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1. The result

Let Δ be the open unit disc in $\mathbb{C}$. A holomorphic embedding $f: \Delta \to \mathbb{C}^3$ can grow arbitrarily fast near $b\Delta$. In fact, it is not difficult to see that given a sequence R_n of positive numbers that increases to 1 and a sequence C_n of positive numbers that increases to $+\infty$ there is a holomorphic map g: $\Delta \to \mathbb{C}^2$ such that $|g(\zeta)| \geq C_n$ if $R_n \leq |\zeta| < 1,\ n \in \mathbb{N}$. Taking ζ as the third component produces such an embedding f. Whether the same holds for holomorphic embeddings $f: \Delta \to \mathbb{C}^2$ is a much harder problem since already to find a proper holomorphic embedding $f: \Delta \to \mathbb{C}^2$ without additional properties is not easy [St, A, GS].

In the present note we show that a holomorphic embedding f: $\Delta \to \mathbb{C}^2$ can grow arbitrarily fast near $b\Delta$. We also show that a holomorphic embedding f: $\mathbb{C} \to \mathbb{C}^2$ can grow arbitrarily fast near infinity.

Theorem 1.1 *Let C_n be an increasing sequence of positive numbers converging to $+\infty$.*

(a) Given a strictly increasing sequence R_n of positive numbers converging to 1 there is a holomorphic embedding $f: \Delta \to \mathbb{C}^2$ such that $|f(\zeta)| \geq C_n$ $(R_n \leq |\zeta| < 1,\ n \in \mathbb{N})$.

(b) Given a strictly increasing sequence R_n of positive numbers converging to $+\infty$ there is a holomorphic embedding f: $\mathbb{C} \to \mathbb{C}^2$ such that $|f(\zeta)| \geq C_n$ $(|\zeta| \geq R_n, n \in \mathbb{N})$.

A *holomorphic embedding* is a holomorphic immersion which is one to one. Since C_n above increase to $+\infty$ the maps f in the theorem are proper. So in (a), $f(\Delta)$ is a (closed) complex submanifold of $\mathbb{C}^2$ which is, via the map f, biholomorphically equivalent to Δ, and in (b), $f(\mathbb{C})$ is a (closed) complex submanifold of $\mathbb{C}^2$ which is, via f, biholomorphically equivalent to $\mathbb{C}$.

In the proof we shall use alternating sequences of shear automorphisms of $\mathbb{C}^2$ of the form

$$\begin{aligned} \Theta_n(z,w) &= \big(z, w + L_n(z/\lambda_n)^{N_n}\big) \text{ if } n \text{ is odd} \\ \Theta_n(z,w) &= \big(z + L_n(w/\lambda_n)^{N_n}, w\big) \text{ if } n \text{ is even} \end{aligned} \tag{1.1}$$

where N_n is a sequence of positive integers and L_n, λ_n are sequences of positive numbers. Such sequences (with $\lambda_n = L_{n-1}$) were used in [GS] and later in [S, G1, G2, G3].

2. Proof of (b)

Part 1. We shall start with the embedding $g_1(\zeta) = (0, \zeta)$ and then apply the sequence Θ_n from (1.1). Given $n \in \mathbb{N},\ n \geq 2$, let $g_n = \Theta_n \circ \Theta_{n-1} \circ \cdots \circ \Theta_2 \circ g_1 = (P_n, Q_n)$. Clearly P_n, Q_n are polynomials, $P_{n+1} = P_n$ for even n, $Q_{n+1} = Q_n$ for odd n. Let S_n, T_n be the leading coefficients of P_n, Q_n so that $P_n = S_n p_n,\ Q_n = T_n q_n$ where p_n, q_n are polynomials with the leading coefficients 1. Put $N_1 = 1$. Clearly $\deg(p_n) = N_1 N_2 \cdots N_n$ for even n and $\deg(q_n) = N_1 N_2 \cdots N_n$ for odd n. We shall choose L_n, λ_n and N_n in such a way that our embedding f will be the limit, uniform on compacta in $\mathbb{C}$, of the sequence g_n.

Before we begin with the proof we describe the main point of a typical induction step. Suppose that n is odd. We pass from $(P_n, Q_n) = (S_n p_n, T_n q_n)$ to

$$\begin{aligned}(P_{n+1}, Q_{n+1}) &= \Big(P_n + L_{n+1}\Big(\frac{Q_n}{T_n R_n^{N_1\cdots N_n}}\Big)^{N_{n+1}}, Q_n\Big)\\ &= \Big(P_n + L_{n+1}\Big(\frac{q_n}{R_n^{N_1\cdots N_n}}\Big)^{N_{n+1}}, Q_n\Big).\end{aligned}$$

Suppose for a moment that $q_n(\zeta) = \zeta^{N_1\cdots N_n}$. Then

$$\Big|\frac{q_n(\zeta)}{R_n^{N_1\cdots N_n}}\Big| = 1 \ \text{ iff } \ |\zeta| = R_n.$$

However, our q is a monomial only if $n = 1$. We choose the constants in our process so that

$$\Big\{\zeta\colon \ \Big|\frac{q_n(\zeta)}{R_n^{N_1\cdots N_n}}\Big| = 1\Big\} \subset R_{n+1}\Delta \setminus R_{n-1}\overline{\Delta}.$$

This is achieved mainly with choosing L_{n+1} at each step large enough. The preceding inclusion implies that choosing N_{n+1} large enough, $P_{n+1} - P_n$ will be arbitrarily small on $R_{n-1}\overline{\Delta}$ (which we need for convergence and to achieve that the limit map is one to one and regular) and P_{n+1} will be arbitrarily large on $\mathbb{C} \setminus R_{n+1}\Delta$ (which we need to achieve the prescribed growth).

