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**Preprint series, Vol. 38 (2000), 697**

REGULAR HOMOMORPHISMS  
AND REGULAR MAPS

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ISSN 1318-4865

May 10, 2000

Ljubljana, May 10, 2000

# Regular Homomorphisms and Regular Maps

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## Abstract

Regular homomorphisms of oriented maps essentially arise from a factorization by a subgroup of automorphisms. This kind of map homomorphisms is studied in detail, and generalized to the case when the induced homomorphism of the underlying graphs is not valency preserving. Reconstruction is treated by means of voltage assignments on angles, a natural extension of the common assignments on darts. Lifting and projecting groups of automorphisms along regular homomorphisms is studied in some detail. Finally, the split-extension structure of lifted groups is analyzed.

## 1 Introduction

Regular coverings (well-known from the basic topology course) have their natural analogue in the theory of maps on surfaces – the concept of a *regular homomorphism* of maps. It is surprising that, in spite of their vital importance in construction of highly symmetrical maps [2, 9, 16, 19], no systematic treatment of regular homomorphisms has been undertaken so far.

As with regular coverings we have several equivalent possibilities how to define a regular homomorphism: either in terms of the action of the fibre transformation

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<sup>1</sup>Supported in part by “Ministrstvo za znanost in tehnologijo Slovenije”, proj. no. J1-0496-99, and by “Rektorjev sklad Univerze v Ljubljani”.

<sup>2</sup>Supported in part by VEGA, grant no. 1/6132/99.

<sup>3</sup>Supported in part by VEGA, grant no. 1/6132/99.

group, in terms of factorization by a group of automorphisms, or in terms of certain subgroups in the monodromy group of the covering map. For the purpose of this paper, the first approach seems to be most convenient.

We study regular homomorphism from several aspects. Our main concern, however, is the problem of lifting and projecting map automorphisms [1, 2, 9, 14, 13, 18]. This problem has to be solved in virtually every particular case when a new regular map is constructed from an old one. In this context, it is interesting to note that a homomorphism defined on a regular map is necessarily regular. This result does not have any direct topological counterpart.

In order to be able to provide a convenient combinatorial description of all regular homomorphisms we need an extension of the classical theory of voltage graphs [8] to voltage assignments on angles of a map. This approach proves to be quite versatile both to treat the lifting problem as well as to analyze the structure of lifted groups. As the lifted group is an extension of the fibre transformation group with no essential restrictions, it may be difficult to analyze its structure in general. We show that the case when the extension splits can be well understood. In particular, we characterize those homomorphisms  $\tilde{M} \rightarrow M$  between regular maps for which  $\text{Aut } \tilde{M}$  is a split extension of the fibre transformation group by  $\text{Aut } M$ .

## 2 Preliminaries

Topologically, an oriented map is a finite connected graph (possibly with semiedges) cellularly embedded into an orientable surface with a prescribed global orientation. A map homomorphism is an orientation preserving branched covering of the supporting surfaces with possible branching points at face-centres, vertices, and free ends of semiedges of the embedded graph. A map automorphism is an orientation preserving self-homeomorphism of the surface leaving the embedding invariant.

For technical reasons, however, it is more convenient to deal with maps within a combinatorial framework. Formally, an *oriented map*, or briefly a *map*, is a triple  $M = (D; R, L)$  where  $D$  is a nonempty finite set of *darts*,  $R$  and  $L$  are two permutations on  $D$  such that  $L^2 = 1$ , and the group  $\text{Mon } M = \langle R, L \rangle$ , the *monodromy group* of  $M$ , acts transitively on  $D$ . The *underlying graph* is obtained by taking the orbits of  $R$  and  $L$  as the set of vertices and the set of edges, respectively, with incidence arising from their nonempty intersection. The permutation  $L$  is also called the *arc-reversing involution* and  $R$  is the *rotation* which cyclically permutes the incident edges around vertices in their natural order determined by the chosen global orientation of the supporting surface. Words over the alphabet  $\{R, R^{-1}, L\}$  correspond to walks in the graph, and the assumption that the monodromy group is transitive is nothing but the requirement that the graph be connected.

Let  $M = (D; R, L)$  and  $M' = (D'; R', L')$  be maps. A *map homomorphism*  $\varphi : M \rightarrow M'$  is a function  $\varphi : D \rightarrow D'$  such that  $\varphi R = R'\varphi$  and  $\varphi L = L'\varphi$ . The latter implies that  $\varphi$  induces a group epimorphism  $\varphi_* : \text{Mon } M \rightarrow \text{Mon } M'$  defined by  $R \rightarrow R'$  and  $L \rightarrow L'$ . The notion of a *map automorphism* is self-explanatory. The transitivity of  $\text{Mon } M$  implies that every map automorphism is fully determined by the image of a single dart, and so the *map automorphism group*  $\text{Aut } M$  acts semiregularly on  $D$ . We call a map  $M$  *regular* whenever  $\text{Aut } M$  acts transitively, and hence regularly, on the set of darts of  $M$ . Thus, regular maps are those which enjoy the highest possible degree of symmetry.

It is well-known that  $\text{Aut } M$  is equal to the centralizer of  $\text{Mon } M$  in the symmetric group  $\text{Sym } D$ , and abstractly isomorphic to  $N(\text{Mon } M)_x / (\text{Mon } M)_x$ , where  $(\text{Mon } M)_x$  denotes the stabilizer of a dart  $x$  of  $M$ .

We call two map homomorphisms  $\varphi : M \rightarrow N$  and  $\varphi' : M' \rightarrow N'$  *equivalent* if there exist isomorphisms  $\kappa : M \rightarrow M'$  and  $\lambda : N \rightarrow N'$  such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\kappa} & M' \\ \varphi \downarrow & & \downarrow \varphi' \\ N & \xrightarrow{\lambda} & N' \end{array}$$

commutes.

For a full account on combinatorial representation of maps and their symmetries we refer the reader to Jones and Singerman [12].

### 3 Regular homomorphisms of maps

Let  $\varphi : M \rightarrow M'$  be a homomorphism of maps where  $M = (D; R, L)$  and  $M' = (D'; R', L')$ . Define the *fibre transformation group*  $\text{FT}(\varphi)$  of  $\varphi$  to be the group of all automorphisms  $\tau$  of  $M$  for which the diagram

$$\begin{array}{ccc} M & \xrightarrow{\tau} & M \\ \varphi \downarrow & & \downarrow \varphi \\ M' & \xrightarrow{\text{id}} & M' \end{array}$$

commutes. We will say that  $\varphi : M \rightarrow M'$  is a *regular homomorphism* if  $\text{FT}(\varphi)$  acts transitively on each fibre. Due to the transitivity of the action of the monodromy group of  $M$ , it is sufficient to verify that  $\text{FT}(\varphi)$  is transitive on a single fibre.

