a positive number, $pr_{\mathbf{C}^n} : V = B_n(d) \times \mathbf{C}^q \longrightarrow B_n(d)$ a trivial bundle and $\Sigma \subset V$ a closed analytic subset such that $pr_{\mathbf{C}^n} : V \longrightarrow B_n(d)$ is proper when restricted to Σ. Assume that there exists $k \in \mathbf{N}$ such that each fiber $\Sigma_y := (\Sigma \cap (\{y\} \times \mathbf{C}^q))$ consists of at most k points for each $y \in B_n(d)$. Let a point $x_0 = (x_0', x_0'') \in V \setminus \Sigma$ be given satisfying $\|x_0\| < d$. Put $c = \frac{\|x_0\|}{2}$ and assume that $q \geq 3$. Then there exists a holomorphic section $C : B_n(d) \longrightarrow V \setminus \Sigma$ with the properties*

(1) *$C(x_0') = x_0$, and*

(2) *$\|C(x)\| > c$ for each $x \in B_n(c)$.*

Remark. If U is a Stein manifold, $q \geq 3$ and Σ is a graph of a map $(g_1, g_2) : U \longrightarrow B_n(d') \times \mathbf{C}^q$ such that $g_1 : U \longrightarrow B_n(d')$ is proper, then for each $d < d'$ the assumptions of lemma 3.3. are satisfied for the trivial bundle $B_n(d) \times \mathbf{C}^q$ and $\Sigma|_{B_n(d)}$.

Proof. We have to deal with two problems here: the first one is how to find a section of $V \backslash \Sigma$ through a prescribed point, and the second one is how to make the section big enough over $B_n(c)$. To solve the first problem we will use the h-principle (theorem 3.3., proposition 3.1.). Obviously the set Σ satisfies the assumptions in proposition 3.1., hence we obtain a section which avoids Σ. Unfortunately the h-principle does not give us any estimates of the size of the section which makes it difficult to find a section which is large enough on a prescribed set. Therefore we will look for the section of some 'affine subbundle', i.e., a section of $B_n(c) \times (z + L)$ for a point $z \in \mathbf{C}^q$ and a complex vector subspace $L \subset \mathbf{C}^q$ of codimension 1. When moving to a larger ball we will use the approximative version of the h-principle (theorem 3.3. (b)). We have to consider two cases according to the position of the point x_0 in lemma 3.3.

Case 1. Assume that $x_0' \in \overline{B_n(c)}$. This means that $\|x_0''\|^2 \geq 3c^2$. Set $d' := (c+d)/2$ and let L be the orthogonal complement of x_0'' in $\mathbf{C}^q$. First we construct a section C' over $B_n(d')$ which satisfies (1) and (2). Define $V' := B_n(d') \times (x_0'' + L)$, $\Sigma' := \Sigma \cap V'$. According to 3.1. the submersion $pr_{\mathbf{C}^n} : V' \setminus \Sigma' \longrightarrow B_n(d')$ has a spray since its fibers Σ_x' are (if not empty) finite sets, i.e., algebraic sets of codimension $q - 1 \geq 2$ and $\Sigma' \subset V' \cap (B_n(d') \times B_q(R))$ for some large $R > 0$. Transversality implies that the (analytic) set $pr_{\mathbf{C}^n}(\Sigma')$ is of dimension at most $n - 1$ for almost all choices of L (we can move the chosen L a bit if necessary).

Choose a basis $v_1, \dots, v_{q-1}$ for L and let $g_1, \dots, g_m : B_n(d') \longrightarrow \mathbf{C}$ be holomorphic functions with x_0' as the only common zero. We are looking for a section C of the following form:

$$C'(x') = (x', x_0'' + \sum_{i=1}^{q-1} v_i \cdot \sum_{j=1}^{m} a_{i,j}(x') g_j(x')),$$

where $a = (a_{i,j})$ is a section of the trivial bundle $B_n(d') \times \mathbf{C}^{m \times (q-1)}$. Consider the map