Part 2. It is enough to prove that if $R_n = 5^n$, and if C_n is an increasing sequence of positive numbers then there is a holomorphic embedding $f\colon \mathbb{C} \to \mathbb{C}^2$ such that $|f(\zeta)| \geq C_n$ $(|\zeta| \geq 2R_n,\ n \in \mathbb{N})$.

Assume for a moment that we have already constructed the sequences L_n, λ_n, N_n and a decreasing sequence $\varepsilon_n,\ 0 < \varepsilon_n < 1$, with the following propertiees:

$$\text{If } n \in \mathbb{N} \text{ is odd and } |q_n(\zeta)| = R_n^{N_1\cdots N_n} \text{ then } R_n/2 < |\zeta| < 2R_n, \tag{2.1$'$}$$

$$\text{if } n \in \mathbb{N} \text{ is even and } |p_n(\zeta)| = R_n^{N_1\cdots N_n} \text{ then } R_n/2 < |\zeta| < 2R_n, \tag{2.1$''$}$$

$$|Q_{n+1}(\zeta)| \geq C_n + 1 \text{ if } \ n \text{ is even and } |p_n(\zeta)| \geq R_n^{N_1\cdots N_n} \tag{2.2$'$}$$

$$|P_{n+1}(\zeta)| \geq C_n + 1 \text{ if } \ n \text{ is odd and } |q_n(\zeta)| \geq R_n^{N_1\cdots N_n} \tag{2.2$''$}$$

and

$$\left.\begin{array}{l}|P_{n+2}(\zeta) - P_{n+1}(\zeta)| < \dfrac{\varepsilon_{n+1}}{2^{n+1}} \ \ (|\zeta| \leq 2R_n,\ n \text{ even})\\ \text{- note that } Q_{n+2} = Q_{n+1} \text{ in this case}\end{array}\right\} \tag{2.3$'$}$$

and

$$\left.\begin{array}{l}|Q_{n+2}(\zeta) - Q_{n+1}(\zeta)| < \dfrac{\varepsilon_{n+1}}{2^{n+1}} \ \ (|\zeta| \leq 2R_n,\ n \text{ odd})\\ \text{- note that } P_{n+2} = P_{n+1} \text{ in this case}\end{array}\right\} \tag{2.3$''$}$$

and

$$\left.\begin{array}{l}\text{if } g \text{ is a holomorphic map on } R_{n-1}\Delta \text{ satisfying } |g-g_n|<\varepsilon_n \\ \text{on } R_{n-1}\Delta \text{ then } g \text{ is one to one and regular on } R_{n-2}\Delta.\end{array}\right\} \qquad (2.4)$$

By (2.3) g_n converges, uniformly on compacta in $\mathbb{C}$, to a holomorphic map $f = (P,Q)\colon \mathbb{C}\to\mathbb{C}^2$. Since (2.3) implies that

$$|g_{j+1}(\zeta)-g_j(\zeta)| < \frac{\varepsilon_j}{2^j} \;\; (|\zeta|\le R_{j-1},\; j\in\mathbb{N}),$$

it follows that $|f(\zeta)-g_n(\zeta)|<\varepsilon_n$ ($\zeta\in R_{n-1}\Delta$) which, by (2.4) implies that f is one to one and regular on $R_{n-2}\Delta$. This holds for every n, so f is one to one and regular on $\mathbb{C}$.

Suppose that n is even and that $\zeta\in(2R_{n+1}\overline{\Delta})\setminus 2R_n\Delta$. Since $\zeta\notin 2R_n\Delta$ (2.1) implies that $|p_n(\zeta)|>R_n^{N_1\cdots N_n}$ which, by (2.2) implies that $|Q_{n+1}(\zeta)|\ge C_n+1$. On the other hand, since $\zeta\in 2R_{n+1}\overline{\Delta}\subset 2R_m\overline{\Delta}$ ($m\ge n+1$) it follows by (2.3) that $Q_{j+1}=Q_j$ for odd j and $|Q_{j+1}(\zeta)-Q_j(\zeta)|<\varepsilon_j/2^j$ for even j, $j\ge n+1$, which implies that $|f(\zeta)|\ge|Q(\zeta)|\ge|Q_{n+1}(\zeta)|-\sum_{j\ge n+1}|Q_{j+1}(\zeta)-Q_j(\zeta)|\ge C_n+1-\sum_{j\in\mathbb{N}}2^{-j}=C_n$. Thus, $|f(\zeta)|\ge C_n$ ($\zeta\in(2R_{n+1}\overline{\Delta})\setminus 2R_n\Delta$) for even n. For odd n this is proved in the same way. Since C_n and R_n are increasing it follows that $|f(\zeta)|\ge C_n$ ($|\zeta|\ge 2R_n$).