There is a convenient method of constructing regular homomorphisms from  $M$ , namely factorization by a group of automorphisms of  $M$ . Take a subgroup  $G \leq \text{Aut } M$  and let  $\bar{D}$  be the set of orbits of  $G$  on  $D$ . Denoting by  $[x]$  the orbit of  $x \in D$

under the action of  $G$ , define the quotient map  $M/G = \bar{M} = (\bar{D}; \bar{R}, \bar{L})$  with dart-set  $\bar{D}$  by setting  $\bar{R}([x]) = [R(x)]$  and  $\bar{L}([x]) = [L(x)]$ . The action of  $\bar{R}$  and  $\bar{L}$  is correctly defined. Indeed, if  $[x] = [y]$ , then  $\gamma(x) = y$  for some  $\gamma \in G$ . It follows that  $R(y) = R\gamma(x) = \gamma R(x) = \gamma(R(x))$ , whence  $\bar{R}([y]) = [R(y)] = [\gamma(R(x))] = [R(x)] = \bar{R}([x])$ . Similarly,  $\bar{L}([y]) = \bar{L}([x])$ . Since  $\bar{L}$  is clearly an involution and the group  $\langle \bar{R}, \bar{L} \rangle$  acts transitively on  $\bar{D}$ , we have that  $\bar{M}$  is a map. It is easy to see that the natural projection  $\pi = \pi_G : M \rightarrow \bar{M}$ ,  $x \mapsto [x]$  is a map homomorphism. In fact,  $\pi$  is a regular homomorphism. To see this, it is sufficient to realize that the fibre over any dart  $[x]$  of  $\bar{M}$  is the set  $\{\gamma(x); \gamma \in G\}$  and that  $\text{FT}(\pi_G) = G$ .

We now show that every regular map homomorphism arises by factorization.

**Theorem 3.1** *A map homomorphism  $\varphi : \tilde{M} \rightarrow M'$  is regular if and only if it is equivalent to the natural projection  $\pi : M \rightarrow M/\text{FT}(\varphi)$ .*

PROOF. First assume that  $\varphi$  is equivalent to the natural projection  $\pi : M \rightarrow M/\text{FT}(\varphi)$ . Since  $\pi$  is regular, and a homomorphism equivalent to a regular homomorphism is again regular, so is  $\varphi$ .

For the converse, assume that  $\varphi : M \rightarrow M'$  is regular. We want to show that then  $\varphi$  is equivalent to the natural projection  $\pi : M \rightarrow M/\text{FT}(\varphi)$ . To do this it is sufficient to find an automorphism  $\kappa$  of  $M$  and isomorphism  $\lambda : M' \rightarrow M/\text{FT}(\varphi)$  such that  $\pi\kappa = \lambda\varphi$ . Set  $\kappa = \text{id}$  and define  $\lambda$  by  $\varphi(x) \mapsto [x]$ . As  $\varphi$  is regular, the mapping  $\lambda$  is a bijection. We prove that  $\lambda$  is a map homomorphism.

Let  $M = (D; R, L)$ ,  $M' = (D'; R', L')$  and  $\bar{M} = M/\text{FT}(\varphi) = (\bar{D}; \bar{R}, \bar{L})$ . For  $y = \varphi(x)$  we have  $\lambda R'(y) = \lambda R'\varphi(x) = \lambda\varphi R(x) = [R(x)] = \bar{R}[x] = \bar{R}\lambda\varphi(x) = \bar{R}\lambda(y)$ . Thus,  $\lambda R' = \bar{R}\lambda$ . Similarly,  $\lambda L' = \bar{L}\lambda$ . It follows that  $\lambda$  is a homomorphism, and hence an isomorphism. It is immediate that  $\pi = \lambda\varphi$ , proving that  $\varphi$  is equivalent to  $\pi$ , as required. ■

There is an alternative characterization of regular homomorphisms, somewhat similar to the usual definition of a regular map in topology. A map homomorphism  $\varphi : M \rightarrow M'$  is regular if and only if  $(\text{Mon } M)_x \trianglelefteq \varphi_*^{-1}((\text{Mon } M')_{\varphi(x)})$  (cf. [13]).

**Theorem 3.2** *Let  $\varphi : \tilde{M} \rightarrow M$  be a map homomorphism where  $\tilde{M}$  is regular. Then  $\varphi$  is a regular homomorphism. In particular, any homomorphism between regular maps is regular.*

PROOF. Let  $\tilde{x}$  and  $\tilde{y}$  be two darts in the same fibre of  $\varphi$ . We have to show that there exists an element in  $\text{FT}(\varphi)$  which sends  $\tilde{x}$  to  $\tilde{y}$ . Because  $\tilde{M}$  is regular, we certainly have  $\tilde{y} = \tilde{\tau}(\tilde{x})$  for some  $\tilde{\tau} \in \text{Aut } \tilde{M}$ . All we need is to prove that  $\tilde{\tau}$  belongs to  $\text{FT}(\varphi)$ .

Let  $\tilde{z}$  be an arbitrary dart of  $\tilde{M}$ . Then  $\tilde{z} = w(\tilde{R}, \tilde{L})(x)$  for some element  $w(\tilde{R}, \tilde{L})$  of  $\text{Mon } \tilde{M}$ . Set  $w(R, L) = \varphi_* w(\tilde{R}, \tilde{L})$ . Since  $\tilde{\tau}$  commutes with both  $\tilde{R}$  and  $\tilde{L}$ , while  $\varphi\tilde{R} = R\varphi$  and  $\varphi\tilde{L} = L\varphi$ , we have  $\varphi\tilde{\tau}(\tilde{z}) = \varphi\tilde{\tau}w(\tilde{R}, \tilde{L})(\tilde{x}) = \varphi w(\tilde{R}, \tilde{L})\tilde{\tau}(\tilde{x}) = \varphi w(\tilde{R}, \tilde{L})(\tilde{y}) = w(R, L)\varphi(\tilde{y}) = w(R, L)\varphi(\tilde{x}) = \varphi w(\tilde{R}, \tilde{L})(\tilde{x}) = \varphi(\tilde{z})$ . Thus,  $\varphi\tilde{\tau} = \varphi$ , and  $\tilde{\tau}$  belongs to  $\text{FT}(\varphi)$ , as required.  $\blacksquare$

## 4 Lifting and projecting automorphisms

Let  $\varphi : \tilde{M} \rightarrow M$  be a map homomorphism, and let  $\tau \in \text{Aut } M$  and  $\tilde{\tau} \in \text{Aut } \tilde{M}$ . We say that  $\tau$  *lifts* to  $\tilde{\tau}$ , or that  $\tilde{\tau}$  *projects* to  $\tau$ , whenever the following diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\tau}} & \tilde{M} \\ \varphi \downarrow & & \downarrow \varphi \\ M & \xrightarrow{\tau} & M \end{array}$$

is commutative. If  $A$  is a group of map automorphisms of  $M$  which lift, then the collection  $\tilde{A}$  of all of these lifts constitutes a group. Obviously,  $\tilde{A}$  is isomorphic to an extension of  $\text{FT}(\varphi)$  by  $A$ .

