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INTERPOLATION BY PROPER HOLOMORPHIC EMBEDDINGS OF THE DISC INTO $\mathbb{C}^2$

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Josip Globevnik

Dedicated to the memory of my mother

1. The result

Let Δ be the open unit disc in $\mathbb{C}$. A map $f\colon\ \Delta \to \mathbb{C}^2$ is called *a proper holomorphic embedding* if it is a holomorphic immersion which is one to one and such that the preimage of every compact set is compact. If $f\colon\ \Delta \to \mathbb{C}^2$ is a proper holomorphic embedding then $f(\Delta)$ is a closed submanifold of $\mathbb{C}^2$ which is, via f, biholomorphically equivalent to Δ.

It is not trivial to prove that there are proper holomorphic embeddings from Δ to $\mathbb{C}^2$ [St, A, GS]. It is known that given a discrete set $E \subset \mathbb{C}^2$ there is a proper holomorphic embedding $f\colon\ \Delta \to \mathbb{C}^2$ such that $E \subset f(\Delta)$ [FGS]. In the present paper we prove a stronger result:

Theorem 1.1 *Given a discrete set $S \subset \Delta$ and a proper injection $\varphi\colon\ S \to \mathbb{C}^2$ there is a proper holomorphic embedding $f\colon\ \Delta \to \mathbb{C}^2$ that extends φ.*

In other words, given an injective sequence $\{\zeta_j\} \subset \Delta$ such that $|\zeta_j| \to 1$ and an injective sequence $\{w_j\} \subset \mathbb{C}^2$ such that $|w_j| \to +\infty$ there is a proper holomorphic embedding $f\colon\ \Delta \to \mathbb{C}^2$ such that $f(\zeta_j) = w_j$ $(j \in \mathbb{N})$.

The proof of the Carleman approximation theorem of Buzzard and Forstnerič [BFo] can be adapted to prove such a result for proper holomorphic embeddings $f\colon\ \mathbb{C} \to \mathbb{C}^2$. In the proof there one uses the fact that $\mathbb{C}$ admits particularly simple embeddings into $\mathbb{C}^2$ of the form $\zeta \to (\zeta, a(\zeta))$ where a is an entire function. There are no such embeddings for Δ so a different proof is necessary in our case. In the induction step of our proof we use simultaneous composition by automorphisms on the left and on the right, a novelty introduced by Buzzard and Forstnerič.

2. The scheme of the proof

Suppose that $S \subset \Delta$ is a discrete set and let $\varphi\colon\ S \to \mathbb{C}^2$ be a proper injection. With no loss of generality assume that S is infinite.

Denote by $\mathbb{B}$ the open unit ball in $\mathbb{C}^2$. We shall construct inductively a sequence K_n of compact subsets of Δ, such that bK_n is a smooth Jordan curve for each $n \in \mathbb{N}$ and such that $K_n \subset\subset K_{n+1}$ $(n \in \mathbb{N})$, $\cup_{n=1}^{\infty} K_n = \Delta$, an increasing sequence r_n of positive numbers converging to $+\infty$, a decreasing sequence ε_n of positive numbers and a sequence f_n of holomorphic maps from Δ to $\mathbb{C}^2$ which are one to one and regular and such that the following holds:

(i) $\varphi((\Delta \setminus K_n) \cap S) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}}$

(ii) $f_n(\Delta \setminus \mathrm{Int}K_n) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}}$

(iii) $f_{n+1}(\Delta \setminus K_n) \subset \mathbb{C}^2 \setminus r_{n-1}\overline{\mathbb{B}}$

(iv) $f_n|K_n \cap S = \varphi|K_n \cap S$

(v) $|f_{n+1}-f_n|<\varepsilon_n/2^n$ on K_n

(vi) if h is a holomorphic map on $\mathrm{Int}K_n$ that satisfies $|h-f_n|<\varepsilon_n$ on $\mathrm{Int}K_n$ then h is one to one and regular on K_{n-1}

(vii) $(1-1/n)\Delta\subset K_n$.

Suppose for a moment that we have done this. By (v) and (vii) f_n converges, uniformly on compacta in Δ, to a holomorphic map f. By (v), $|f_n-f|\leq\sum_{j=n}^{\infty}|f_{j+1}-f_j|\leq\sum_{j=n}^{\infty}\varepsilon_j/2^j<\varepsilon_n$ on K_n which implies by (vi) that f is regular and one to one on K_{n-1}. As this holds for every n it follows that f is regular and one to one on Δ . By (iv), f extends φ. Let $\zeta\in K_{n+1}\setminus K_n$. By (v), $|f_{j+1}(\zeta)-f_j(\zeta)|<\varepsilon_j/2^j$ $(j\geq n+1)$ which, by (iii) implies that $|f(\zeta)|\geq|f_{n+1}(\zeta)|-\sum_{j=n+1}^{\infty}|f_{j+1}(\zeta)-f_j(\zeta)|\geq r_{n-1}-\sum_{j=n+1}^{\infty}\varepsilon_j/2^j\geq r_{n-1}-\varepsilon_{n+1}$. This holds for every n. Since r_n increase to $+\infty$ and since ε_n are decreasing it follows that the map f is proper. Thus, f has all the required properties.

In the process we shall also construct two sequences S_n,T_n of positive numbers such that $S_{n+1}=S_n$ for even n and $T_{n+1}=T_n$ for odd n. Each map f_n will be of the form $f_n=A_n\circ g_n$ where A_n is a holomorphic automorphism of $\mathbb{C}^2$ and g_n is a one to one and regular holomorphic map from an open neighbourhood U_n of $\overline{\Delta}$ to $\mathbb{C}^2$ which, for even n is transverse to $\{(z,w)\colon\ |z|=S_n\}$ and satisfies $g_n^{-1}(\{|z|=S_n\})=b\Delta$, and for odd n, is transverse to $\{(z,w)\colon\ |w|=T_n\}$ and satisfies $g_n^{-1}(\{|w|=T_n\})=b\Delta$.