It remains to prove the existence of the sequences Θ_n and ε_n with the above properties. To do this we need

Lemma 2.1 *Let $S_0>S>0$. Let p be a nonconstant polynomial and let q be a polynomial. Given ε, $0<\varepsilon<1$, there are T_0 and N_0 such that if $T\ge T_0$, $N\ge N_0$, and $S_1\ge S_0$, then*

$$q(\zeta)+T\Big(\frac{p(\zeta)}{S}\Big)^N = T\Big(\frac{S_1}{S}\Big)^N$$

implies that

$$(1-\varepsilon)S_1<|p(\zeta)|<(1+\varepsilon)S_1.$$

Proof of (b), Part 3. We now prove the existence of Θ_n and ε_n with the required properties.

Choose a decreasing sequence δ_n, $0<\delta_n<1/2$, such that

$$\prod_{n=1}^{\infty}(1-\delta_n)>\frac{1}{2},\;\; \prod_{n=1}^{\infty}(1+\delta_n)<2. \qquad (2.5)$$

We shall show that one can construct our Θ_n in such a way that, in addition to the properties above, our polynomials satisfy the following:

$$\left.\begin{array}{l}\text{if } R\ge R_n/2 \text{ then } R\prod_{j=1}^{n}(1-\delta_j)<|\zeta|<R\prod_{j=1}^{n}(1+\delta_j) \text{ provided that} \\ \text{either } n \text{ is even and } |p_n(\zeta)|=R^{N_1\cdots N_n} \text{ or } n \text{ is odd and } |q_n(\zeta)|=R^{N_1\cdots N_n}.\end{array}\right\} \qquad (2.6)$$

Note that this, together with (2.5), implies (2.1).

To begin the induction, put $g_1(\zeta) = (0, \zeta),\ R_0 = 1,\ R_{-1} = 1/5$. Choose $\varepsilon_1,\ 0 < \varepsilon_1 < 1/2$, such that (2.4) holds for $n = 1$. Let $\Theta_2(z, w) = (z + L_2(w/R_1)^{N_2}, w)$ where L_2 is so large that (2.2) holds for $n = 1$ and for every $N_2 \in \mathbb{N}$ and then choose N_2 so large that (2.3″) holds for $n = 0$. Note that (2.6) holds for $n = 1$.

Suppose that n is odd and that we have already constructed ε_{n-1} and $(P_n, Q_n) = (S_n p_n, T_n q_n)$ such that (2.6) holds. Choose $\varepsilon_n,\ 0 < \varepsilon_n < \varepsilon_{n-1}$ so small that (2.4) holds. We shall choose L_{n+1} and N_{n+1} such that Θ_{n+1} given by

$$\Theta_{n+1}(z, w) = \Big(z + L_{n+1}\Big(\frac{w}{T_n R_n^{N_1 \cdots N_n}}\Big)^{N_{n+1}}, w\Big)$$

will have the required properties. We have

$$Q_{n+1} = Q_n,\ P_{n+1} = P_n + L_{n+1}\left(\frac{q_n}{R_n^{N_1 \cdots N_n}}\right)^{N_{n+1}}.$$

By (2.6), $|q_n(\zeta)/R_n^{N_1 \cdots N_n}| = 1$ implies that $R_n \prod_{j=1}^n (1 - \delta_j) < |\zeta|$. Since, by (2.5), $2R_{n-1} < R_n/2 < R_n \prod_{j=1}^n (1 - \delta_j)$ it follows that there is an $\eta,\ 0 < \eta < 1$, such that

$$\left|\frac{q_n(\zeta)}{R_n^{N_1 \cdots N_n}}\right| \le \eta \quad (|\zeta| \le 2R_{n-1}). \tag{2.7}$$

Since $R_n < R_{n+1}/2$ Lemma 2.1 provides $A_0 < \infty$ and $M_0 \in \mathbb{N}$, $M_0 \ge 2$, such that whenever $L_{n+1} \ge A_0$ and $N_{n+1} \ge M_0$, for every $R \ge R_{n+1}/2$ the set

$$\begin{aligned}\mathcal{S}(R, L_{n+1}, N_{n+1}) &= \left\{\zeta : \left|P_n(\zeta) + L_{n+1}\left(\frac{q_n(\zeta)}{R_n^{N_1 \cdots N_n}}\right)^{N_{n+1}}\right| = L_{n+1}\left(\frac{R^{N_1 \cdots N_n}}{R_n^{N_1 \cdots N_n}}\right)^{N_{n+1}}\right\} \\ &= \{\zeta : |p_{n+1}(\zeta)| = R^{N_1 \cdots N_{n+1}}\}\end{aligned}$$

is contained in the set

$$\{\zeta : ((1 - \delta_{n+1})R)^{N_1 \cdots N_n} < |q_n(\zeta)| < ((1 + \delta_{n+1})R)^{N_1 \cdots N_n}\}$$

which, by (2.6), is contained in the set $\{\zeta : R\prod_{j=0}^{n+1}(1 - \delta_j) < |\zeta| < R\prod_{j=0}^{n+1}(1 + \delta_j)\}$. Thus (2.6) with n replaced by $n + 1$, will hold whenever $L_{n+1} \ge A_0$ and $N_{n+1} \ge M_0$.