The problem of lifting and projecting automorphisms naturally occurs in the study of highly symmetric maps as well as in a broader context of graphs [3, 7, 14, 13, 20]. In particular, the next theorem lies in the background of many constructions of new regular maps from old ones [1, 2, 9, 11, 16, 18]. A concrete environment in which this theorem can be applied will be developed in Section 6.

**Theorem 4.1** *Let  $\varphi : \tilde{M} \rightarrow M$  be a regular homomorphism such that  $\text{Aut } M$  lifts. If  $M$  is a regular map, so is  $\tilde{M}$ .*

**PROOF.** It is sufficient to realize that  $\text{Aut } M$  and  $\text{FT}(\varphi)$  are transitive (on the respective maps), and the lift of  $\text{Aut } M$  is transitive as well.  $\blacksquare$

It is part of the folklore in this area that, on the other hand, a factorization of a regular map by a normal subgroup of its automorphisms results in a regular map.

Passing to a slightly more general context, it is interesting to observe that the regularity of either the base or the covering map alone implies that the whole map automorphism group lifts or, respectively, projects.

**Theorem 4.2** *If  $\tilde{M} \rightarrow M$  is a map homomorphism with  $\tilde{M}$  regular, then  $\text{Aut } M$  lifts.*

PROOF. Consider an arbitrary automorphism  $\tau \in \text{Aut } M$ . Choose a base dart  $x$  in  $M$ , and let  $\tilde{x}$  and  $\tilde{y}$  be arbitrarily chosen darts in  $\tilde{M}$  over  $x$  and  $\tau(x)$ , respectively. Since the map  $\tilde{M}$  is regular, there exist (a unique) automorphism  $\tilde{\tau} \in \text{Aut } \tilde{M}$  sending  $\tilde{x}$  to  $\tilde{y}$ . We claim that  $\tilde{\tau}$  is a lift of  $\tau$ .

Indeed, let  $\tilde{z}$  be an arbitrary dart of  $\tilde{M}$ , and let  $z$  be its projection. Let  $w(\tilde{R}, \tilde{L})$  be an element of  $\text{Mon } \tilde{M}$  taking  $\tilde{x}$  to  $\tilde{z}$ . Because  $\tilde{\tau}\tilde{R} = \tilde{R}\tilde{\tau}$ ,  $\tilde{\tau}\tilde{L} = \tilde{L}\tilde{\tau}$ ,  $\tau R = R\tau$ ,  $\tau L = L\tau$  and  $\varphi\tilde{R} = R\varphi$ ,  $\varphi\tilde{L} = L\varphi$ , we have  $\varphi\tilde{\tau}(\tilde{z}) = \varphi\tilde{\tau}w(\tilde{R}, \tilde{L})(\tilde{x}) = \varphi w(\tilde{R}, \tilde{L})\tilde{\tau}(\tilde{x}) = \varphi w(\tilde{R}, \tilde{L})(\tilde{y}) = w(R, L)\varphi(\tilde{y}) = w(R, L)\tau(x) = \tau w(R, L)(x) = \tau(z) = \tau\varphi(\tilde{z})$ .

We conclude that  $\varphi\tilde{\tau} = \tau\varphi$ , as required.  $\blacksquare$

**Theorem 4.3** *If  $\tilde{M} \rightarrow M$  is a map homomorphism with  $M$  regular, then  $\text{Aut } \tilde{M}$  projects.*

PROOF. Consider an arbitrary automorphism  $\tilde{\tau} \in \text{Aut } \tilde{M}$ . Choose a base dart  $\tilde{x}$  in  $\tilde{M}$ , and let  $x$  and  $y$  be the projections of  $\tilde{x}$  and  $\tilde{\tau}(\tilde{x})$ , respectively. Since the map  $M$  is regular, there exist (a unique) automorphism  $\tau \in \text{Aut } M$  sending  $x$  to  $y$ . We claim that  $\tau$  is the projection of  $\tilde{\tau}$ . The fact that  $\varphi\tilde{\tau} = \tau\varphi$  is established in the same way as in Theorem 4.2.  $\blacksquare$

**Corollary 4.4** *If  $\varphi : \tilde{M} \rightarrow M$  is a homomorphism of regular maps, then  $\text{Aut } \tilde{M}$  projects onto the full automorphism group  $\text{Aut } M$  (and  $\text{Aut } M$  lifts to the full automorphism group  $\text{Aut } \tilde{M}$ ). In particular,  $\text{Aut } \tilde{M}$  is isomorphic to an extension of  $\text{FT}(\varphi)$  by  $\text{Aut } M$ .*  $\blacksquare$

In general, neither the lifted group nor the projected group has to coincide with the full automorphism group of the respective map. We present two examples which also illustrate Theorem 4.2 and Theorem 4.3.

**Example 4.5** Let  $\tilde{M}$  be the embedding of the complete graph  $K_4$  into the 2-sphere  $S^2$ . It is well known that  $\text{Aut } \tilde{M}$  is isomorphic to the alternating group  $\mathbb{A}_4$ . Let  $G \leq \text{Aut } \tilde{M}$  be the stabilizer of an arbitrary vertex of  $K_4$ , and set  $M = \tilde{M}/G$ . Then  $M$  is isomorphic to an embedding in  $S^2$  of  $K_2$  with an added loop. Now  $\text{Aut } M$  is trivial and lifts to the fibre transformation group  $G \cong \mathbb{Z}_3$ , which is not the whole group  $\text{Aut } \tilde{M} \cong \mathbb{A}_4$ .

**Example 4.6** Let  $M$  and  $\tilde{M}$  be the embeddings of the cartesian product of cycles  $C_3 \times C_3$  and  $C_3 \times C_6$  in the torus, respectively. There is an obvious regular homomorphism  $\tilde{M} \rightarrow M$  with the fibre transformation group isomorphic to  $\mathbb{Z}_2$ . The group  $\text{Aut } \tilde{M}$  projects by Theorem 4.3. In contrast, the automorphism of  $M$  which rotates one of the faces by  $\pi/2$  does not lift.