$$S_1 : B_n(d') \times \mathbf{C}^{m \times (q-1)} \longrightarrow V',$$

defined by

$$S_1(x', a) := (x', x_0'' + \sum_{i=1}^{q-1} v_i \cdot \sum_{j=1}^{m} a_{i,j} g_j(x'))$$

and put $\Sigma_1 := S_1^{-1}(\Sigma')$. The map $S_1 : B_n(d') \times \mathbf{C}^{m \times (q-1)} \setminus \Sigma_1 \longrightarrow V' \setminus \Sigma'$ is a surjective submersion everywhere except over the point x_0'. Therefore the map

$$S := pr_{\mathbf{C}^n} \circ S_1 : W := B_n(d') \times \mathbf{C}^{m \times (q-1)} \setminus \Sigma_1 \longrightarrow B_n(d')$$

is a submersion with spray over a neighbourhood of every point $x \neq x_0'$; if $U \subset B_n(d')$ is a neighbourhood of x such that there exists a spray $s : (V' \setminus \Sigma')|_U \times \mathbf{C}^N \longrightarrow (V' \setminus \Sigma')|_U$ (for some $N \in \mathbf{N}$) then the map $W|_U \times \mathbf{C}^N \longrightarrow W|_U$, defined by $(y, z, t) \mapsto s(S_1(y, z), t)$ for $(y, z) \in W|_U,\ t \in \mathbf{C}^N$, is a spray on $W|_U$.

The zero section a_0 over $B_n(d') \times \mathbf{C}^{m\times(q-1)}$ is a section of W over a small neighbourhood of x_0' (because x_0 is not in Σ). Now we are in the following situation: $dim(pr_{\mathbf{C}^n}(\Sigma_1)) \leq n-1$ and the fibers of Σ_1 are (if not empty) algebraic sets of codimension $q-1$, which means that the fibers of W are at least $(n-2)$-connected. There is a stratification of $B_n(d') = B_n \supset B_{n-1} \ldots \supset B_0 \neq \emptyset$ by analytic sets such that for each $j = 1, \ldots, n$ the set $B_j \setminus B_{j-1}$ is a j-dimensional manifold (or empty) and such that the number of points in $\Sigma_{1,x}$ is constant on each connected component of $B_j \setminus B_{j-1}$, $x_0 \in B_0$, while $\Sigma_{1,x}$ is empty for each $x \in B_n \setminus B_{n-1}$.

We first construct a continuous section by induction over the strata. As a first step we extend our section a_0 to a continuous section a_0' over a neighbourhood of B_0 which is easy since the set Σ is closed. Now we can apply lemma 3.2. (a) inductively over the strata. The induction step is the following. Assume that we have already constructed a continuous section a_j' of W over a neighbourhood of B_j which is holomorphic in a neighbourhood of x_0'. Lemma 3.2. provides a continuous section a_{j+1} of $W|_{B_{j+1}}$ which coincides with a_j' over a neighbourhood of B_j. Since Σ is closed, the section a_{j+1} can be extended to a continuous section a_{j+1}' of W over some neighbourhood of B_{j+1} which is holomorphic in a neighbourhood of x_0'.

The result is a continuous section a_n' of W over $B_n(d')$ which is holomorphic in a neighbourhood of x_0'. The h-principle (theorem 3.3.) gives us the desired holomorphic section a (and C').

Case 2. Now we have $x_0' \in B_n(d) \setminus \overline{B_n(c)}$. Put $d' = (\|x_o'\| + c)/2$. Since $\Sigma \cap V|_{B_n(d')}$ is a subset of $B_n(d') \times B_q(R)$ for some large $R > c$, we can simply define $C'(x) = (x, R+1)$ for $x \in B_n(d')$.

In both cases the section C' satisfies $\|C'(x')\| > c$ and in Case 1 we also have $C'(x_0') = x_0$.