Passing to a larger L_{n+1} if necessary we may assume that

$$L_{n+1}\left|\frac{q_n(\zeta)}{R_n^{N_1 \cdots N_n}}\right|^{N_{n+1}} - |P_n(\zeta)| > C_{n+1}$$

whenever $N_{n+1} \ge 2$ and $|q_n(\zeta)| \ge R_n^{N_1 \cdots N_n}$ which implies (2.2). Choosing N_{n+1} so large that $L_{n+1}\eta^{N_{n+1}} < \varepsilon_n$ will, together with (2.7), imply (2.3). This completes the proof of the induction step for odd n. The proof for even n is similar. The proof of (b) is complete.

Proof of Lemma 2.1. The equality $q + T(p(\zeta)/S)^N = T(S_1/S)^N$ implies that

$$q/T + (p/S)^N = (S_1/S)^N. \tag{2.8}$$

Choose N_0 so large that $(S_0/S)^{N_0} \geq 2$. Let $S_1 \geq S_0$. Put $\mathcal{T} = \{\zeta : |p(\zeta)/S| \leq 1\}$. Since p is not a constant $\mathcal{T}$ is a compact set. Choose T_0 so large that $|q/T_0| \leq 1/2$ on $\mathcal{T}$. Let $N \geq N_0$. Since $(S_1/S)^N \geq (S_0/S)^{N_0} \geq 2$ it follows that (2.8) has no solution in $\mathcal{T}$ whenever $T \geq T_0$ and $N \geq N_0$.

If $T \geq T_0$ and $\zeta \in \mathbb{C} \setminus \mathcal{T}$ satisfies (2.8) then

$$(p(\zeta)/S)^N \big(1 + \frac{q(\zeta)/T}{(p(\zeta)/S)^N}\big) = (S_1/S)^N. \tag{2.9}$$

Passing to a larger N_0 we may assume that $\deg(q) < \deg(p^{N_0})$. Fix such an N_0. One can choose T_0 so large that

$$\Big|\frac{q(\zeta)/T}{(p(\zeta)/S)^N}\Big| < \omega \quad \text{whenever} \quad T \geq T_0, \; N \geq N_0 \text{ and } \; \zeta \in \mathbb{C} \setminus \mathcal{T}$$

where $\omega > 0$ is so small that $1 - \varepsilon < (1+\omega)^{-1/N_0}$, $1 + \varepsilon > (1-\omega)^{-1/N_0}$. Now (2.9) implies that

$$|(p(\zeta)/S)^N|(1-\omega) \leq (S_1/S)^N \leq |(p(\zeta)/S)^N|(1+\omega)$$

so

$$(S_1/S)(1+\omega)^{-1/N} \leq |p(\zeta)/S| \leq (S_1/S)(1-\omega)^{-1/N}.$$

It follows that $(1-\varepsilon)S_1 \leq |p(\zeta)| \leq (1+\varepsilon)S_1$. This completes the proof.

3. Proof of (a)

We shall need the following lemma.

Lemma 3.1 *Let p, q be polynomials. Assume that p, as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is transverse to $b\Delta$ and assume that $p^{-1}(b\Delta) = \{\rho_p(\zeta)\zeta\colon \zeta \in b\Delta\}$ where ρ_p is a smooth positive function on $b\Delta$. Given $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < t < \delta$ and $N \in \mathbb{N}$, the polynomial $h = p^N + tq$ is transverse to $b\Delta$ and $h^{-1}(b\Delta) = \{\rho_h(\zeta)\zeta\colon \zeta \in b\Delta\}$ where $\|\rho_h - \rho_p\|_{\mathcal{C}^1(b\Delta)} < \varepsilon$.*

Lemma 3.1 follows from [G2, Lemma 5.1].

Proof of (a), Part 1 We observe first that given R_n and C_n as in the theorem it is enough to find f such that for a subsequence $m(n)$ of $\mathbb{N}$, $n(1) = 1$, we have $|f(\zeta)| \geq C_{n(m)+1}$ $(R_{n(m)} \leq |\zeta| < 1)$.