With no loss of generality assume that $0\notin S$.

To begin the induction, let $f_1(\zeta)=(0,\zeta)$ and let $r_1,\ 0<r_1<1/2$ be such that $2r_1\overline{\Delta}$ contains no point of S. Put $K_0=r_1\overline{\Delta},\ K_1=2r_1\overline{\Delta}$. Then (i), (ii) and (vii) are satisfied for $n=1$ and (iv) is vacuously satisfied for $n=1$. Put $S_1=T_1=1$ and $A_1=\mathrm{Id}$ so that $f_1=A_1\circ g_1$ where $g_1(\zeta)=(0,\zeta)$ and $U_1=\mathbb{C}$. Clearly g_1 is transverse to $\{|w|=T_1\}$ and $g_1^{-1}(\{|w|=T_1\})=b\Delta$. Put $r_0=r_1/2$. Then $A_1(\{|w|>T_1/2\})$ misses $2r_0\mathbb{B}$. Put $\varepsilon_0=\min\{1,r_1/2\}$.

Given $f_n=A_n\circ g_n$ we shall have $f_{n+1}=A_{n+1}\circ g_{n+1}$ with $A_{n+1}=\Psi_{n+1}\circ\Theta_{n+1}\circ A_n$ where Θ_{n+1} and Ψ_{n+1} are holomorphic automorphisms of $\mathbb{C}^2$ and with $g_{n+1}=G_{n+1}\circ g_n\circ p_{n+1}$ where p_{n+1} is a conformal map from a neighbourhood U_{n+1} of $\overline{\Delta}$ to $p_{n+1}(U_{n+1})\subset\mathbb{C}$ which is a slight perturbation of the identity on $\overline{\Delta}$ and G_{n+1} is an automorphism of $\mathbb{C}^2$ of the form

$$G_{n+1}(z,w)=\Big(z+S_{n+1}\Big(\frac{w}{T_n}\Big)^{M_{n+1}},\ w\Big)\ \text{ if } n \text{ is odd,}\qquad(2.1')$$

$$G_{n+1}(z,w)=\Big(z,\ w+T_{n+1}\Big(\frac{z}{S_n}\Big)^{M_{n+1}}\Big)\ \text{ if } n \text{ is even.}\qquad(2.1'')$$

3. The induction step, Part 1

Suppose for a moment that we have constructed $f_n=A_n\circ g_n,\ K_n,\ S_n,\ T_n,\ r_n$ and ε_{n-1}. We want to show how to obtain $\varepsilon_n,\ K_{n+1},\ S_{n+1},\ T_{n+1},\ r_{n+1}$ and $f_{n+1}=A_{n+1}\circ g_{n+1}$. Suppose that n is odd so that $g_n\colon U_n\to\mathbb{C}^2$ is transverse to $\{(z,w)\colon\ |w|=T_n\}$ and satisfies $g_n^{-1}(\{|w|=T_n\})=b\Delta$. Put $T_{n+1}=T_n$. Since g_n is transverse to $\{|w|=T_n\}$ and since S is discrete one can, after shrinking U_n if necessary, choose $T_{n1},\ T_{n2},\ T_{n3}$ such that

$$\frac{T_n}{2}<T_{n3}<T_{n2}<T_{n1}<T_n$$

where T_{n3} is so close to T_n that for all $T,\ T_{n3} \le T \le T_n,\ g_n$ is transverse to $\{|w| = T\}$ and $g_n^{-1}(\{|w| = T\})$ is a smooth Jordan curve, that

$$K_n \subset g_n^{-1}\{|w| < T_{n3}\},$$

that $g_n^{-1}(\{T_{n3} \le T \le T_{n1}\})$ contains no point of S, and that $g_n^{-1}(\{|w| < T_{n1}\})$ contains a point in S that does not belong to K_n. Put

$$P_{n+1} = g_n^{-1}(\{|w| \le T_{n3}\}),\ Q_{n+1} = g_n^{-1}(\{|w| \le T_{n2}\}),\ K_{n+1} = g_n^{-1}(\{|w| \le T_{n1}\}).$$

With no loss of generality assume that T_{n3} has been chosen so close to T_n that (vii) holds with n replaced by $n+1$. We have

$$K_n \subset\subset P_{n+1} \subset\subset Q_{n+1} \subset\subset K_{n+1}.$$

Clearly bK_{n+1} is a smooth Jordan curve.

By (i), $r_n < \min\{|\varphi(w)|\colon\ w \in (\Delta \setminus K_n) \cap S\}$. Thus, one can choose $r_{n+1} > r_n$ such that

$$\min\{|\varphi(w)|\colon w \in (\Delta \setminus K_{n+1}) \cap S\} - 1 < r_{n+1} < \min\{|\varphi(w)|\colon w \in (\Delta \setminus K_{n+1}) \cap S\}. \quad (3.1)$$

Then (i) is satisfied with n replaced by $n+1$. Choose $\varepsilon_n,\ 0 < \varepsilon_n < \varepsilon_{n-1}$, such that

$$\varepsilon_n < r_n - r_{n-1}, \quad \varepsilon_n < r_{n-1} \quad (3.2)$$

and such that (vi) holds. Since f_n is one to one and regular on Δ this is possible by a lemma of Narasimhan [Na, p. 926].