Now we are looking for a holomorphic section $C : B_n(d) \longrightarrow V \setminus \Sigma$ through the prescribed point and approximating C' over $\overline{B_n(c)}$. The idea is similar as in Case 1. Let $g_1, \ldots, g_m : B_n(d) \longrightarrow \mathbf{C}$ be holomorphic functions with x_0' as the only common zero. We are looking for a section C of the form

$$C(x') = (0, x_0'') + (x', \sum_1^m a_{1,j}(x')g_j(x'), \dots, \sum_1^m a_{q,j}(x')g_j(x')),$$

$a = (a_{i,j})$ being a section of the trivial bundle $B_n(d) \times \mathbf{C}^{m\times q}$.

Consider the map

$$S_1 : B_n(d) \times \mathbf{C}^{m\times q} \longrightarrow V$$

defined by

$$S_1(x', a) := (0, x_0'') + (x', \sum_1^m a_{1,j}(x')g_j(x'), \dots, \sum_1^m a_{q,j}(x')g_j(x'))$$

and put $\Sigma_1 := S_1^{-1}(\Sigma)$. The map $S_1 : B_n(d) \times \mathbf{C}^{m\times(q-1)} \setminus \Sigma_1 \longrightarrow V \setminus \Sigma$ is a surjective submersion everywhere except over the point x_0'. As in Case 1 the map $S := pr_{\mathbf{C}^n} \circ S_1 : W := B_n(d) \times \mathbf{C}^{m\times q} \setminus \Sigma_1 \longrightarrow B_n(d)$ is a submersion with a spray over a neighbourhood of every point $x' \neq x_0'$.

Because C' is a holomorphic section over $B_n(d')$, there is a holomorphic section b' of $W|_{B_n(d')}$ such that $C'(x') = (0, x_0'') + (x', \sum_1^m b'_{1,j}(x')g_j(x'), \dots, \sum_1^m b'_{q,j}(x')g_j(x'))$. The section b' is a section of $W|_{B_n(d')}$ provided that U is small enough. Similarly as in Case 1 let $B_n := B_n(d) \supset B_{n-1} \ldots \supset B_0 \neq \emptyset$ be a stratification according to the number of points in $\Sigma_{x'}$, except that in this case $\Sigma_{x'}$ is not empty for $x' \in B_n \setminus B_{n-1}$. Since B_0 is a discrete set, it is trivial to extend the section b' to a holomorphic section b_0 over a neighbourhood of $B_0 \cup \overline{B_n((c+d')/2)}$. By applying lemma 3.2. (b) inductively we obtain a continuous section b_n of W which coincides with b' over a neighbourhood of $B_0 \cup \overline{B_n((c+d')/2)}$. So the section b_n is holomorphic in a neighbourhood of $B_0 \cup B_n(c)$. By theorem 3.3. (b) there is a holomorphic section $b : B_n(d) \longrightarrow W$ which approximates b' over $B_n(c)$. The section

$$C(x') = (0, x_0'') + (x', \sum_1^m b_{1,j}(x')g_j(x'), \dots, \sum_1^m b_{q,j}(x')g_j(x'))$$

has all the desired properties. ♠

4. Proof of the main theorem

Proof of theorem 1.1. We are first going to prove statement (a) in theorem 1.1. Assume $q \geq N$, where N is as in the statement of theorem 1.1. Write $Y = \{y_i\}_{i\in\mathbf{N}}$, $\varphi' = (\varphi_1, \dots, \varphi_n)$ and choose an exhaustion $\{P_i\}_{i\in\mathbf{N}}$ of X by special analytic polyhedra such that $\partial P_i \cap Y = \emptyset$ for each $i \in \mathbf{N}$ and $(P_{i+1} \setminus P_i) \cap Y = \{y_{i+1}\}$ for each $i \geq 2$ (modulo a rearrangement of the y_i-s). Put $m_i := \|\varphi(y_i)\|$ and choose a sequence of numbers $\{d_i\}_{i\in\mathbf{N}}$ satisfying

- $d_1 > \max\{1, m_1, m_2\} + 1$,
- $d_{i+1} > \max\{m_{i+2}, d_i\} + 1$ for $i \in \mathbf{N}$.