We shall construct our L_m, λ_m and N_m as in the proof of (b) and will obtain our map f as the limit of the sequence $f_m = g_m \circ \varphi_m$ where $\Omega_m = \{\zeta \in \mathbb{C}\colon |p_m(\zeta)| < 1\}$ (m even), $\Omega_m = \{\zeta \in \mathbb{C}\colon |q_m(\zeta)| < 1\}$ (m odd) and $\varphi_m\colon \Delta \to \Omega_m$ is the conformal map satisfying $\varphi_m(0) = 0$, $\varphi_m'(0) > 0$. Each $b\Omega_m$, $m \in \mathbb{N}$, will be of the form $\{\rho_m(\zeta)\zeta\colon \zeta \in b\Delta\}$ where ρ_m is a sequence of smooth positive functions converging in $\mathcal{C}^1(b\Delta)$ to a positive function ρ such that $b\Omega = \{\rho(\zeta)\zeta\colon \zeta \in b\Delta\}$. In fact, we shall construct g_m (and φ_m) in such a way that g_m will converge, uniformly on compacta in Ω, the limit of the sequence Ω_n, to a holomorphic embedding g of Ω to $\mathbb{C}^2$, and φ_m will converge, uniformly on Δ to φ, the conformal map from Δ to Ω, satisfying $\varphi(0)=0,\ \varphi'(0)>0$, so that $f=g\circ\varphi$ will have the required growth properties.

Before we begin the proof we describe the main points of a typical induction step. Suppose that m is even. In the induction step we pass from $(P_m,Q_m)=(S_mp_m,T_mq_m)$ to (P_{m+1},Q_{m+1}). The set Ω_m will be a small perturbation of Δ, the map p_m will be transverse to $b\Delta$ and $b\Omega_m=\{\zeta\colon\ |p_m(\zeta)|=1\}$ will be a small perturbation of $b\Delta$. Let φ_m be as above and let $D_{mn}=\varphi_m(R_n\Delta)$ $(m,n\in\mathbb{N})$. We have also constants $t_m,\ 0<t_m<1$, and $\varepsilon_m>0$ such that if $\Sigma_m=\{\zeta\in\mathbb{C}\colon\ |p_m(\zeta)|<t_m^{N_1\cdots N_m}\}$ then we have $D_{m-1,n(m-1)}\subset\Sigma_m\subset\Sigma_m+2\varepsilon_m\Delta\subset D_{m,n(m)}$. Let $W_m=b\Omega_m+2^{-m}\Delta$. Choose $t_{m+1},\ t_m<t_{m+1}<1$, and $\eta>0$ such that $t_{m+1}^{N_1\cdots N_m}(1+\eta)<1$ and that the set $\mathcal{S}_m=\{\zeta\colon\ t_{m+1}^{N_1\cdots N_m}(1-\eta)<|p_m(\zeta)|<t_{m+1}^{N_1\cdots N_m}(1+\eta)\}$ is contained in W_m. Now choose $n(m+1)>n(m)$ and $\varepsilon_{m+1},\ 0<\varepsilon_{m+1}<\varepsilon_m$, such that $D_{m,n(m)}\subset\subset\big\{\zeta\colon\ |p_m(\zeta)|<t_{m+1}^{N_1\cdots N_m}(1-\eta)\big\}$, and $\{\zeta\colon\ |p_m(\zeta)|<t_{m+1}^{N_1\cdots N_m}(1+\eta)\}+3\varepsilon_{m+1}\Delta\subset D_{m,n(m+1)}$. Now pass from P_m to $P_{m+1}=P_m$ and from Q_m to

$$\begin{aligned}
Q_{m+1}&=Q_m+L_{m+1}\Big(\frac{P_m}{S_mt_m^{N_1\cdots N_m}}\Big)^{N_{m+1}}\\
&=Q_m+L_{m+1}\Big(\frac{p_m}{t_m^{N_1\cdots N_m}}\Big)^{N_{m+1}}\\
&=\frac{L_{m+1}}{t_m^{N_1\cdots N_mN_{m+1}}}\Big[\frac{t_m^{N_1\cdots N_{m+1}}}{L_{m+1}}Q_m+p_m^{N_{m+1}}\Big]\\
&=\frac{L_{m+1}}{t_m^{N_1\cdots N_mN_{m+1}}}q_{m+1}\ .
\end{aligned}$$

By Lemma 3.1 $b\Omega_{m+1}=\{\zeta\colon\ |q_{m+1}(\zeta)|=1\}=\{\rho_{m+1}(\zeta)\zeta\colon\ \zeta\in b\Delta\}$ is an arbitrarily small $\mathcal{C}^1$-perturbation of $b\Omega_m$ provided that L_{m+1} is large enough, independently of N_{m+1}; in particular, this will imply that $b\Omega_{m+1}\subset W_m$ and that $|\varphi_{m+1}-\varphi_m|<\varepsilon_{m+1}$ on Δ provided that L_{m+1} is large enough [P, p.286] and thus $\{\zeta\colon\ |p_m(\zeta)|<t_{m+1}^{N_1\cdots N_m}(1+\eta)\}+2\varepsilon_{m+1}\Delta\subset D_{m+1,n(m+1)}$. If in all subsequent steps the constants L_{n+1} will be chosen large enough $\varphi-\varphi_m$ will be very small on Δ. By choosing L_{m+1} sufficiently large we get also that $|Q_{m+1}(\zeta)|\geq C_{n(m+1)}+1$ outside Σ_m which will be necessary to achieve the prescribed growth.

