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Abstract

An action graph is a combinatorial representation of a group acting on a set. Comparing two group actions by a morphism of actions induces a covering projection of the respective graphs. This simple observation generalizes and unifies many well-known results in graph theory, with applications ranging from the theory of maps on surfaces and group presentations to theoretical computer science, among others. Reconstruction of action graphs from smaller ones is considered, some results on lifting and projecting the equivariant group of automorphisms are proved, and a special case of the split-extension structure of lifted groups is studied. Action digraphs in connection with group presentations are also discussed.

1 Introduction

With a group G acting on a set Z we can naturally associate, relative to a subset $S \subset G$, a certain (di)graph called the *action (di)graph*. Its vertices are the elements of the set Z, with adjacencies being induced by the action of the elements of S on Z. The definition adopted here is such that a connected action (di)graph corresponds to a Schreier coset (di)graph, with "repeated generators" and semiedges allowed. However, to think of an action (di)graph actually as a Schreier coset (di)graph is much too rigid in many instances. For similar concepts dealing with (di)graphs and group actions see [1, 2, 3, 8, 12, 17, 18, 21, 22, 28, 43, 45]. Some of them, although conceptually different, bare the same name [45], and some of them, quite close to our definition, are referred to by a variety of other names [1, 3]. It seems that the term action (di)graph should be attributed to T. Parsons [43]. For a computer implementation of (a variant of) action graphs see [44].

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Group actions are compared by morphisms. The starting observation of this paper is that an epimorphism between two actions invokes a covering projection of the respective action graphs. Surprisingly enough, this simple result does not seem to have been explicitly stated so far, although there are many well-known special cases with numerous applications.

For instance, it is generally known that Schreier coset (di)graphs are actually covering (di)graphs, and that a Schreier coset (di)graph is regularly covered by its corresponding Cayley (di)graph. These facts are commonly used as background results in the theory of group presentations [12, 13, 28, 31, 32, 14, 48, 52], and have recently been applied in the design and analysis of interconnection networks and parallel architectures [1, 2, 3, 22], among others. Coverings of Cayley graphs are frequently employed to construct new graphs with variuos types of symmetry and other graph-theoretical properties [9, 19, 34, 47] as well as to prove that a subgroup cannot have genus greater than the group itself [7, 19]. As for the maps on surfaces [4, 5, 10, 16, 19, 20, 25, 26, 27, 30, 35, 36, 40, 41, 46], one of the many combinatorial approaches to this topic is by means of a Schreier representation [25, 40]. Such a representation is actually a certain action graph in disguise, and homomorphisms of maps correspond to covering projections of the respective action graphs. Some important basic facts can be elegantly derived along these lines.

We here give a unified approach to all these diverse topics, and in addition, we derive certain results which appear to be new. Section 2 is preliminary. Action graphs are introduced formally in Section 3, and covering projections induced by morphisms of actions in Section 4. Further basic properties of such coverings are discussed in Sections 5 and 6. In Section 7 we briefly consider automorphism groups of action graphs. Section 8 is devoted to lifting and projecting automorphisms, with focus on the equivariant group. In Section 9 we determine the group of covering transformations, and apply some results of [35] to obtain conditions for a natural splitting of a lifted group of map automorphisms (valid also if the map homomorphism is not valency preserving [35, 36]). In Section 10 we treat action digraphs in connection with group presentations. The lifting problem along a regular covering (of graphs as well as of general topological spaces) is reduced to a question about action digraphs.

2 Preliminaries: Group actions, Graphs and Coverings

By an ordered pair (Z, G) we denote a group G *acting on the right* on a nonempty set Z. (For convenience we omit the dot sign indicating the action.) A *morphism of actions* is an ordered pair $(\phi, \psi) : (Z, G) \to (Z', G')$, where $\phi : Z \to Z'$ is a function and $\psi : G \to G'$ is a homomorphism such that $\phi(u \cdot g) = \phi(u) \cdot \psi(g)$.

Morphisms are composed on the left. Left actions and their morphisms are defined similarly. Morphisms of the form $(\phi, \mathrm{id}) : (Z, G) \to (Z', G)$ are called *equivariant*, and morphisms of the form $(\mathrm{id}, \psi) : (Z, G) \to (Z, G')$ are called *invariant*. Invariant epimorphisms formalize the intuitive notion of "groups, acting in the same way on a given set".