Choose $R,\ R > 2r_{n+1},\ R > 2r_n + \varepsilon_n,$ so large that $f_n(K_n) + \mathbb{B} \subset R\mathbb{B}$ and that $\varphi(K_{n+1} \cap S) \subset R\mathbb{B}$. We need the following lemma.

Lemma 3.1. *Let $R > 0$ and let $w_1, w_2, \cdots w_n \in R\mathbb{B},\ w_i \ne w_j\ (i \ne j)$. Given $\gamma > 0$ there is a $\delta > 0$ such that whenever $q_1, q_2, \cdots q_n \in \mathbb{C}^2$ satisfy $|q_i - w_i| < \delta,\ 1 \le i \le n$, there is a holomorphic automorphism Ψ of $\mathbb{C}^2$ such that*

(i) $\Psi(q_i) = w_i\ \ (1 \le i \le n)$

(ii) $|\Psi(w) - w| < \gamma\ \ (w \in R\mathbb{B})$.

Lemma 3.1 provides a $\theta_n,\ 0 < \theta_n < \varepsilon_n/2^{n+2}$, such that

$$\left.\begin{array}{l}\text{whenever } \psi\colon\ K_{n+1} \cap S \to \mathbb{C}^2 \text{ satisfies } |\psi - \varphi| < 3\theta_n \text{ on} \\ K_{n+1} \cap S \text{ there is a holomorphic automorphism } \Psi \text{ of } \mathbb{C}^2 \text{ such that} \\ \Psi \circ \psi = \varphi | K_{n+1} \text{ and such that } |\Psi - \mathrm{Id}| < \varepsilon_n/2^{n+1} \text{ on } R\mathbb{B}.\end{array}\right\} \quad (3.3)$$

By (3.2) we may assume that

$$r_n - 3\theta_n > r_{n-1} + \varepsilon_n + \theta_n, \quad 2r_{n-1} - \theta_n > r_{n-1} + \varepsilon_n + \theta_n. \quad (3.4)$$

4. Proof of Lemma 3.1

Sublemma 4.1 *Suppose that* $R>0$ *and let* $\alpha_1,\cdots\alpha_n\in R\Delta,\ \alpha_i\neq\alpha_j\ (i\neq j)$. *There are* $\eta>0$ *and* $L<\infty$ *such that whenever* $\beta_1,\cdots\beta_n$ *satisfy* $|\beta_i-\alpha_i|<\eta,\ 1\leq i\leq n$, *then for every* $j,\ 1\leq j\leq n$, *there is a polynomial* Q_j *such that*

(i) $Q_j(\beta_i)=\delta_{ji}\ (1\leq i,j\leq n)$

(ii) $|Q_j(\zeta)|\leq L\ (\zeta\in 2R\Delta)$.

Proof. Choose $\eta>0$ so small that $\alpha_i+\eta\Delta\subset R\Delta\ (1\leq i\leq n)$ and let $|\beta_i-\alpha_i|<\eta\ (1\leq i\leq n)$. For each $j,\ 1\leq j\leq n$, the polynomial

$$Q_j(\zeta)=\prod_{k=1,k\neq j}^{n}\frac{\zeta-\beta_k}{\beta_j-\beta_k}$$

satisfies (i). If $|\zeta|<2R$ then

$$|Q_j(\zeta)|\leq\frac{(3R)^{n-1}}{\left(\min_{j\neq k}|\beta_j-\beta_k|\right)^{n-1}}.$$

Now, let $\gamma=\min_{j\neq k}|\alpha_j-\alpha_k|$. Passing to a smaller η we may assume that $0<\eta<\gamma/2$. If $|\alpha_i-\beta_i|<\eta,\ 1\leq i\leq n$, then $\min_{j\neq k}|\beta_j-\beta_k|\geq\gamma-2\eta>0$ so Q_j satisfies (ii) with $L=[3R/(\gamma-2\eta)]^{n-1}$. This completes the proof.

Proof of Lemma 3.1. Choose a coordinate system in $\mathbb{C}^2$ such that if $w_i=(w_i^1,w_i^2)$ then $w_i^1\neq w_j^1,\ w_i^2\neq w_j^2$ if $i\neq j,\ 1\leq i,j\leq n$. By Sublemma 5.1 there are $\eta>0$ and $L<\infty$ such that whenever β_i^1 satisfy $|\beta_i^1-w_i^1|<\eta$ and β^2 satisfy $|\beta_i^2-w_i^2|<\eta,\ 1\leq i\leq n$, then for each $j,\ 1\leq j\leq n$, there are polynomials Q_j^1 and Q_j^2 such that $Q_j^1(\beta_j^1)=1,\ Q_j^1(\beta_i^1)=0\ (i\neq j)$, $Q_j^2(\beta_j^2)=1,\ Q_j^2(\beta_i^2)=0\ (i\neq j)$ and $|Q_j^1|<L,\ |Q_j^2|<L$ on $2R\Delta$.

Let $|z_j-w_j|<\eta,\ 1\leq j\leq n$. Our map Φ will be of the form $\Phi=T\circ S$ where T,S are the automorphisms of $\mathbb{C}^2$

$$T(\xi,\zeta)=(\xi,\zeta+Q_1(\xi)),\quad S(\xi,\zeta)=(\xi+Q_2(\zeta),\zeta)$$

such that

$$S(R\Delta\times R\Delta)\subset(2R\Delta)\times(R\Delta),\tag{4.1}$$

$$|S(\xi,\zeta)-(\xi,\zeta)|<\gamma/2\quad((\xi,\zeta)\in(R\Delta)^2),\tag{4.2}$$

$$|T(\xi,\zeta)-(\xi,\zeta)|<\gamma/2\quad((\xi,\zeta)\in(2R\Delta)\times(R\Delta))\tag{4.3}$$

and

$$S(z_i^1,z_i^2)=(w_i^1,z_i^2),\quad T(w_i^1,z_i^2)=(w_i^1,w_i^2)\ \ (1\leq i\leq n).\tag{4.4}$$

By (4.1)-(4.4) the map Φ satisfies (i) and (ii) in Lemma 3.1.