Since φ is proper the sequence $\{m_i\}_{i\in\mathbf{N}}$ goes to infinity (as i goes to infinity) and so does the sequence $\{d_i\}_{i\in\mathbf{N}}$. By choosing an appropriate coordinate system in $\mathbf{C}^{n+q}$ we may assume that $m_i \geq 2$ for each i. Fix an almost proper extension $H : X \longrightarrow \mathbf{C}^n$ of φ' from corollary 2.1., satisfying

(1) $|H|_{\partial P_i} > d_i$ for each $i \in \mathbf{N}$, and

(2) $dim\{x \in X,\ rank_x H \leq n - i\} < 2([\frac{n+1}{2}] - i + 1)$ for $i = 1, \ldots, n$.

For each $j \in \mathbf{N}$ choose constants a_j, b_j, c_j and e_j satisfying $\max\{d_{j-1}, m_{j+1}\} < e_j < a_j < b_j < c_j < d_j$. Define Q_j to be the union of (finitely many) connected components of the set $H^{-1}(B_n(d_j))$ contained in P_j. Then $\{Q_j\}_{j\in\mathbf{N}}$ is a normal exhaustion of X with $(Q_j \setminus Q_{j-1}) \cap Y = \{y_j\}$. Note that the map $H : Q_j \longrightarrow B_n(d_j)$ is proper for each $j \in \mathbf{N}$. Put

$$\begin{aligned} K_j &:= \{x \in Q_j,\ \|H(x)\| \leq a_j\}, \\ U_j &:= \{x \in Q_j,\ \|H(x)\| < c_j\}, \\ K'_j &:= \{x \in Q_{j+1} \setminus Q_j,\ \|H(x)\| \leq a_j\}, \\ U'_j &:= \{x \in Q_{j+1} \setminus Q_j,\ \|H(x)\| < c_j\}, \\ L_j &:= \{x \in Q_{j+1} \setminus Q_j,\ \|H(x)\| \leq \frac{m_{j+1}}{2}\}. \end{aligned}$$

Obviously U_j and U'_j are disjoint open neighbourhoods of K_j and K'_j respectively, $y_{j+1} \in K'_j$ and $L_j \subset K'_j$ for each $j \in \mathbf{N}$.

We are looking for a holomorphic map $G : X \longrightarrow \mathbf{C}^q$ such that $(H, G) : X \longrightarrow \mathbf{C}^{n+q}$ is a proper holomorphic embedding which extends φ. To make the map (H, G) proper, it suffices to make the map G big enough on the sets L_j (where $\|H\|$ is too small), say $\|G(x)\| > \frac{m_{j+1}}{2}$ for each $x \in L_j$ and $j \in \mathbf{N}$. This implies $\|(H, G)(x)\| \geq \frac{m_{j+1}}{2}$ for each $x \in Q_{j+1} \setminus Q_j$; since the sequence m_j goes to infinity, it follows that the map (H, G) is proper.

We shall construct a sequence of holomorphic maps

$$G^{(j)} : \{x \in U_j,\ \|H(x)\| < b_j\} \longrightarrow \mathbf{C}^q,\ j \in \mathbf{N}$$

and a sequence of positive real numbers $\{\varepsilon_j\}_{j\in\mathbf{N}}$ such that the following hold:

(1) $G^{(j)}$ satisfies $\alpha(b_j)$,$\beta(b_j)$, $\gamma(b_j)$ and $\delta(b_j)$ from theorem 3.1.;

(2) if $\tilde{G} : \{x \in U_j,\ \|H(x)\| < b_j\} \longrightarrow \mathbf{C}^q$ is any holomorphic map such that $\|\tilde{G}(x) - G^{(j)}(x)\| \leq \varepsilon_j$ for each $x \in K_j$, then $\tilde{G}$ satisfies $\alpha(e_j)$, $\beta(e_j)$ and $\delta(e_j)$;

(3) $1 \geq \varepsilon_{j-1} \geq \varepsilon_j > 0$ for $j \geq 2$;

(4) for $G^{(j+1)} = (G_1^{(j+1)}, \ldots, G_q^{(j+1)})$ we have $\|G^{(j+1)}(x)\| > \frac{m_{j+1}}{2} - 2^{-j}$ for each $x \in L_j$;

(5) $\|G^{(j+1)}(x) - G^{(j)}(x)\| \leq 2^{-j}\varepsilon_j$ for each $x \in K_j$.