As we have seen, factorization by a subgroup of the map automorphism group is a regular homomorphism, and moreover, every regular homomorphism arises in this way. We now examine the case when the factorization is done by a normal subgroup. The result we obtain is a variation on a theorem of Macbeath [15] which holds for regular covering projections between topological spaces.

**Theorem 4.7** *Let  $\varphi : \tilde{M} \rightarrow M$  be a regular homomorphism. Then  $\text{Aut } \tilde{M}$  projects if and only if  $G = \text{FT}(\varphi)$  is normal in  $\text{Aut } \tilde{M}$ .*

PROOF. Let  $\tilde{\tau} \in \text{Aut } \tilde{M}$ . If  $\varphi(\tilde{x}) = \varphi(\tilde{y})$ , then  $\tilde{y} = \psi(\tilde{x})$  for some  $\psi \in \text{FT}(\varphi)$ . Since  $\text{FT}(\varphi)$  is normal in  $\text{Aut } \tilde{M}$ , there exists an element  $\psi' \in \text{FT}(\varphi)$  such that  $\tilde{\tau}(\tilde{y}) = \tilde{\tau}\psi(\tilde{x}) = \psi'\tilde{\tau}(\tilde{x})$ . Hence  $\varphi\tilde{\tau}(\tilde{x}) = \varphi\psi'\tilde{\tau}(\tilde{x}) = \varphi\tilde{\tau}(\tilde{x})$ . Thus,  $\tilde{\tau}$  is fibre preserving, as required. ■

## 5 Reconstruction of regular homomorphisms by voltages

Concrete constructions of maps require a practical method of description of coverings of a given base map. Such a method is provided by voltage assignments. Usually, voltages are assigned to darts. However, in order to encompass all regular homomorphisms (including those which are not valency preserving), we need to employ voltage assignments on angles.

Let  $M = (D; R, L)$  be an oriented map. An (oriented) *angle* of  $M$  is an ordered pair  $\mathbf{a} = (x, y) = \overrightarrow{xy}$ , where  $x$  and  $y$  are darts of  $M$  such that  $y \in \{R(x), R^{-1}(x), L(x)\}$ . We always consider the angles  $(x, R(x))$  and  $(x, R^{-1}(x))$  to be distinct. The darts  $x$  and  $y$  are called the *initial* and the *terminal* dart of  $\mathbf{a}$ , respectively. The angle  $\mathbf{a}^{-1} = (y, x)$  is the *inverse* of  $\mathbf{a} = (x, y)$ . We denote by  $\mathbf{A}(M)$  the set of all angles of  $M$ ; obviously,  $|\mathbf{A}(M)| = 3|D|$ .

An *angle-walk* (or briefly a *walk*) is a sequence  $W = \mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n$  of angles of  $M$  such that the terminal dart of  $\mathbf{a}_i$  coincides with the initial dart of  $\mathbf{a}_{i+1}$ , for each index  $i = 1, 2, \dots, n-1$ . The initial dart of  $\mathbf{a}_1$  and the terminal dart of  $\mathbf{a}_n$  are called the *initial* and the *terminal* dart of  $W$ , respectively. The term *closed angle-walk* has the obvious meaning. The *empty walk* is considered to be closed. If  $W = \mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n$  is a walk originating at  $x$  and terminating at  $y$ , then  $W^{-1} = \mathbf{a}_n^{-1}\mathbf{a}_{n-1}^{-1} \dots \mathbf{a}_1^{-1}$  is a walk originating at  $y$  and terminating at  $x$ , called the *inverse* of  $W$ .

For a dart  $x$  of  $M$  we denote by  $\Pi(M, x)$  the set of all closed walks based at  $x$ . Adopting, without loss of generality, an additional assumption that consecutive angles in a walk are not mutually inverse, the set  $\Pi^x = \Pi(M, x)$  can be endowed with a group structure, called the *fundamental group* at  $x$ , with the empty walk as the identity.



Let  $M$  be a map and let  $G$  be a finite group. A *voltage assignment* on  $M$  valued in  $G$  is a function  $\alpha : \mathbf{A}(M) \rightarrow G$  such that for any angle  $\mathbf{a}$  one has  $\alpha(\mathbf{a}^{-1}) = \alpha(\mathbf{a})^{-1}$ . The group  $G$  is called the *voltage group*. Note that the voltage  $\alpha(\overrightarrow{xLx})$  is necessarily an involution.

A voltage assignment can be extended to walks in the obvious way. Let  $W = \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n$  be an angle-walk on  $M$ . The *voltage* of  $W$  is defined to be the product  $\alpha(W) = \alpha(\mathbf{a}_1) \alpha(\mathbf{a}_2) \dots \alpha(\mathbf{a}_n)$ . The mapping  $\Pi^x \rightarrow G$ ,  $W \mapsto \alpha(W)$  is obviously a group homomorphism; in particular,  $\alpha(W^{-1}) = \alpha(W)^{-1}$ . The image  $G^x = \{\alpha(W); W \in \Pi^x\}$  is called the *local voltage group* at  $x$ .

Given a voltage assignment  $\alpha$  on  $M = (D; R, L)$  valued in  $G$ , set  $D^\alpha = D \times G$ , and define the permutations of  $D^\alpha$  by

$$\begin{aligned} R^\alpha(x, h) &= (R(x), h \alpha(\overrightarrow{xRx})) \\ L^\alpha(x, h) &= (L(x), h \alpha(\overrightarrow{xLx})). \end{aligned}$$

If the group  $\langle R^\alpha, L^\alpha \rangle$  is transitive, then  $M^\alpha = (D^\alpha; R^\alpha, L^\alpha)$  is the *derived map* determined by  $M$  and  $\alpha$ . The criterion for the group  $\langle R^\alpha, L^\alpha \rangle$  to be transitive is that  $G^x = G$ , and is, basically, well known (the argument valid for voltage assignments on graphs [8] goes through without any difficulty). In what follows we assume this property to be automatically satisfied.

It is easy to see that the *natural projection*  $\pi_\alpha : M^\alpha \rightarrow M$  erasing the second coordinate is a map homomorphism. Observe, however, that for each element  $a \in G$  the mapping  $\tau_a : (x, g) \mapsto (x, ag)$  is an automorphism of  $M^\alpha$  and the group  $\tilde{G} = \{\tau_a; a \in G\}$ , which we call the *copy* of  $G$  in  $\text{Aut } M^\alpha$ , is isomorphic to  $G$ . Moreover, the projection  $\pi_{\tilde{G}} : M^\alpha \rightarrow M^\alpha / \tilde{G}$  is clearly equivalent to  $\pi_\alpha$ . Therefore,  $\pi_\alpha$  is a regular map homomorphism. We show that the converse holds as well.