To construct S, put $\beta_j^2=z_j^2,\ 1\leq j\leq n$, and let $Q_j^2,\ 1\leq j\leq n$, be as above. In particular, $Q_j^2(z_i^2)=\delta_{ji},\ 1\leq i,j\leq n$. Put

$$Q_2(\zeta)=\sum_{j=1}^{n}(w_j^1-z_j^1)Q_j^2(\zeta).$$

We have $Q_2(z_j^2) = \sum_{i=1}^n (w_i^1 - z_i^1) Q_i^2(z_j^2) = w_j^1 - z_j^1$ and so $S(z_i^1, z_i^2) = (z_i^1 + w_i^1 - z_i^1, z_i^2) = (w_i^1, z_i^2)$. We have $|Q_2(\zeta)| \leq n.\max_{1\leq j\leq n}|w_j^1 - z_j^1|.L \quad (|\zeta| < R)$ which implies that $|S(\xi,\zeta) - (\xi,\zeta)| = |(Q_2(\zeta), 0)| \leq n.L.\max_{1\leq j\leq n}|w_j - z_j| \quad (|\zeta| < R)$. In particular, if $\eta > 0$ is small enough then $|Q_2(\zeta)| < R$ $(|\zeta| < R)$ so that (4.1) and (4.2) hold.

To construct T, put $\beta_j^1 = w_j^1$, $1 \leq j \leq n$, and let Q_j^1, $1 \leq j \leq n$, be as above. Put

$$Q_1(\zeta) = \sum_{j=1}^n (w_j^2 - z_j^2) Q_j(\zeta).$$

We have $Q_1(w_j^1) = w_j^2 - z_j^2 \quad (1 \leq j \leq n)$ so $T(w_i^1, z_i^2) = (w_i^1, z_i^2 + w_i^2 - z_i^2) = (w_i^1, w_i^2)$ $(1 \leq i \leq n)$. Again, $|Q_1(\zeta)| \leq n.\max_{1\leq j\leq n}|w_j^2 - z_j^2|.L \quad (|\zeta| < 2R)$, which implies that $|T(\xi,\zeta) - (\xi,\zeta)| = |(0, Q_1(\xi))| \leq n.\max_{1\leq j\leq n}|w_j - z_j|.L \quad (|\xi| < 2R)$. In particular, if $\delta = \eta$ is small enough then (4.3) holds. (4.4) is clear. This completes the proof.

Remark. Lemma 3.1 holds for $\mathbb{C}^N$, $N \geq 2$. The proof is an easy elaboration of the proof above.

5. The induction step, Part 2

We need the following

Lemma 5.1 *Let $r > 0$ and let $\Phi\colon \mathbb{C} \to \mathbb{C}^2$ be a proper holomorphic embedding. Let $\Sigma \subset\subset \mathbb{C}$ be a domain bounded by a smooth Jordan curve and assume that $\Phi(b\Sigma) \subset \mathbb{C}^2 \setminus r\overline{\mathbb{B}}$. Then the set $(r\overline{\mathbb{B}}) \cup \Phi(\overline{\Sigma})$ is polynomially convex.*

Proof. Since Σ is a Jordan domain with smooth boundary it is easy to see that if $K \subset \mathbb{C}\setminus\overline{\Sigma}$ is a compact set, if $a, b \in (\mathbb{C}\setminus\overline{\Sigma})\setminus K$, and if p is a path in $\mathbb{C}\setminus K$ joining a and b then there is a path $\tilde{p}$ in $(\mathbb{C}\setminus\overline{\Sigma})\setminus K$ joining a and b. Let $K = \{\zeta \in \mathbb{C}\setminus\overline{\Sigma}\colon |\Phi(\zeta)| \leq r\}$. Since $\Phi(b\Sigma) \subset \mathbb{C}^2 \setminus r\overline{\mathbb{B}}$ and since $|\Phi(\zeta)| \to +\infty$ as $|\zeta| \to +\infty$, the set K is compact. Suppose for a moment that $(\mathbb{C} \setminus \overline{\Sigma}) \setminus K$ is not connected. The preceding discussion implies that $\{\zeta \in \mathbb{C}\colon |\Phi(\zeta)| > r\}$ has a bounded component which contradicts the maximum principle. Thus, $(\mathbb{C}\setminus\overline{\Sigma})\setminus K$ is connected which implies that for each $q \in \Phi(\mathbb{C}) \setminus (\Phi(\overline{\Sigma}) \cup r\overline{B})$ there is a path $\eta\colon [0,1) \to \Phi(\mathbb{C}) \setminus (\Phi(\overline{\Sigma}) \cup r\overline{\mathbb{B}})$ such that $\eta(0) = q$ and $|\eta(t)| \to +\infty$ as $t \to 1$. The statement of the lemma now follows from [BF, Lemma 3.1]. This completes the proof.

Remark. It is easy to see that the proof of Lemma 3.1 in [BF] works for $\mathbb{C}^N$, $N \geq 2$, and so Lemma 5.1. holds for proper holomorphic embeddings $\Phi\colon \mathbb{C} \to \mathbb{C}^N$, $N \geq 2$.