By (5) the sequence $G^{(j)}$ converges uniformly on compacts in X to a holomorphic map $G: X \longrightarrow \mathbf{C}^q$. At the end of the section we shall verify that G satisfies the required conditions.

We construct the sequence $G^{(j)}$ by induction on j.
$\underline{j = 1.}$ We have to choose a map $G^{(1)}$ and ε_1 satisfying (1) and (3). From theorem 3.1., applied with $(r, R) = (b_1, c_1)$ and $X^{c_1} := U_1$, we obtain a holomorphic map $G^{(1)} : \{x \in U_1,\ \|H(x)\| < b_1\} \longrightarrow \mathbf{C}^q$ satisfying $\alpha(b_1) - \delta(b_1)$. Since the conditions of 'being regular and injective on a compact set' and 'avoiding a discrete set over a compact set' are open, there is an $\varepsilon_1 \in (0, 1)$ such that (2) holds.
$\underline{j \longrightarrow j + 1.}$ Suppose ε_j and $G^{(j)}$ have already been chosen. To construct $G^{(j+1)}$ we shall proceed in two steps. In the first step we construct a holomorphic map $\tilde{G} : \{x \in U_j', \|H(x)\| < b_j\} \longrightarrow \mathbf{C}^q$ such that $(H, \tilde{G})$ embeds this set in $V := B_n(b_j) \times \mathbf{C}^q$ in such a way that the image does not intersect the set $\Sigma = (H, G^{(j)})(\{x \in U_j, \|H(x)\| < b_j\})$ (the image of the previous embedding). The maps $\tilde{G}$ and $G^{(j)}$ together give an embedding over $\{x \in Q_{j+1},\ \|H(x)\| < b_j\}$. In the second step we apply theorem 3.2. to approximate this map by a map $G^{(j+1)} : \{x \in U_{j+1},\ \|H(x)\| < b_{j+1}\} \longrightarrow \mathbf{C}^q$ with the desired properties.

First we explain the construction of the map $\tilde{G}$. We begin by applying theorem 3.1. with $(r, R) = (b_j, c_j)$ and X^{c_j} the union of the finitely many connected components of the set U_j' containing the set $\{x \in U_j',\ \|H(x)\| \leq b_j\}$ to get a holomorphic map $\overline{G} : \{x \in U_j',\ \|H(x)\| < b_j\} \longrightarrow \mathbf{C}^q$ satisfying $\alpha(b_j)$ and $\beta(b_j)$. The problem with $\overline{G}$ is that the image of the map $(H, \overline{G}) : \{x \in U_j',\ \|H(x)\| < b_j\} \longrightarrow \mathbf{C}^{n+q}$ may intersect $\Sigma := (H, G^{(j)})(\{x \in U_j,\ \|H(x)\| < b_j\})$. To pull them apart, an immediate idea would be to push $\overline{G}$ away by adding a sufficiently large constant to one of its components. This however cannot be done since we have to satisfy the interpolation condition. Instead we construct a holomorphic section $C : B_n(b_j) \longrightarrow V \setminus \Sigma$ by using lemma 3.3. Once we have such a section, we squeeze the image of $\overline{G}$ into some small tubular neighbourhood of C which avoids Σ; for instance, we can (and will) take a convex combination $\tilde{G}(x) := C(H(x)) + \delta(\overline{G}(x) - \overline{G}(y_{j+1}))$.

Obviously, the bundle $pr_{\mathbf{C}^n} : V \longrightarrow B_n(b_j)$, the set Σ and the point $x_0 = (x_0', x_0'') := \varphi(y_{j+1})$ fulfill the assumptions of the lemma 3.3., hence there is a holomorphic section

$C : B_n(b_j) \longrightarrow V \setminus \Sigma$ with the properties

- $C(x_0') = x_0$ and
- $\|C(x)\| > \frac{\|x_0\|}{2} = \frac{m_{j+1}}{2}$ for each $x \in B_n(\frac{m_{j+1}}{2})$.