After choosing L_{m+1} we choose N_{m+1} so large that $Q_{m+1}-Q_m$ will be very small on Σ_{m-1} which we need for the convergence and for the fact that the limit map f is regular and one to one.

Finally, we pass to an even larger N_{m+1} to achieve that $\{\zeta\colon\ |q_{m+1}(\zeta)|=t_{m+1}^{N_1\cdots N_{m+1}}\}$ is contained in $\mathcal{S}_m$ so that if $\Sigma_{m+1}=\{\zeta\colon\ |q_{m+1}(\zeta)|<t_{m+1}^{N_1\cdots N_{m+1}}\}$ then $D_{m,n(m)}\subset\Sigma_{m+1}\subset\Sigma_{m+1}+2\varepsilon_{m+1}\Delta\subset D_{m+1,n(m+1)}$ and $b\Sigma_{m+1}\subset W_m$.

Part 2. Assume for a moment that we have already constructed L_m,λ_m and N_m. Suppose that there are an increasing sequence $t_m,\ 0<t_m<1$, converging to 1, a decreasing sequence ε_n, $0 < \varepsilon_n < 1$, and a subsequence $n(m) \in \mathbb{N}$, $n(1) = 1$, such that if

$$\Sigma_m = \{\zeta \in \mathbb{C}: |p_m(\zeta)| < t_m^{N_1 \cdots N_m}\} \ (m \text{ even}), \ \Sigma_m = \{\zeta \in \mathbb{C}: |q_m(\zeta)| < t_m^{N_1 \cdots N_m}\} \ (m \text{ odd})$$

then

(i') $|Q_{m+1}(\zeta)| \geq C_{n(m+1)} + 1$ $(\zeta \in \mathbb{C} \setminus \Sigma_m)$ if m is even

(i") $|P_{m+1}(\zeta)| \geq C_{n(m+1)} + 1$ $(\zeta \in \mathbb{C} \setminus \Sigma_m)$ if m is odd

(ii') $|Q_{m+1} - Q_m| < \varepsilon_m/2^m$ on Σ_{m-1} if m is even (recall that $P_{m+1} = P_m$ in this case)

(ii") $|P_{m+1} - P_m| < \varepsilon_m/2^m$ on Σ_{m-1} if m is odd (recall that $Q_{m+1} = Q_m$ in this case)

(iii) if $h: \Sigma_{m-1} \to \mathbb{C}^2$ is a holomorphic map such that $|h - g_m| < \varepsilon_m$ then h is one to one and regular on Σ_{m-2}

(iv) the map $\zeta \mapsto p_m(\zeta)$ is transverse to $b\Delta$ if m is even and the map $\zeta \mapsto q_m(\zeta)$ is transverse to $b\Delta$ if m is odd

(v) $\Sigma_m + 2\varepsilon_m\Delta \subset D_{m,n(m)} \subset \Sigma_{m+1}$

(vi) $|\varphi_{m+1} - \varphi_m| < \varepsilon_m/2^m$ on Δ

(vii) $b\Sigma_{m+1} \subset b\Omega_m + 2^{-m}\Delta$

(viii) $b\Omega_{m+1} \subset b\Omega_m + 2^{-m}\Delta$

(ix) the domains $\Omega_m = \{\rho_m(\zeta)\zeta: \zeta \in b\Delta$ converge to $\Omega = \{\rho(\zeta)\zeta: \zeta \in b\Delta\}$ in the sense that ρ_m converge to ρ in $\mathcal{C}^1(b\Delta)$.

By (v), $\Sigma_m \subset\subset \Sigma_{m+1}$ $(m \in \mathbb{N})$ and by (vii), (viii),and (ix), $\cup_{m=1}^{\infty}\Sigma_m = \Omega$. By (ii), g_m converge uniformly on compacta in Ω to a holomorphic map $g = (P, Q)$. If $\zeta \in \Sigma_{m-1}$ then by (ii), $|g(\zeta) - g_m(\zeta)| \leq \sum_{j=m}^{\infty} |g_{j+1}(\zeta) - g_j(\zeta)| \leq \varepsilon_m/2^m + \varepsilon_{m+1}/2^{m+1} + \cdots < \varepsilon_m$ and consequently by (iii), g is one to one and regular on Σ_{m-2}. Since this holds for every m it follows that g is one to one and regular on Ω. By (ix) the maps φ_n converge, uniformly on Δ to the conformal map $\varphi: \Delta \to \Omega$ such that $\varphi(0) = 0$, $\varphi'(0) > 0$ [P, p.286].