Proof of the induction step continued. We have already mentioned that for each m, $f_{m+1} = (\Psi_{m+1} \circ \Theta_{m+1} \circ A_m) \circ (G_{m+1} \circ g_m \circ p_{m+1}) = A_{m+1} \circ g_{m+1}$. Thus, $f_n = H_n \circ g_1 \circ (p_2 \circ \cdots \circ p_n)$ where $H_n = (\Psi_n \circ \Theta_n) \circ \cdots \circ (\Psi_2 \circ \Theta_2) \circ (G_n \circ \cdots \circ G_2)$ is a holomorphic automorphism of $\mathbb{C}^2$. It follows that $f_n(K_n)$ is a compact subset of $(H_n \circ g_1)(\mathbb{C})$, a closed submanifold of $\mathbb{C}^2$ biholomorphically equivalent to $\mathbb{C}$, whose boundary $f_n(bK_n)$ is a smooth Jordan curve which is, by (ii), contained in $\mathbb{C}^2 \setminus r_n\overline{\mathbb{B}}$. By Lemma 5.1 the set $f_n(K_n) \cup r_n\overline{\mathbb{B}}$ is polynomially convex.

By (ii) $f_n(K_n) \cup r_n\overline{\mathbb{B}}$ contains no point of $f_n((K_{n+1} \setminus K_n) \cap S)$. Since f_n is one to one it follows that $f_n(\xi) \neq f_n(\eta)$ if $\xi, \eta \in (K_{n+1} \setminus K_n) \cap S$. By (i), $\varphi((K_{n+1} \setminus K_n) \cap S)$ does not meet $r_n\overline{\mathbb{B}}$. However, some points of $\varphi((K_{n+1} \setminus K_n) \cap S)$ may lie in $f_n(K_n)$. Since

$f_n(K_n)$ is contained in $(H_n\circ g_1)(\mathbb{C})$, a closed one dimensional complex submanifold of $\mathbb{C}^2$, one can change φ slightly on $K_{n+1}\cap S$ to $\tilde\varphi$ so that

$$|\tilde\varphi-\varphi|<\theta_n \ \text{ on } K_{n+1}\cap S,$$

so that $\tilde\varphi$ is one to one on $K_{n+1}\cap S$ and that $f_n(K_n)\cup r_n\overline{\mathbb{B}}$ contains no point of $\tilde\varphi((K_{n+1}\setminus K_n)\cap S)$.

By [FGS] there is an automorphism Θ_{n+1} of $\mathbb{C}^2$ which fixes each point of $f_n(K_n\cap S)$, that moves each point $f_n(\zeta),\ \zeta\in(K_{n+1}\setminus K_n)\cap S$ to $\tilde\varphi(\zeta)$, and that satisfies

$$|\Theta_{n+1}-\mathrm{Id}|<\theta_n \ \text{ on } f_n(K_n)\cup r_n\overline{\mathbb{B}}. \tag{5.2}$$

By (iv) we have $f_n|K_n\cap S=\varphi|K_n\cap S$. Almost the same equality holds for $\Theta_{n+1}\circ f_n$ in place of f_n since $\Theta_{n+1}\circ f_n|K_{n+1}\cap S=\tilde\varphi|K_{n+1}\cap S$. Applying on both sides on the left an automorphism Ψ provided by Lemma 3.1 which satisfies $\Psi\circ\tilde\varphi=\varphi$ on $K_{n+1}\cap S$, would produce a map from Δ to $\mathbb{C}^2$ that would satisfy (iv) with n replaced by $n+1$. However, such a map does not necessarily satisfy (ii) with $n+1$ in place of n or (iii) since we have no control over what Θ_{n+1} does with $f_n(\Delta\setminus K_n)$.

6. The induction step, Part 3

We perform our induction process in such a way that

$$A_n(\{(z,w)\colon\ |w|>T_n/2\}) \text{ misses } 2r_{n-1}\mathbb{B} \ \text{ if } n \text{ is odd.} \tag{6.1'}$$

and

$$A_n(\{(z,w)\colon\ |z|>S_n/2\}) \text{ misses } 2r_{n-1}\mathbb{B} \ \text{ if } n \text{ is even.} \tag{6.1''}$$

Recall that (6.1') holds for $n=1$. We are describing the induction step for odd n so assume that (6.1') holds. To handle the problem described at the end of the previous section we replace g_n in $\Theta_{n+1}\circ A_n\circ g_n=\Theta_{n+1}\circ f_n$ by $G_{n+1}\circ g_n$ where G_{n+1} is an automorphism of $\mathbb{C}^2$ of the form (2.1').

Passing to a slightly smaller U_n if necessary we may assume that $g_n(U_n)$ is bounded. We want that G_{n+1} changes g_n only slightly on K_n and on $K_{n+1}\cap S$ while it maps $g_n(U_n\setminus \mathrm{Int}Q_{n+1})$ so far from the origin that

$$(\Theta_{n+1}\circ A_{n+1})\circ(G_{n+1}\circ g_n)(U_n\setminus\mathrm{Int}Q_{n+1})\subset\mathbb{C}^2\setminus 2r_{n+1}\overline{\mathbb{B}} \tag{6.2}$$

which, since $g_n(U_n)$ is bounded, and since $\Theta_{n+1}\circ A_{n+1}$ is an automorphism of $\mathbb{C}^2$, holds if

$$\left|S_{n+1}\left(\frac{w}{T_n}\right)^{M_{n+1}}\right|\ge\rho_n \ \ (|w|\ge T_{n2}) \tag{6.3}$$

provided that ρ_n is sufficiently large. Choose $\tau_n>0$ so small that

$$|(\Theta_{n+1}\circ A_n)(p)-(\Theta_{n+1}\circ A_n)(q)|<\theta_n \ \ (q\in g_n(P_{n+1}),\ |p-q|<2\tau_n). \tag{6.4}$$

We want that

$$\left| S_{n+1} \left(\frac{w}{T_n} \right)^{M_{n+1}} \right| \leq \tau_n \quad (|w| \leq T_{n1}) \tag{6.5}$$

which will imply that G_{n+1} changes g_n on P_{n+1} for at most τ_n.