By perturbing C a bit we can achieve that $(C(B_n(b_j)) \setminus \{x_0\}) \cap \varphi(Y \setminus \{y_1, \dots, y_{j+1}\}) = \emptyset$. Choose a number $r_j \in (a_j, b_j)$. There is $\delta > 0$ such that for

$$\tilde{G}(x) := C(H(x)) + \delta(\overline{G}(x) - \overline{G}(y_{j+1}))$$

the following hold:

$$\begin{aligned}
&\Big[(H, \tilde{G})(\{x \in U_j',\ \|H(x)\| < r_j\})\Big] \cap \Big[(H, G^{(j)})(\{x \in U_j,\ \|H(x)\| < r_j\})\Big] = \emptyset, \\
&\|\tilde{G}(x)\| > \frac{m_{j+1}}{2} \text{ for each } x \in L_j, \text{ and} \\
&(H, \tilde{G})(\{x \in U_j' \setminus \{y_{j+1}\},\ \|H(x)\| \le b_j\}) \cap \varphi(Y \setminus \{y_1, \dots, y_{j+1}\}) = \emptyset.
\end{aligned}$$

Write $A = \{x \in U_{j+1},\ \|H(x)\| < r_j\}$ and let $G' : A \longrightarrow \mathbf{C}^q$ be the map defined by

$$G'(x) := \begin{cases} G^{(j)}(x), & x \in U_j \cap A, \\ \tilde{G}(x), & x \in U_j' \cap A. \end{cases}$$

It is clear that the map (H, G') is injective and regular on A (by construction of $\tilde{G}$) and that $\|G'(x)\| > \frac{m_{j+1}}{2}$ for each $x \in L_j$. It follows from the definition of $\tilde{G}$ that the map (H, G') extends φ from $A \cap Y$ to A and $((H, G')(A \setminus Y)) \cap (\varphi(Y \setminus A)) = \emptyset$, which means that G' fulfills the assumptions of theorem 3.2. for $(r, r', R) := (r_j, b_{j+1}, c_{j+1})$ and the set U_{j+1} as the union of connected components of the set $H^{-1}(B_n(c_{j+1}))$. Therefore the map G' can be approximated arbitrarily well on the set $\{x \in U_{j+1},\ \|H(x)\| < r_j\}$ by a map $G^{(j+1)} : \{x \in U_{j+1},\ \|H(x)\| < b_{j+1}\} \longrightarrow \mathbf{C}^q$ satisfying $\alpha(b_{j+1})$, $\beta(b_{j+1})$, $\gamma(b_{j+1})$ and $\delta(b_{j+1})$. Choose $G^{(j+1)}$ so close to G' that for each $x \in U_{j+1}$ with $\|H(x)\| \le a_j$ we have $\|G^{(j+1)}(x) - G'(x)\| \le 2^{-j}\varepsilon_j$. Since G' and $G^{(j)}$ coincide on the set $A \cap U_j$, the map $G^{(j+1)}$ satisfies (5). For each $x \in L_j$ we have

$$|G^{(j+1)}(x)| \ge |G'(x)| - |G^{(j+1)}(x) - G'(x)| > \frac{m_{j+1}}{2} - 2^{-j}\varepsilon_j > \frac{m_{j+1}}{2} - 2^{-j},$$

(recall that $\varepsilon_j \le 1$) which means that (4) holds. Similarly as in the case $j = 1$ there is $\varepsilon_{j+1} \le \varepsilon_j$ such that (2) holds as well.