We say that an action $(\tilde{Z}, \tilde{G})$ *covers* an action (Z, G) whenever there exists an epimorphism $(\phi, q) : (\tilde{Z}, \tilde{G}) \to (Z, G)$. This terminology is justified by the fact that the cardinality $|\phi^{-1}(z)|$ depends just on the orbit of G to which $z \in Z$ belongs. A covering of actions $(\phi, q) : (\tilde{Z}, \tilde{G}) \to (Z, G)$ can be decomposed into an equivariant covering $(\phi, \mathrm{id}) : (\tilde{Z}, \tilde{G}) \to (Z, \tilde{G})$ followed by an invariant covering $(\mathrm{id}, q) : (Z, \tilde{G}) \to (Z, G)$, where the action of $\tilde{G}$ on Z is defined by $z \cdot \tilde{g} = z \cdot q(\tilde{g})$.

Proposition 2.1 *There exists a covering $(\tilde{Z}, \tilde{G}) \to (Z, G)$ of transitive actions if and only if there exists, for a fixed chosen $\tilde{b} \in \tilde{Z}$, a group epimorphism $q : \tilde{G} \to G$ such that $q(\tilde{G}_{\tilde{b}}) \leq G_b$ for some $b \in Z$. The corresponding onto mapping of sets is then given by $\phi_{\tilde{b},b}(\tilde{b} \cdot \tilde{g}) := b \cdot q(\tilde{g})$. In particular, two transitive actions are isomorphic if and only if there exists an isomorphism between the respective groups mapping a stabilizer onto a stabilizer.* ■

Example 2.2 Let $H \leq H' \leq G$ and $K \lhd G$. The group G acts by right multiplication on the set of right cosets $H|G$. Similarly, the quotient group G/K acts on the set of right cosets $H'K|G$. There is an obvious covering of actions $(H|G, G) \to (H'K|G, G/K)$. In particular, the regular action $(G, G)_r$ of G on itself by right multiplication covers any transitive action of G/K. ■

Example 2.3 There is an equivariant isomorphism representing a transitive action of a group G as an action on the cosets of a stabilizer. Moreover, all transitive and faithful quotient actions of G can be treated in a similar fashion.

Indeed, a conjugacy class $\mathcal{C}$ of subgroups in G determines the action of $G/\mathrm{core}(\mathcal{C})$ on the cosets of an element of $\mathcal{C}$. This action is transitive and faithful. Conversely, let $q : G \to Q$ be a group epimorphism and let (Z, Q) be transitive and faithful. Define the action (Z, G) such that $(\mathrm{id}, q) : (Z, G) \to (Z, Q)$ is an invariant covering, and let G_b^Q be a stabilizer of (Z, G). Let $(c, \mathrm{id}) : (Z, G) \to (G_b^Q|G, G)$ be the standard representation of (Z, G). Then $(c, \mathrm{id}) : (Z, Q) \to (G_b^G|G, Q)$ is the standard representation of (Z, Q). It follows that (Z, Q) determines a conjugacy class $\mathcal{C}^Q$ in G with $\mathrm{core}(\mathcal{C}^Q) = \mathrm{Ker}\, q$. Thus, the isomorphism classes of transitive and faithful actions of quotient groups of G are in natural correspondence with conjugacy classes of subgroups in G. Moreover, $(\phi, r) : (Z, Q) \to (Z', Q')$ is a covering satisfying $rq = q'$ if and only if $(\phi, \mathrm{id}) : (Z, G) \to (Z', G)$ is a morphism. That is, such a covering exists if and only if G_b^Q is contained in a conjugate subgroup of $G_b^{Q'}$. See Examples 4.7 and 4.8 for an application. ■

By $\mathrm{Aut}(Z,G)$ we denote the automorphism group of (Z,G). An automorphism ψ of G is called *admissible* whenever there exists a bijection ϕ on Z such that $(\phi,\psi) \in \mathrm{Aut}(Z,G)$. The group of admissible automorphisms is denoted by $\mathrm{Adm}_Z\, G$. By $\mathrm{Aut}(Z)_G$ we denote the *equivariant group* of the action, formed by all bijections ϕ on Z for which $(\phi, \mathrm{id}) \in \mathrm{Aut}(Z,G)$. If (Z,G) is transitive, then $\mathrm{Aut}(Z)_G$ can be computed explicitly relative to a point of reference $b \in Z$ as $\mathrm{Aut}(Z)_G = \{\tau \mid \tau(b \cdot g) = b \cdot ag,\ a \in \mathrm{N}(G_b) \bmod G_b\} \cong \mathrm{N}(G_b)/G_b$. Also, the left action of $\mathrm{Aut}(Z)_G$ on Z is fixed-point free, and is transitive if and only G acts with a normal stabilizer. Hence if G is, in addition, faithful, then $\mathrm{Aut}(Z)_G$ is regular if and only if G is regular. In this case, (Z,G) is essentially the right multiplication $(G,G)_r$, whereas $(\mathrm{Aut}(Z)_G, Z)$ is essentially the left multiplication $(G,G)_l$.