Let

$$S_{n+1} = \rho_n \left(\frac{T_n}{T_{n2}} \right)^{M_{n+1}}.$$

Notice that S_{n+1} is arbitrarily large provided that M_{n+1} is large enough. The choice of S_{n+1} implies (6.3) while (6.5) becomes equivalent to

$$\rho_n \left(\frac{T_{n1}}{T_{n2}} \right)^{M_{n+1}} < \tau_n \tag{6.7}$$

which will hold provided that M_{n+1} is large enough. Choose M_{n+1} so large that S_{n+1} becomes so large that

$$(\Theta_{n+1} \circ A_n)(\{|z| > S_{n+1}/2\}) \text{ misses } (2r_n + \varepsilon_n)\overline{B}. \tag{6.8}$$

Notice that if an automorphism G: $\mathbb{C}^2 \to \mathbb{C}^2$ satisfies $|G(z) - z| < \tau$ $(z \in R\mathbb{B})$ where $0 < \tau < R$ then $(R - \tau)\overline{\mathbb{B}} \subset G(R\overline{\mathbb{B}})$. Choose a compact set $K_n' \subset \mathrm{Int}K_n$ such that $f_n(\Delta \setminus K_n') \subset f_n(\Delta \setminus K_n) + \theta_n\mathbb{B}$. Now, (ii) implies that $A_n(g_n(\Delta \setminus K_n')) = f_n(\Delta \setminus K_n')$ misses $(r_n - \theta_n)\overline{\mathbb{B}}$ and (5.2) implies that

$$(\Theta_{n+1} \circ A_n \circ g_n)(\Delta \setminus K_n') \subset \mathbb{C}^2 \setminus (r_n - 2\theta_n)\overline{\mathbb{B}}. \tag{6.9}$$

By (6.5), $|G_{n+1} \circ g_n - g_n| \leq \tau_n$ on P_{n+1} so by (6.4)

$$|(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(\zeta) - (\Theta_{n+1} \circ A_n \circ g_n)(\zeta)| \leq \theta_n \quad (\zeta \in P_{n+1})$$

which, by (6.9) gives

$$(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(P_{n+1} \setminus K_n') \subset \mathbb{C}^2 \setminus (r_n - 3\theta_n)\overline{\mathbb{B}} \tag{6.10}$$

Let $\zeta \in Q_{n+1} \setminus P_{n+1}$. Since $g_n(Q_{n+1} \setminus P_{n+1}) \subset \{|w| > T_n/2\}$ and since G_{n+1} does not change the w coordinate we have $(G_{n+1} \circ g_n)(\zeta) \in \{|w| > T_n/2\}$ and so $(A_n \circ G_{n+1} \circ g_n)(\zeta) \in A_n(\{|w| > T_n/2\})$. By (6.1') $A_n(\{|w| > T_n/2\})$ misses $2r_{n-1}\mathbb{B}$ which implies that $(A_n \circ G_{n+1} \circ g_n)(\zeta) \in \mathbb{C}^2 \setminus 2r_{n-1}\mathbb{B}$. By (5.2) it follows that $(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(\zeta) \in \mathbb{C}^2 \setminus s\overline{\mathbb{B}}$ where $s = \min\{r_n - \theta_n, 2r_{n-1} - \theta_n\}$, by (3.4), satisfies $s > r_{n-1} + \theta_n + \varepsilon_n$. By (6.10), (6.2) and (3.4) it follows that

$$(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(U_n \setminus K_n') \subset \mathbb{C}^2 \setminus (r_{n-1} + \theta_n + \varepsilon_n)\overline{\mathbb{B}}. \tag{6.11}$$

7. The induction step, Part 4

Note first that $\Theta_{n+1}\circ A_n\circ g_n|K_{n+1}\cap S=\tilde\varphi|K_{n+1}\cap S$. This does not necessarily hold if we replace g_n by $G_{n+1}\circ g_n$. However, since all points of $K_{n+1}\cap S$ lie in P_{n+1}, since $|G_{n+1}\circ g_n-g_n|<\tau_n$ on P_{n+1} and since $|\varphi-\tilde\varphi|<\theta_n$ on $K_{n+1}\cap S$ it follows by (6.4) that

$$|\Theta_{n+1}\circ A_n\circ G_{n+1}\circ g_n-\varphi|<2\theta_n \quad \text{on } K_{n+1}\cap S. \tag{7.1}$$

The problem now is that $(G_{n+1}\circ g_n)^{-1}(\{(z,w)\colon\ |z|=S_{n+1}\})$ is not necessarily equal to $b\Delta$ so we cannot use $\Theta_{n+1}\circ A_n\circ G_{n+1}\circ g_n$ as f_{n+1} even after composing with a correction automorphism provided by Lemma 3.1. However, $(G_{n+1}\circ g_n)^{-1}(\{|z|=S_{n+1}\})$ is a real analytic curve that is arbitrarily small $\mathcal{C}^1$ perturbation of $b\Delta$ independently of M_{n+1} if only S_{n+1} is large enough [G, Sec. 5]; in our case this means if only M_{n+1} is large enough.