**Theorem 5.1** *Let  $\varphi : \tilde{M} \rightarrow M = \tilde{M}/G$  be a regular map homomorphism. Then there exists a voltage assignment  $\alpha$  on  $M$  valued in  $G$  such that the natural projection  $\pi_\alpha : M^\alpha \rightarrow M$  is equivalent to  $\varphi$ .*

PROOF. Let  $\tilde{M} = (\tilde{D}; \tilde{R}, \tilde{L})$ ,  $M = (D; R, L)$ , and let  $D = \{X_1, X_2, \dots, X_n\}$ . Then each  $X_i$  is equal to some  $[x]$ , where  $x \in D(\tilde{M})$ . In every dart  $X_i$  of  $M$  choose a fixed representative  $x_i \in X_i$ . If  $y \in X_i$  is arbitrary, then there exists an automorphism  $\sigma_y \in G$  such that  $\sigma_y(x_i) = y$ . This automorphism is uniquely determined for  $\text{Aut } \tilde{M}$  (and hence  $G$ ) acts freely on  $D(\tilde{M})$ . It follows that  $X_i = \{\sigma_y(x_i); y \in X_i\} = \{\sigma(x_i); \sigma \in G\}$ . Thus we can denote the dart  $\sigma(x_i)$  by  $(X_i, \sigma)$  and thereby identify  $D(\tilde{M})$  with  $D(M) \times G$ .

Now, let  $R(X_i) = X_j$ . The definition of  $R$  implies that for each  $x \in X_i$  there exists a unique dart  $y \in X_j$  such that  $\tilde{R}(x) = y$ . Let  $x = (X_i, \sigma)$  and  $y = (X_j, \tau)$ .

Then  $x = \sigma(x_i)$  and  $y = \tau(x_j)$ , whence

$$\tilde{R}(X_i, \sigma) = \tilde{R}(x) = y = (X_j, \tau) = (R(X_i), \tau) = (R(X_i), \sigma(\sigma^{-1}\tau)). \quad (1)$$

We show that the automorphism  $\sigma^{-1}\tau$  only depends on the angle  $\overrightarrow{X_i X_j}$  and not on the choice of  $x \in X_i$ . To see this, pick up an element  $a \in X_i$  and let  $b = \tilde{R}(a) \in X_j$ . Assume that  $a = \eta(x_i)$  and  $b = \theta(x_j)$ . Since  $\theta(x_j) = b = \tilde{R}(a) = \tilde{R}\eta(x_i) = \eta\tilde{R}(x_i)$ , we have

$$\eta^{-1}\theta(x_j) = \tilde{R}(x_i). \quad (2)$$

Similarly,  $\tau(x_j) = y = \tilde{R}(x) = \tilde{R}\sigma(x_i) = \sigma\tilde{R}(x_i)$ , and hence

$$\sigma^{-1}\tau(x_j) = \tilde{R}(x_i). \quad (3)$$

Combining (2) and (3) we obtain  $\eta^{-1}\theta(x_j) = \sigma^{-1}\tau(x_j)$ . Realizing that  $G$  acts freely on  $\tilde{D}$  we deduce that  $\eta^{-1}\theta = \sigma^{-1}\tau$ . In other words, the automorphism  $\sigma^{-1}\tau$  in (1) is independent of the choice of  $x \in X_i$  and depends only on the angle  $\overrightarrow{X_i X_j}$ . This conclusion enables us to define the voltage on the angle  $\overrightarrow{X_i X_j}$  to be

$$\alpha(\overrightarrow{X_i R X_i}) = \sigma^{-1}\tau. \quad (4)$$

A similar computation can be done for angles of the form  $\overrightarrow{X_i L X_i}$  by replacing  $R$  with  $L$ . Thus,

$$\alpha(\overrightarrow{X_i L X_i}) = \sigma^{-1}\tau. \quad (5)$$

Now we can rewrite (1) using (4) as

$$\tilde{R}(X_i, \sigma) = (R(X_i), \sigma\alpha(\overrightarrow{X_i R X_i})),$$

and similarly we get

$$\tilde{L}(X_i, \sigma) = (L(X_i), \sigma\alpha(\overrightarrow{X_i L X_i})).$$

But this shows that  $\tilde{R} = R^\alpha$  and  $\tilde{L} = L^\alpha$ , and Theorem 5.1 follows.  $\blacksquare$

## 6 Invariance of voltage assignments

Following [2, 3, 9, 14, 16] we study conditions, expressed in terms of voltage assignments, which guarantee that a given map automorphism group lifts along a regular homomorphism. As we shall see, the lift exists if and only if the distribution of voltages is well behaved with respect to the action of the automorphism group.

To be more precise, let  $A$  be a group of automorphisms of a map  $M$ . We say that a voltage assignment  $\alpha$  in a group  $G$  is *locally  $A$ -invariant* at a dart  $x$  if, for every  $\tau \in A$  and for every walk  $W \in \Pi^x$ , we have

$$\alpha(W) = 1 \Rightarrow \alpha(\tau W) = 1. \quad (6)$$

We remark that we can speak of local invariance without specifying the dart, as local invariance at some dart implies the same at all darts. A voltage assignment is *locally  $\tau$ -invariant* for an individual automorphism  $\tau$  of  $M$  if it is locally invariant with respect to the group  $\langle \tau \rangle$  generated by  $\tau$ .

Assume that the voltage assignment  $\alpha$  on  $M$  is locally  $A$ -invariant. Since with each  $\tau \in A$  its inverse  $\tau^{-1}$  must also satisfy condition (6), local  $A$ -invariance is equivalent to the requirement that for each  $\tau \in A$  there exists an induced isomorphism  $\tau^{\#x} : G^x \rightarrow G^{\tau x}$  of local voltage groups such that the following diagram

$$\begin{array}{ccc} \Pi^x & \xrightarrow{\tau^*} & \Pi^{\tau x} \\ \alpha \downarrow & & \downarrow \alpha \\ G^x & \xrightarrow{\tau^{\#x}} & G^{\tau x} \end{array}$$

is commutative; in other words,  $\tau^{\#x}(\alpha(W)) = \alpha(\tau W)$  where  $W \in \Pi^x$ .

**Theorem 6.1** *Let  $\varphi : \tilde{M} \rightarrow M$  be a regular homomorphism of maps and let  $A \leq \text{Aut } M$ . Then the following statements hold.*

(a) *If  $A$  lifts along  $\varphi$ , then each voltage assignment  $\alpha$  associated with  $\varphi$  is locally  $A$ -invariant.*

(b) *If there is a voltage assignment  $\alpha$  associated with  $\varphi$  which is locally  $A$ -invariant, then  $A$  lifts to  $\tilde{M}$ .*

PROOF. (a) Let  $\alpha$  be a voltage assignment on  $M$  reconstructing  $\varphi$ . In our considerations we can henceforth replace  $\tilde{M}$  by  $M^\alpha$ .