Let $f = g \circ \varphi$. The map $f: \Delta \to \mathbb{C}^2$ is a one to one, regular holomorphic map. We show that it has the required growth properties. Suppose that $\zeta \in \Sigma_{m+1} \setminus \Sigma_m$. If m is even then (i') implies that $|Q_{m+1}(\zeta)| \geq C_{n(m+1)} + 1$. Since $\zeta \in \Sigma_{m+1}$ (ii') implies that either $Q_{m+j+1}(\zeta) - Q_{m+j}(\zeta) = 0$ when $j \in \mathbb{N}$ is odd or $|Q_{m+j+1}(\zeta) - Q_{m+j}(\zeta)| < \varepsilon_{m+j}/2^{m+j}$ when $j \in \mathbb{N}$ is even. Thus, $|Q(\zeta)| \geq |Q_{m+1}(\zeta)| - \sum_{j=1}^{\infty} |Q_{m+j+1}(\zeta) - Q_{m+j}(\zeta)| \geq |Q_{m+1}(\zeta)| - \sum_{j=1}^{\infty} \varepsilon_{m+j}/2^j \geq C_{n(m+1)} + 1 - 1 = C_{n(m+1)}$. Since the sequence C_j is increasing it follows that $|Q(\zeta)| \geq C_{n(m+1)}$ $(\zeta \in \Omega \setminus \Sigma_m)$. In the same way, using (ii"), we get $|P(\zeta)| \geq C_{n(m+1)}$ $(\zeta \in \Omega \setminus \Sigma_m,\ m$ odd$)$. Thus, $|g(\zeta)| \geq C_{n(m+1)}$ $(\zeta \in \Omega \setminus \Sigma_m;\ m \in \mathbb{N})$.

Suppose that $\zeta \in \Delta \setminus R_{n(m)}\Delta$. Then $\varphi_m(\zeta) \notin D_{m,n(m)}$. By (vi), $|\varphi_{m+j+1}(\zeta) - \varphi_{m+j}(\zeta)| < \varepsilon_{m+j}/2^{m+j}$ $(j \geq 0)$ so $|\varphi(\zeta) - \varphi_m(\zeta)| \leq \sum_{j=0}^{\infty} |\varphi_{m+j+1}(\zeta) - \varphi_{m+j}(\zeta)| \leq \varepsilon_m/2^m + \varepsilon_{m+1}/2^{m+1} + \cdots < \varepsilon_m$ which, by (v), implies that $\varphi(\zeta) \notin \Sigma_m + \varepsilon_m\Delta$. The preceding discussion implies that $|f(\zeta)| = g(\varphi(\zeta)) \geq C_{n(m+1)}$.

Part 3. It remains to prove the existence of $\Theta_m, t_m, \varepsilon_m$ and $n(m)$ with the properties above. To start the induction, put $g_1(\zeta) = (P_1(\zeta), Q_1(\zeta) = (0, \zeta)$, $\Sigma_{-1} = \Sigma_0 = \emptyset$, $n(1) = 1$, and $t_1 = R_1/2$ so that $\Sigma_1 = (R_1/2)\Delta$. Clearly $\Omega_1 = \Delta$, $\varphi_1 = \text{id}$, $D_{11} = R_1\Delta$. Choose ε_1, $0 < \varepsilon_1 < 1$ so that the left inclusion in (v) for $m = 1$ holds. Obviously (iii) and (iv) are satisfied for m=1.

Suppose that m is even and that we have already constructed $\Theta_1, \cdots, \Theta_m$, t_1, $\cdots$, t_m, ε_{m-1}, and $n(m)$such that

$$\overline{\Sigma_m} \subset D_{m,n(m)} \tag{3.1}$$

and such that (iv) holds and (v) holds for $m-1$ in the place of m. Choose ε_m, $0<\varepsilon_m<\varepsilon_{m-1}$, so small that (iii) and the left inclusion in (v) hold. We shall choose L_{m+1} and N_{m+1} such that Θ_{m+1} given by

$$\Theta_{m+1}(z,w) = \Big(z, w + L_{m+1}\Big(\frac{z}{S_m t_m^{N_1\cdots N_m}}\Big)^{N_{m+1}}\Big)$$

will have the required properties.

Let $W_m = b\Omega_m + 2^{-m}\Delta$. Since $\overline{D_{m,n(m)}}$ is a compact subset of Ω_m we can choose $n(m+1) > n(m)$, t_{m+1}, $t_m < t_{m+1} < 1$, and $\eta > 0$ and pass to a smaller ε_m if necessary such that $t_{m+1}^{N_1\cdots N_m}(1+\eta) < 1$, that

$$D_{m,n(m)} \subset \{\zeta\colon |p_m(\zeta)| < (1-\eta)t_{m+1}^{N_1\cdots N_m}\}, \tag{3.2}$$

that

$$\{\zeta\colon |p_m(\zeta)| < (1+\eta)t_{m+1}^{N_1\cdots N_m}\} + 2\varepsilon_m\Delta \subset D_{m,n(m+1)}, \tag{3.3}$$

and that the set $\mathcal{S}_m = \{\zeta\colon\ t_{m+1}^{N_1\cdots N_m}(1-\eta) < |p_m(\zeta)| < t_{m+1}^{N_1\cdots N_m}(1+\eta)\}$ is contained in W_m. We have

$$\begin{aligned} Q_{m+1} &= Q_m + L_{m+1}\Big(\frac{p_m}{t_m^{N_1\cdots N_m}}\Big)^{N_{m+1}} \\ &= \frac{L_{m+1}}{t_m^{N_1\cdots N_m N_{m+1}}}\Big[\frac{t_m^{N_1\cdots N_{m+1}}}{L_{m+1}} Q_m + p_m^{N_{m+1}}\Big] \end{aligned}$$