A *graph* is an ordered 4-tuple $X = (D, V; \mathrm{beg}, \mathrm{inv})$ where D and V are disjoint nonempty sets of *darts* and *vertices*, respectively, $\mathrm{beg} : D \to V$ is an onto mapping which assigns to each dart x its *initial vertex* $\mathrm{beg}\, x$, and $\mathrm{inv} : D \to D$ is an involution which interchanges every dart x and its *inverse* $x^{-1} = \mathrm{inv}\, x$. For notational convenience we use *beg* and *inv* just as symbolic names denoting the actual concrete functions. The *terminal vertex* $\mathrm{end}\, x$ of a dart x is the initial vertex of x^{-1}. The orbits of beg are called *edges*. An edge is called a *semiedge* if $x^{-1} = x$, a *loop* if $x^{-1} \neq x$ and $\mathrm{end}\, x = \mathrm{beg}\, x^{-1} = \mathrm{beg}\, x$, and it is called a *link* otherwise. *Walks* are defined as sequences of darts in the obvious way. By $\mathcal{W} = \mathcal{W}(X)$ and $\mathcal{W}^u = \mathcal{W}(X,u)$ we denote the set of all walks and the set of all u-based closed walks of a graph X, respectively. By recursively deleting all consecutive occurrences of a dart and its inverse in a given walk we obtain its *reduction*. Two walks with the same reduction are called *homotopic*. The naturally induced operation in set of all reduced u-based closed walks defines the *fundamental group* $\pi^u = \pi(X,u)$. A *morphism of graphs* $f : (D, V; \mathrm{beg}, \mathrm{inv}) \to (D', V'; \mathrm{beg}, \mathrm{inv})$ is a function $f : V \cup D \to V' \cup D'$ such that $fV \subseteq V'$, $fD \subseteq D'$ and $f\,\mathrm{beg} = \mathrm{beg}\, f$, $f\,\mathrm{inv} = \mathrm{inv}\, f$. For convenience we write $f = f_V + f_D : V \cup D \to V' \cup D'$, where $f_V = f|_V : V \to V'$ and $f_D = f|_D : D \to D'$ are the appropriate restrictions. Graph morphisms are composed on the left.

A graph epimorphism $p : \tilde{X} \to X$ is called a *covering projection* if, for every vertex $\tilde{u} \in \tilde{X}$, the set of darts with $\tilde{u}$ as the initial vertex is bijectively mapped onto the set of darts with the initial vertex $p(\tilde{u})$. The graph X is called the *base graph* and $\tilde{X}$ the *covering graph*. By $\mathrm{fib}_u = p^{-1}(u)$ and $\mathrm{fib}_x = p^{-1}(x)$ we denote the *fibre* over the vertex u and the dart x of X, respectively. A *morphism of covering projections* $p \to p'$ is an ordered pair $(f, \tilde{f})$ of graph morphisms $f : X \to X'$ and $\tilde{f} : \tilde{X} \to \tilde{X}'$ such that $fp = p'\tilde{f}$. An *equivalence* of covering projections p and p' of the same base graph is a morphism of the the form $(\mathrm{id}, \tilde{f})$, where $\tilde{f}$ is graph isomorphism. Equivalence of covering projections defined on the same covering graph is defined similarly. An automorphism of $p : \tilde{X} \to X$ is of course a pair of automorphisms $(\tilde{f}, f)$ satisfying $fp = p\tilde{f}$. The automorphism $\tilde{f}$ is called a *lift* of f, and f the *projection* of $\tilde{f}$. In particular, all lifts of the identity automorphism form the *group* $\mathrm{CT}(p)$ *of covering transformations*. If the covering graph (and hence the base graph) is connected, then $\mathrm{CT}(p)$ acts semiregularly on vertices and on darts of $\tilde{X}$. The covering projection of connected graphs is *regular* whenever $\mathrm{CT}(p)$ acts regularly on each fibre.