Let  $W = \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n$  be a closed walk in  $M$  based at  $u$ . For any angle  $\mathbf{a} = \overrightarrow{xy}$  of  $M$  and for an arbitrary element  $h \in H = \text{FT}(\varphi)$  denote by  $(\mathbf{a}, h)$  the angle in  $M^\alpha$  with initial dart  $(x, h)$  and terminal dart  $(y, h \alpha(\mathbf{a}))$ . Then for every  $h \in H$ , the walk

$$W_h = (\mathbf{a}_1, h)(\mathbf{a}_2, h \alpha(\mathbf{a}_1)) \dots (\mathbf{a}_n, h \alpha(\mathbf{a}_1) \alpha(\mathbf{a}_2) \dots \alpha(\mathbf{a}_{n-1}))$$

is a lift of  $W$  originating at  $(u, h)$ . Since  $W$  is closed, the terminal dart of  $W_h$  is

$$(u, h \alpha(\mathbf{a}_1) \alpha(\mathbf{a}_2) \dots \alpha(\mathbf{a}_{n-1}) \alpha(\mathbf{a}_n)) = (u, h \alpha(W)) = (u, h).$$

Thus  $W_h$  is a closed walk in  $M^\alpha$ . Now, let  $\sigma \in A$  and let  $\sigma(W) = \mathbf{b}_1 \mathbf{b}_2 \dots \mathbf{b}_n$ , where  $\mathbf{b}_i = \sigma(\mathbf{a}_i)$ ,  $i = 1, 2, \dots, n$ . If  $\tilde{\sigma} \in \text{Aut } M^\alpha$  is an arbitrary lift of  $\sigma$ , then  $\pi_\alpha \tilde{\sigma} = \sigma \pi_\alpha$  implying that  $\sigma(W) = \sigma \pi_\alpha(W_h) = \pi_\alpha \tilde{\sigma}(W_h)$ . It follows that  $\tilde{\sigma}(W_h)$  is a lift of the walk  $\sigma(W)$ . Our preceding consideration about lifts of  $W$  implies that there exists an element  $h' \in H$  such that

$$\tilde{\sigma}(W_h) = (\mathbf{b}_1, h')(\mathbf{b}_2, h' \alpha(\mathbf{b}_1)) \dots (\mathbf{b}_n, h' \alpha(\mathbf{b}_1) \alpha(\mathbf{b}_2) \dots \alpha(\mathbf{b}_{n-1})).$$

Since  $\tilde{\sigma}$  is an automorphism of  $M^\alpha$  and  $W_h$  is a closed walk in  $M^\alpha$ , so is  $\tilde{\sigma}(W_h)$ . But then  $\alpha(\sigma W) = \alpha(\mathbf{b}_1)\alpha(\mathbf{b}_2)\dots\alpha(\mathbf{b}_n) = 1$ , as required. This proves (a).

(b) Throughout the proof, let  $M = (D; R, L)$ . Let  $u \in D(M)$  be a fixed base dart, and let  $\alpha$  be a locally  $A$ -invariant voltage assignment reconstructing  $\varphi$ . Of course, it is sufficient to deal with  $M^\alpha$  instead of  $\tilde{M}$ . Clearly, for every element  $g \in H$  and for an arbitrary dart  $x$  of  $M$  there exists an angle walk  $W$  from  $u$  to  $x$  such that  $\alpha(W) = g$ . Now, for every automorphism  $\sigma \in \text{Aut } M$  and for every element  $h \in H$  we can define a mapping

$$\sigma_h(x, \alpha(W)) = (\sigma(x), h \alpha(\sigma W)), \quad (7)$$

where  $W$  is an arbitrary  $u - x$  walk with  $\alpha(W) = g$ . We show that the equation (7) correctly defines an automorphism of the derived map  $M^\alpha$ .

If  $W_1$  and  $W_2$  are two  $u - x$  walks with  $\alpha(W_1) = g = \alpha(W_2)$  then  $W_1 W_2^{-1}$  is a closed walk based at  $u$  with net voltage equal to 1. Then  $W_1 W_2^{-1}$  is a closed walk at  $u$  such that  $\alpha(W_1 W_2^{-1}) = \alpha(W_1)\alpha(W_2)^{-1} = 1$ . Since the voltage assignment  $\alpha$  is locally  $A$ -invariant, we obtain

$$\begin{aligned} 1 &= \alpha(\sigma(W_1 W_2^{-1})) = \alpha(\sigma(W_1)\sigma(W_2^{-1})) = \alpha(\sigma(W_1))\alpha(\sigma(W_2^{-1})) \\ &= \alpha(\sigma(W_1))\alpha(\sigma(W_2))^{-1} = \alpha(\sigma(W_1))\alpha(\sigma(W_2))^{-1}. \end{aligned}$$

Hence,  $\alpha(\sigma(W_1)) = \alpha(\sigma(W_2))$ , which means that  $\sigma_h$  is correctly defined.

Next, we have to prove that  $\sigma_h$  is a map automorphism, i. e., that it commutes with both  $R^\alpha$  and  $L^\alpha$ . This is verified by the following computations, wherein  $x \in D$ ,  $W$  is a  $u - x$  walk and  $W' = W\mathbf{a}(x)$  is a  $u - R(x)$  walk obtained from  $W$  by appending the angle  $\mathbf{a}(x) = \overrightarrow{xR(x)}$  at the end of  $W$ :

$$\begin{aligned} \sigma_h R^\alpha(x, \alpha(W)) &= \sigma_h(R(x), \alpha(W\mathbf{a}(x))) = \sigma_h(R(x), \alpha(W')) \\ &= (\sigma R(x), h \alpha(\sigma(W'))) = (R\sigma(x), h \alpha(\sigma(W)\overrightarrow{\sigma(x)\sigma R(x)})) \\ &= (R\sigma(x), h \alpha(\sigma W)\alpha(\overrightarrow{\sigma(x)R\sigma(x)})) = (R\sigma(x), h \alpha(\sigma W)\alpha(\mathbf{a}(\sigma(x)))) \\ &= R^\alpha(\sigma(x), h \alpha(\sigma W)) = R^\alpha \sigma_h(x, \alpha(W)). \end{aligned}$$

By similar calculations we get  $\sigma_h L^\alpha(x, \alpha(W)) = L^\alpha \sigma_h(x, \alpha(W))$ , and Theorem 6.1 is proved.  $\blacksquare$

Note that for two automorphisms  $\sigma_h$  and  $\tau_k$  of  $M^\alpha$  we have  $\sigma_h = \tau_k$  if and only if  $\sigma = \tau$  and  $h = k$ . The first assertion follows from the commutativity of the diagram and the fact that  $\varphi$  is onto. The second is the consequence of the semiregularity of the action of  $\text{FT}(\varphi)$ .