so that $q_{m+1} = p_m^{N_{m+1}} + (t_m^{N_1\cdots N_{m+1}}/L_{m+1})Q_m$ and $T_{m+1} = L_{m+1}/t_m^{N_1\cdots N_{m+1}}$. By (iv) and by Lemma 3.1 $b\Omega_{m+1} = \{\zeta\colon\ |q_{m+1}(\zeta)| = 1\} = \{\rho_{m+1}(\zeta)\zeta\colon \zeta\in b\Delta\}$ will be arbitrarily small $\mathcal{C}^1$-perturbation of $b\Omega_m = \{\zeta\colon\ |p_m(\zeta)| = 1\} = \{\rho_m(\zeta)\zeta\colon\ \zeta\in b\Delta\}$ independently of N_{m+1} provided that L_{m+1} is sufficiently large; in particular, $b\Omega_{m+1} \subset b\Omega_m + 2^{-m}\Delta$ so that (viii) is satisfied. Moreover, (iv) will hold with m replaced by $m+1$. Choose L_{m+1} so large that (vi) and (vii) hold for every N_{m+1}. Passing to an even larger L_{m+1} if necessary we may assume that (i') holds whenever $N_{m+1} \geq 2$.

By (v) with m replaced by $m-1$, $\overline{\Sigma_{m-1}}$ is a compact subset of Σ_m. It follows that there is a t'_m, $0 < t'_m < t_m$, such that $\overline{\Sigma_{m-1}} \subset \{\zeta\colon |p_m(\zeta)| \leq (t'_m)^{N_1\cdots N_m}\}$ which implies that

$$\Big|\frac{p_m(\zeta)}{t_m^{N_1\cdots N_m}}\Big| \leq \Big(\frac{t'_m}{t_m}\Big)^{N_1\cdots N_m} \quad (\zeta\in\overline{\Sigma_{m-1}}).$$

Choosing N_{m+1} so large that

$$L_{m+1}\Big(\frac{t'_m}{t_m}\Big)^{N_1\cdots N_m N_{m+1}} < \frac{\varepsilon_m}{2^m}.$$

implies (ii'). Now, $|q_{m+1}(\zeta)| = t_{m+1}^{N_1\cdots N_{m+1}}$ iff

$$\left| \frac{p_m(\zeta)^{N_{m+1}}}{t_{m+1}^{N_1\cdots N_{m+1}}} + \left(\frac{t_m}{t_{m+1}}\right)^{N_1\cdots N_{m+1}} \frac{Q_m(\zeta)}{L_{m+1}} \right| = 1. \tag{3.4}$$

It is easy to see that there is a compact set K such that all solutions of (3.4) belong to K for each $N_{m+1} \geq 2$. There is a constant $\gamma < \infty$ such that $|Q_m(\zeta)|/L_{m+1} < \gamma$ $(\zeta \in K)$. Passing to a larger M_{n+1} we may assume that

$$1 - \eta < (1-\gamma)^{1/N_{m+1}}, \quad 1 + \eta < (1+\gamma)^{1/N_{m+1}}. \tag{3.5}$$

Since every solution ζ of (3.4) satisfies $1-\gamma < |p_m(\zeta)/t_m^{N_1\cdots N_m}|^{N_{m+1}} < 1+\gamma$, (3.5) implies that $(1-\eta)t_m^{N_1\cdots N_m} < |p_m(\zeta)| < (1+\eta)t_m^{N_1\cdots N_m}$. Thus, $\{\zeta\colon |q_{m+1}(\zeta)| = t_{m+1}^{N_1\cdots N_{m+1}}\} \subset \mathcal{S}_m$.

Since $q_{m+1}(0) = p_m(0) = 0$ it follows that $D_{m,n(m)}$, a connected set containing 0 and contained in $\{\zeta\colon |p_m(\zeta)| < (1-\eta)t_m^{N_1\cdots N_m}\}$ is contained in the component of Σ_{m+1} containing 0. In particular, $D_{m,n(m)} \subset \Sigma_{m+1}$ so the right inclusion in (v) holds. Further, the unbounded component of $\mathbb{C}\setminus\overline{\Sigma}_{m+1}$ has its boundary in $\{\zeta\colon |p_m(\zeta)| < (1+\eta)t_{m+1}^{N_1\cdots N_m}\}$, whose $2\varepsilon_m$-neighbourhood is contained in $D_{m,n(m+1)}$, a Jordan domain. It follows that $\Sigma_{m+1} + 2\varepsilon_m\Delta \subset D_{m,n(m+1)}$ which, by (vi), gives $\Sigma_{m+1} + \varepsilon_m\Delta \subset D_{m+1,n(m+1)}$ so (3.1) holds for $m+1$ in place of m. This completes the proof of the induction step when m is even. The proof for odd m is similar, with the roles of the coordinates interchanged.

It is clear that at each step ρ_{m+1} can be made so small $\mathcal{C}^1$-perturbation of ρ_m that (ix) is satisfied. The proof of (a) is complete.

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