Because of (3) and (5) the map $G = (G_1, \dots, G_q) : X \longrightarrow \mathbf{C}^q$,

$$G(x) := \lim_{j \to \infty} (G^{(j)}(x))$$

is holomorphic and $G|_Y = (\varphi_{n+1}, \ldots, \varphi_{n+q})$. Recall that the maps $G^{(j)}$ satisfy $\gamma(b_j)$ which means that the map $(H, G^{(j)})$ extends φ to $U_j \cap H^{-1}(B_n(b_j))$. It follows from (3) and (5) that

$$\|G(x) - G^{(j)}(x)\| \leq \varepsilon_j, \quad x \in U_j,\ \|H(x)\| \leq a_j,$$

for each $j \in \mathbf{N}$. Because of (2) the map G satisfies the conditions $\alpha(e_j)$, $\beta(e_j)$ for each $j \in \mathbf{N}$. In other words this means that (H, G) is an injective immersion. The conditions $(3) - (5)$ imply that $\|G(x)\| > \frac{m_{j+1}}{2} - 1$ for each $x \in L_j$ and this makes the map (H, G) proper, which proves (a) from theorem 1.1.

The proof of (b) is based on a similar but much simpler idea. In this case we do not have to take care of injectivity, i.e., we do not have to deal with the conditions $\alpha(r)$ and $\delta(r)$, and for each $j \in \mathbf{N}$ we can find a holomorphic section C (the one that is big enough over L_j and goes through a prescribed point as in lemma 3.3.) by approximating an appropriate constant over L_j. ♠

5. * References

[ABT] Aquistapace, F., Broglia F., Tognolli, A.: A Relative Embedding Theorem for Stein Spaces. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 507-522 (1975)

[Bi] Bishop, E.: Mappings of partially analytic spaces. Amer. J. Math. 83, 209-242 (1961)

[BFo] Buzzard, G., Forstnerič, F.: An interpolation theorem for holomorphic automorphisms and embeddings in $\mathbf{C}^n$. J. Geom. Anal. 10, (2000)

[EG1] Eliashberg, Y. in Gromov, M.: Nonsingular mappings of Stein manifolds. Funct. Anal. and Appl. 5, 156-157 (1971)

[EG2] Eliashberg, Y. in Gromov, M.: Embeddings of Stein manifolds. Annals of Math. 136, 123-135 (1992)

[Fs] Forster, O.: Plongements des varits de Stein. Comment. Math. Helv. 45, 170–184 (1970)

[Fo] Forstnerič, F.: Interpolation by holomorphic automorphisms and ebmeddings in $\mathbf{C}^n$. J. Geom. Anal. 9, 93–118 (1999)

[FP1] Forstnerič, F., Prezelj, J.: The Oka's principle for sections of holomorphic fiber bundles with sprays. Math. Ann., to appear

[FP2] Forstnerič. F., Prezelj, J.: The Oka's principle for holomorphic submersions with sprays. Preprint, 1999

[GG] Golubitsky, M., Guillemin, V.: Stable mappings and their singularities. Grad. Texts in Math. 14, Springer, New York, 1973

[Gr] Gromov, M.: Oka's principle for holomorphic sections of elliptic bundles. J. of the Amer. Math. Soc. 2, 851–897 (1989)

[GR] Gunning, C.R., Rossi, H.: Analytic functions of several complex variables. Prentice–Hall (1965)

[Ha1] Hamm, H.: Zum Homotopietyp Steinscher Räume. J. Reine Angew. Math. 338, 121–135 (1983)

[Ha2] Hamm, H.: Zum Homotopietyp q-vollständiger Räume. J. Reine Angewandte Math. 364, 1–9 (1986)

[HL] Henkin, G.M., Leiterer, J.: The Oka-Grauert principle without induction over the basis dimension. Math. Ann. 311, 71–93 (1998)

[Na] Narasimhan, R.: Imbedding of holomorphically complete spaces. Amer. J. Math. 82, 917–934 (1960)

[RR] Rosay, J.-P., Rudin, W.: Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. Trans. Amer. Math. Soc. 310, 47–86 (1988)

[Sch1] Schürmann, J.: Einbettungen Steischer Räume in affine Räume minimaler Dimension. Schriftenreihe des Math. Inst. Univ. Münster, 3. Serie, Heft 7 (1992)

[Sch2] J. Schürmann: Embeddings of Stein spaces into affine spaces of minimal dimension. Math. Ann. 307, 381–399 (1997)