**Corollary 6.2** *Let  $M$  and  $\tilde{M}$  be regular maps. Then:*

(a) *If  $\alpha$  is a locally invariant voltage assignment on  $M$ , then the derived map  $M^\alpha$  is regular;*

(b) *If  $\varphi : \tilde{M} \rightarrow M$  is a map homomorphism, then there exists locally invariant voltage assignment such that the natural projection  $M^\alpha \rightarrow M$  is equivalent to  $\varphi$ . ■*

Note that part (a) of Corollary 6.2 is a generalization of Gvozđjak and Širáň [9, Theorem 3], and part (b) is the converse to (a). Note further that there is a one-to-one correspondence between equivalent classes of homomorphisms of regular maps and equivalence classes of locally invariant voltage spaces.

## 7 Split extensions

Let  $\varphi : \tilde{M} \rightarrow M$  be a map homomorphism, and let  $A$  be a group of automorphisms of  $M$  which lifts along  $\varphi$  to  $\tilde{A}$ . In general, the extension  $\text{FT}(\varphi) \rightarrow \tilde{A} \rightarrow A$  is difficult to analyze. Our aim is to derive conditions in terms of voltage assignments under which these extension splits. This question was also addressed in [3, 9, 14, 20].

To this end we first extend the notion of local  $A$ -invariance of an associated voltage assignment as follows. Following [14], let  $\Omega \subseteq D(M)$  be a nonempty subset of darts of  $M$  which is invariant under the action of  $A$ , that is,  $A(\Omega) = \Omega$ . Denote by  $\Pi^\Omega$  the set of all angle walks in  $M$  having both enddarts in  $\Omega$ . An associated voltage assignment in a group  $G$  is said to be  $(A, \Omega)$ -invariant if the condition

$$\alpha(W) = 1 \Rightarrow \alpha(\tau W) = 1 \quad (8)$$

is satisfied for every  $\tau \in A$  and every walk  $W \in \Pi^\Omega$ .

**Theorem 7.1** *Let  $M$  and  $\tilde{M}$  be maps, let  $A \leq \text{Aut } M$ , and let  $\Omega$  be a set of darts of  $M$  invariant under the action of  $A$  on  $D(M)$ .*

(a) *If  $\alpha$  is a  $(A, \Omega)$ -invariant voltage assignment on  $M$  taking values in a group  $G$ , then  $A$  lifts to  $M^\alpha$  as a split extension of  $\text{FT}(\varphi)$  isomorphic to  $G \rtimes_{\#} A$ , where  $\# : A \rightarrow \text{Aut } G$  is defined by  $\varphi^{\#}(\alpha(W)) = \alpha(\varphi(W))$ , with  $W$  having enddarts in  $\Omega$ .*

(b) *If  $A \leq \text{Aut } M$  lifts along a regular homomorphism  $\varphi : \tilde{M} \rightarrow M$  such that  $\text{FT}(\varphi) \rightarrow \tilde{A} \rightarrow A$  splits, then there exists a  $(A, \Omega)$ -invariant voltage assignment on  $M$  such that  $\pi : M^\alpha \rightarrow M$  is equivalent to  $\varphi$ .*

PROOF. First of all, invariance on  $\Omega$  implies local  $A$ -invariance. Hence  $A$  lifts, by Theorem 6.1. Furthermore, there exist automorphisms  $\tau^{\#x} \in \text{Aut } G$  for each  $\tau \in A$ , see the remarks preceding Theorem 6.1.

We now show that  $\tau^{\#x}$  does not depend on  $x \in \Omega$ . Indeed, let  $x, y \in \Omega$ . There exists a walk  $C$  from  $x$  to  $y$  with  $\alpha(C) = 1$ . Invariance on  $\Omega$  implies that, for an arbitrary closed walk  $W$  at  $y$ , we have

$$\begin{aligned}\tau^{\#x}(\alpha(W)) &= \tau^{\#x}(\alpha(C)\alpha(W)\alpha(C^{-1})) = \tau^{\#x}(\alpha(CWC^{-1})) = \alpha(\tau(CWC^{-1})) \\ &= \alpha(\tau(C)\tau(W)\tau(C^{-1})) = \alpha(\tau(C))\alpha(\tau(W))\alpha(\tau(C^{-1})) = \alpha(\tau(W)) \\ &= \tau^{\#y}(\alpha(W)).\end{aligned}$$

Since the local group equals the whole group  $G$ , we have  $\tau^{\#x} = \tau^{\#y}$ . So we can define  $\tau^{\#} = \tau^{\#x}$ , where  $x$  is just any dart in  $\Omega$ . We leave the reader to show that there is an alternative way how to define  $\tau^{\#}$ , namely,

$$\tau^{\#}(\alpha(W)) = \alpha(\tau(W)), \quad \text{where } W \in \Pi^{\Omega}.$$

Next, we show that  $\# : A \rightarrow \text{Aut } G$  is a homomorphism. Indeed, for an arbitrary  $W \in \Pi^{\Omega}$  we have  $(\tau\psi)^{\#}(\alpha(W)) = \alpha(\tau\psi(W)) = \tau^{\#}(\alpha(\psi(W))) = \tau^{\#}\psi^{\#}(\alpha(W))$ , and the claim follows.

So far we have proved that there exist the semidirect product  $G \rtimes_{\#} A$ . To finish the proof of (a), all that remains is to show that it is naturally isomorphic to  $\tilde{A}$ . This is left as an exercise to the reader.

Now we are going to prove part (b). By Theorem 6.1(b) we already know that there exists a locally  $A$ -invariant voltage assignment  $\alpha$  on  $M$  with values in some group  $G$  such that the natural projection  $M^{\alpha} \rightarrow M$  is equivalent to  $\varphi$ . We consider the lifted group  $\tilde{A}$  as acting on  $M^{\alpha}$ . We are going to modify the existing voltage assignment to an equivalent one satisfying the required property as stated in the theorem.

Let  $\bar{A} \cong A$  be a complement to  $\text{FT}(\varphi) \cong G$  in  $\tilde{A}$ . Such a group  $\bar{A}$  exists by assumption. Observe that each orbit of  $\bar{A}$  intersects each fibre of darts over darts in  $\Omega$  in at most one dart. This is true because  $A$  acts freely. Now consider an arbitrary orbit  $\Omega_0$  in  $\Omega$ . Choose a base dart  $u$  in  $\Omega_0$ , and choose a dart over  $u$  as a point of reference. Label this dart by  $1 \in G$ , and consequently, relabel the darts over  $u$  using the action of  $G$ . Now extend this labelling from the base fibre to fibres over  $\Omega_0$  using the action of  $\bar{A}$  such that darts in the same orbit receive the same label. This is repeated in every orbit in  $\Omega$ . Labellings of fibres over darts not in  $\Omega$  remain unchanged.