Thus, provided that M_{n+1} is large enough the conformal map p_{n+1} mapping Δ to the domain $(G_{n+1}\circ g_n)^{-1}(\{|z|<S_{n+1}\})$ and satisfying $p_{n+1}(0)=0,\ p'_{n+1}(0)>0$, is arbitrarily close to the identity on Δ provided that M_{n+1} is sufficiently large [P, p. 286]. Once we have chosen M_{n+1} the map p_{n+1} extends holomorphically to a neighbourhood $U_{n+1}\subset U_n$ of $\overline\Delta$ so that the extended map p_{n+1} maps U_{n+1} biholomorphically onto $p_{n+1}(U_{n+1})$ and so that the map $g_{n+1}=G_{n+1}\circ g_n\circ p_{n+1}\colon U_{n+1}\to\mathbb{C}^2$ is transverse to $\{(z,w)\colon\ |z|=S_{n+1}\}$ and satisfies $g_{n+1}^{-1}(\{|z|=S_{n+1}\})=b\Delta$ [G].

Passing to a larger M_{n+1} if necessary we may assume that p_{n+1} is so close to the identity on $\overline\Delta$ that

$$|g_n\circ p_{n+1}-g_n|<\tau_n \quad \text{on } \overline\Delta \tag{7.2}$$

and that

$$\left.\begin{aligned} &K_n\subset p_{n+1}^{-1}(P_{n+1}),\quad K_{n+1}\cap S\subset p_{n+1}^{-1}(P_{n+1})\\ &p_{n+1}^{-1}(Q_{n+1})\subset \mathrm{Int}K_{n+1},\quad p_{n+1}^{-1}(K'_n)\subset K_n.\end{aligned}\right\} \tag{7.3}$$

Since $|G_{n+1}\circ g_n-g_n|\le\tau_n$ on P_{n+1} it follows that $|G_{n+1}\circ g_n\circ p_{n+1}-g_n\circ p_{n+1}|\le\tau_n$ on $p_{n+1}^{-1}(P_{n+1})$ which, by (7.2) and (7.3) implies that

$$|G_{n+1}\circ g_n\circ p_{n+1}-g_n|<2\tau_n \quad \text{on}\quad K_n\cup(K_{n+1}\cap S).$$

Since $K_n\cup(K_{n+1}\cap S)\subset P_{n+1}$, (6.4) implies that

$$|(\Theta_{n+1}\circ A_n\circ G_{n+1}\circ g_n\circ p_{n+1})(\zeta)-(\Theta_{n+1}\circ A_n\circ g_n)(\zeta)|<\theta_n \quad (\zeta\in K_n\cup(K_{n+1}\cap S)).$$

By (5.2), $|\Theta_{n+1}f_n(\zeta)-f_n(\zeta)|<\theta_n$ $(\zeta\in K_n)$ so it follows that

$$|(\Theta_{n+1}\circ A_n\circ G_{n+1}\circ g_n\circ p_{n+1})(\zeta)-f_n(\zeta)|<2\theta_n \quad (\zeta\in K_n). \tag{7.4}$$

Further, since $(\Theta_{n+1}\circ A_n\circ g_n)|K_{n+1}\cap S=\tilde\varphi$ and since $|\tilde\varphi-\varphi|<\theta_n$ on $K_{n+1}\cap S$ it follows also that

$$|(\Theta_{n+1}\circ A_n\circ G_{n+1}\circ g_n\circ p_{n+1})(\zeta)-\varphi(\zeta)|<3\theta_n \quad (\zeta\in K_{n+1}\cap S). \tag{7.5}$$

The choice of R and (3.3) imply that there is a holomorphic automorphism Ψ_{n+1} of $\mathbb{C}^2$ such that

$$|\Psi_{n+1}-\mathrm{Id}|<\varepsilon_n/2^{n+1} \quad \text{on}\quad R\mathbb{B} \tag{7.6}$$

and such that

$$(\Psi_{n+1}\circ\Theta_{n+1}\circ A_n\circ G_{n+1}\circ g_n\circ p_{n+1})(\zeta)=\varphi(\zeta)\ \ (\zeta\in K_{n+1}\cap S). \tag{7.7}$$

Put $f_{n+1}=A_{n+1}\circ g_{n+1}$ where $A_{n+1}=\Psi_{n+1}\circ\Theta_{n+1}\circ A_n$ and $g_{n+1}=G_{n+1}\circ g_n\circ p_{n+1}$. By (7.7), (iv) is satisfied with $n+1$ in place of n. Since $\theta_n<\varepsilon_n/2^{n+2}$ and since $f_n(K_n)+\mathbb{B}\subset R\mathbb{B}$, (7.4) and (7.6) imply that $|f_{n+1}(\zeta)-f_n(\zeta)|<2\theta_n+\varepsilon_n/2^{n+1}<\varepsilon_n/2^n$ $(\zeta\in K_n)$ so that (v) is satisfied.

By (7.3), $\zeta\in\Delta\setminus\mathrm{Int}K_{n+1}$ implies that $p_{n+1}(\zeta)\in U_n\setminus Q_{n+1}$ which, by (6.2) implies that $(\Theta_{n+1}\circ A_n\circ g_{n+1})(\zeta)\in\mathbb{C}^2\setminus 2r_{n+1}\overline{\mathbb{B}}$. By (7.6), by the fact that $R>2r_{n+1}$ and by (3.2) it follows that $f_{n+1}(\zeta)\in\mathbb{C}^2\setminus(2r_{n+1}-\varepsilon_n/2^{n+1})\mathbb{B}\subset\mathbb{C}^2\setminus(2r_{n+1}-r_1)\mathbb{B}\subset\mathbb{C}^2\setminus(r_{n+1}\overline{\mathbb{B}})$. Thus (ii) holds with n replaced by $n+1$.