Once we have labelled the darts, we can determine the new voltages on angles of  $M$ . This new voltage assignment is clearly equivalent to the original one.

It remains to prove that the new voltage assignment is  $(A, \Omega)$ -invariant. Consider a walk  $W$  with enddarts in  $\Omega$  whose voltage is trivial. Each lift  $\tilde{W}$  of such a walk has enddarts being labelled by the same element of  $G$ , that is, belonging to the same

orbit of  $\bar{A}$ . If  $\varphi \in A$  let  $\bar{\varphi} \in \bar{A}$ . Now  $\bar{\varphi}(\bar{W})$  is a walk over  $\varphi(W)$  and has enddarts labelled by the same label. Consequently, the voltage of  $\varphi(W)$  is trivial. ■

An  $(\text{Aut } M, D(M))$ -invariant voltage assignment is called *invariant on  $M$*  for short. As a corollary we obtain the following characterization of those homomorphisms between regular maps for which the automorphism group of the covering map is a natural split extension of the fibre transformation group.

**Corollary 7.2** *Let  $M$  and  $\tilde{M}$  be regular maps. Then:*

(a) *If  $\alpha$  is an invariant voltage assignment on  $M$  taking values in a group  $G$ , then the derived map  $M^\alpha$  is regular. Moreover, the extension  $\text{FT}(\varphi) \rightarrow \text{Aut } M^\alpha \rightarrow \text{Aut } M$  splits, and we have  $\text{Aut } M^\alpha \cong G \times_{\#} \text{Aut } M$ , where  $\# : \text{Aut } M \rightarrow \text{Aut } G$  is given by  $\varphi^{\#}(\alpha(W)) = \alpha(\varphi(W))$ .*

(b) *If  $\varphi : \tilde{M} \rightarrow M$  is a map homomorphism such that  $\text{FT}(\varphi) \rightarrow \text{Aut } M^\alpha \rightarrow \text{Aut } M$  splits, then there exists an invariant voltage assignment on  $M$  such that the natural projection  $M^\alpha \rightarrow M$  is equivalent to  $\varphi$ .* ■

**Example 7.3** A reader familiar with *Cayley maps* [4, 5, 17, 10, 19] will recognize Cayley maps as those for which there exists a regular homomorphism onto an embedded monopole (a connected graph with one vertex). Moreover, the Cayley map is *balanced* if and only if the embedded monopole is a regular map.

Suppose  $M$  is a regular balanced Cayley map. Then it can be reconstructed by a voltage assignment valued in the Cayley group. This assignment is necessarily invariant. With the help of Corollary 4.4 and Theorem 7.1 we can deduce that  $\text{Aut } M$  is isomorphic to a split extension of the Cayley group by the cyclic group of automorphisms of the embedded monopole [19].

In contrast to this, a locally invariant voltage assignment on a regularly embedded dipole can give rise to a regular map whose automorphism group does not split.

**Example 7.4** Let  $G_n$  be the group given by the presentation

$$(x, y; x^{2n} = y^n = 1, yx = x^3y^{-1}, yx^2 = x^2y).$$

It readily follows from the defining relations that every element of  $G_n$  can be uniquely expressed as a word of the form  $x^i y^j$ , where  $0 \leq i \leq 2n - 1$  and  $0 \leq j \leq n - 1$ . Thus,  $G_n$  has order  $2n^2$ . Consider the subgroup  $C = \langle x^2 \rangle \leq G_n$ . Since  $x^2$  belongs to the centre of  $G_n$ , it follows that  $C$  is normal in  $G_n$ . It is easy to see that  $C \cong \mathbb{Z}_n$  and that  $G_n/C$  is isomorphic to the dihedral group  $\mathbb{D}_n$  of order  $2n$ .

The group  $G_n$  can also be generated by the elements  $y$  and  $x^{-1}y$ , the latter being an involution. This enables us to construct the map  $M_n = (D; R, L)$  with  $D = G_n$ ,  $R = y$  and  $L = x^{-1}y$  where  $R$  and  $L$  act on  $D$  by the left multiplication. Routine calculations reveal that this map is isomorphic to the well-known regular embedding of the complete bipartite graph  $K_{n,n}$  in which each face is bounded by a Hamiltonian cycle. Obviously,  $\text{Aut } M_n$  is isomorphic to  $G_n$ . In fact, we can identify  $\text{Aut } M_n$  with  $G_n$  acting on  $D$  by the right multiplication. In the same fashion, we can view the subgroup  $C \leq G_n$  as a group of automorphisms of  $M_n$  and construct the quotient map  $M'_n = M_n/C$ . It is easy to see that  $M'_n$  is isomorphic to the regular embedding of the  $n$ -dipole in the 2-sphere, with  $\text{Aut } M'_n \cong \mathbb{D}_n$ .

The covering map  $M_n$  can be obtained from  $M'_n$  via the voltage assignment  $\alpha$  valued in  $\mathbb{Z}_n \cong C$  as follows. The angles, oriented consistently with the orientation of  $S^2$ , carry the voltage 1 at one of the vertices of  $M'_n$ , and the voltage 0 at the other vertex. The voltage assignment  $\alpha$  is clearly locally invariant (in fact, every voltage assignment on a regular map which produces a regular covering map must be locally invariant).

If  $n \geq 2$  is even,  $G_n$  cannot be decomposed into a semidirect product of  $C$  by an appropriate subgroup of  $G_n$ . Indeed, the semidirect product would actually be a direct product because  $C$  belongs to the centre. Thus,  $G_n$  would be isomorphic to  $\mathbb{D}_n \times \mathbb{Z}_n$ , which is impossible as  $G_n$  contains an element of order  $2n$  whereas  $\mathbb{D}_n \times \mathbb{Z}_n$  does not. We conclude that  $M_n$ ,  $n \geq 2$  even, is an example of a regular map obtained from another regular map  $M'_n$  by means of a locally invariant voltage assignment having the property that the automorphism group of  $M_n$  is not isomorphic to a semidirect product of the voltage group by the automorphism group of  $M'_n$ .

Both maps  $M_n$  and  $M'_n$  are reflexible. The same map covering  $M_n \rightarrow M'_n$  can be used to show a similar property of the extended automorphism groups (comprising map automorphisms plus reflections) of  $M_n$  and  $M'_n$ . Details are analogous to those presented above and therefore left to the reader. To summarize, these examples show that Theorem 5 in [9] is false.

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