By (6.11)

$$(\Theta_{n+1}\circ A_n\circ G_{n+1}\circ g_n)(U_n\setminus K_n')\subset\mathbb{C}^2\setminus(r_{n-1}+\theta_n+\varepsilon_n)\overline{\mathbb{B}}.$$

If $\zeta\in\Delta\setminus K_n$ then, by (7.3), $p_{n+1}(\zeta)\in p_{n+1}(\Delta)\setminus K_n'\subset U_n\setminus K_n'$ so

$$(\Theta_{n+1}\circ A_n\circ g_{n+1})(\Delta\setminus K_n)\subset\mathbb{C}^2\setminus(r_{n-1}+\theta_n+\varepsilon_n)\overline{\mathbb{B}}$$

and since $R_{n-1}+\theta_n+\varepsilon_n<R$ it follows by (7.6) that $f_{n+1}(\Delta\setminus K_n)\subset\mathbb{C}^2\setminus r_{n-1}\overline{\mathbb{B}}$, that is, (iii) is satisfied.

Finally, (6.8) implies that

$$(\Psi_{n+1}\circ\Theta_{n+1}\circ A_n)(\{|z|>S_{n+1}/2\})\ \text{misses}\ \Psi_{n+1}((2r_n+\varepsilon_n)\overline{\mathbb{B}}).$$

Since $2r_n+\varepsilon_n<R$, (7.6) implies that $2r_n\mathbb{B}\subset\Psi_{n+1}((2r_n+\varepsilon_n)\mathbb{B})$ so $A_{n+1}(\{|z|>S_{n+1}/2\})$ misses $2r_n\mathbb{B}$, that is, (6.1") holds with n replaced by $n+1$.

This completes the proof of the induction step.

Since the map φ is proper, (vii) and the fact that (3.1) holds for every n imply that $r_n\to+\infty$ as $n\to\infty$.

The proof of Theorem 1.1 is complete.

8. Remarks

Theorem 1.1 holds with $\mathbb{C}^2$ replaced by $\mathbb{C}^N,\ N\geq 2$:

Theorem 8.1 *Let $N\geq 2$. Given a discrete set $S\subset\Delta$ and a proper injection φ: $S\to\mathbb{C}^N$ there is a proper holomorphic embedding f: $\Delta\to\mathbb{C}^N$ that extends φ.*

If $N\geq 3$ then Theorem 8.1. can be proved by first constructing a proper holomorphic immersion f: $\Delta\to\mathbb{C}^2$ that interpolates and which is one to one off a discrete subset of Δ and then remove the selfintersection points by small perturbations. Alternatively, one can use the same proof as in the case $N=2$ with a slight modification: Let ι: $\mathbb{C}^2\to\mathbb{C}^N$ be the standard embedding $\iota(\zeta_1,\zeta_2)=(\zeta_1,\zeta_2,0,\cdots,0)$. In the proof we replace $f_n=A_n\circ g_n$ by $f_n=A_n\circ\iota\circ g_n$ where A_n is a holomorphic automorphism of $\mathbb{C}^N$ and g_n, as in the proof in the case $N=2$, is a one to one and regular holomorphic map from an open neighbourhood U_n of $\overline{\Delta}$ to $\mathbb{C}^2$ which, for even n is transverse to $\{(z,w)\colon\ |z| = S_n\}$ and satisfies $g_n^{-1}(\{|z| = S_n\}) = b\Delta$, and for odd n, is transverse to $\{(z,w)\colon\ |w| = T_n\}$ and satisfies $g_n^{-1}(\{|w| = T_n\}) = b\Delta$. Also, in the induction step, the maps Θ_{n+1} and Ψ_{n+1} are automorphisms of $\mathbb{C}^N$ and G_{n+1} is an automorphism of $\mathbb{C}^2$. In (6.1') and (6.1") we replace A_n by $A_n \circ \iota$.

We say that two proper holomorphic embeddedings $f_1, f_2\colon \Delta \to \mathbb{C}^N$ are $Aut(C^N)$-*equivalent* if there is an automorphism $\Psi\colon \mathbb{C}^N \to \mathbb{C}^N$ such that $f_2 = \Psi \circ f_1$.

Corollary 8.2 *Let $N \geq 2$. The set of $Aut(C^N)$-equivalence classes of proper holomorphic embeddedings of Δ into $\mathbb{C}^N$ is uncountable.*

Proof. [BFo] It is known [RR, Remark 5.2] that there is an uncountable family E of discrete injective sequences in $\mathbb{C}^N$ such that if $\{z_n,\ n \in \mathbb{N}\}$, $\{w_n,\ n \in \mathbb{N}\}$ are distinct elements of E then there is no automorphism Ψ of $\mathbb{C}^N$ such that $\Psi(z_n) = w_n$ $(n \in \mathbb{N})$. Let $\{\zeta_n\} \subset \Delta$ be an injective sequence, $\lim_{n\to\infty} |\zeta_n| = 1$, and let $\{z_n,\ n \in \mathbb{N}\}$, $\{w_n,\ n \in \mathbb{N}\}$ be distinct elements of E. By Theorem 8.1 there are proper holomorphic embeddings $f_1, f_2\colon \Delta \to \mathbb{C}^N$ such that $f_1(\zeta_j) = z_j,\ f_2(\zeta_j) = w_j$ $(j \in \mathbb{N})$. Every automorphism Ψ of $\mathbb{C}^N$ such that $f_2 = \Psi \circ f_1$ would have to satisfy $\Psi(z_n) = w_n$ $(n \in \mathbb{N})$ and there is no such Ψ. Thus, in this way, each element of E produces a proper holomorphic embedding of Δ into $\mathbb{C}^N$ and the embeddings associated with distinct elements of E are not $Aut(\mathbb{C}^N)$-equivalent. This completes the proof.

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