There exists an *action* of the set of walks $\mathcal{W}$ on the vertex-set of $\tilde{X}$ defined by $\tilde{u} \cdot W = \mathrm{end}\, \tilde{W}$, where $\tilde{W}$ is the unique lift of W such that $\mathrm{beg}\, W = \tilde{u}$. In other words, we have $(\tilde{u} \cdot W_1) \cdot W_2 = \tilde{u} \cdot W_1 W_2$ and $\tilde{u} \cdot WW^{-1} = \tilde{u}$. The mapping $\tilde{u} \mapsto \tilde{u} \cdot W$ defines a bijection $\mathrm{fib}_{\mathrm{beg}\, W} \to \mathrm{fib}_{\mathrm{end}\, W}$. Homotopic walks have the same action. In particular, W^u and π^u have the same action on fib_u. The walk-action implies that coverings (of connected graphs) can be studied from a purely combinatorial point of view [35]. A *voltage space* $(F, \Gamma; \xi)$ on a connected graph $X = (D, V; \mathrm{beg}, \mathrm{inv})$ is defined by an action of a *voltage group* Γ on a set F, called the *abstract fibre*, and by an assignment $\xi : D \to \Gamma$ such that $\xi_{x^{-1}} = (\xi_x)^{-1}$. This assignment extends to a homomorphism $\xi : \mathcal{W} \to \Gamma$, with homotopic walks carrying the same voltage. The group $\mathrm{Loc}^u = \Gamma^u = \xi(\mathcal{W}^u) = \xi(\pi^u)$ is called the *local group* at the vertex u. As the graph is assumed connected, the local groups at distinct vertices are conjugate subgroups, and if any of them is transitive we call such a voltage space *locally transitive*. With every voltage space $(F, \Gamma; \xi)$ on a connected graph $X = (D, V; \mathrm{beg}, \mathrm{inv})$ we can associate a covering $p_\xi : \mathrm{Cov}(F, \Gamma; \xi) \to X$. The graph $\mathrm{Cov}(F, \Gamma; \xi)$ has $\tilde{V} = V \times F$ as the vertex-set and $\tilde{D} = D \times F$ as the dart-set. The incidence function is $\mathrm{beg}(x, i) = (\mathrm{beg}\, x, i)$ and the switching involution inv is given by $(x, i)^{-1} = (x^{-1}, i \cdot \xi_x)$. The covering graph is connected if and only if the voltage space is locally transitive. In particular, the *Cayley voltage space* $(\Gamma, \Gamma; \xi)$, where Γ acts on itself by right multiplication, gives rise to a regular covering. Conversely, each covering of a connected base graph is associated with some voltage space, and each regular covering is associated with a Cayley voltage space $(\Gamma, \Gamma; \xi)$, where $\Gamma \cong \mathrm{CT}(p)$ [19, 35].

3 Action graphs

Let G be a (nontrivial) group acting (on the right) on a nonempty set Z and let $S \subset G$ be a *Cayley set*, that is, $\emptyset \neq S = S^{-1}$ and $1 \notin S$. With the triple (Z, G, S) we naturally associate the *action graph*

$$\mathrm{Act}(Z, G; S) = (Z \times S, Z; \mathrm{beg}, \mathrm{inv}),$$

where $\mathrm{beg}(z, s) = z$ and $\mathrm{inv}(z, s) = (z \cdot s,\ s^{-1})$. We shall actually need to consider *Cayley multisets*, that is, S has repeated elements (where for each $s \in S$ the elements

s and s^{-1} have the same multiplicity). Our definition of a graph must then be extended accordingly.

Example 3.1 The action graph of a group G, acting on a one-element set relative to a Cayley (multi)set S, is called a *monopole* and denoted by $\mathrm{mnp}(S)$. ■

Example 3.2 The graph $\mathrm{Act}(H|G, G; S)$ is the *Schreier coset graph* $\mathrm{Sch}(G, H; S)$. By taking $H = 1$ and $H = G$ we get the *Cayley graph* $\mathrm{Cay}(G; S)$ and a monopole, respectively. ■

Example 3.3 Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be permutations of a finite set Z. By representing each of these permutations pictorially in the obvious way we obtain the action graph for the permutation group $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ relative to the symmetrized generating (multi)set $\{\alpha_1^{\pm 1}, \alpha_2^{\pm 1}, \ldots, \alpha_n^{\pm 1}\}$. ■

Example 3.4 A *finite oriented map* M is a finite graph, cellularly embedded into a closed orientable surface endowed with a global orientation. Let D be the dart-set of the embedded graph, L its dart-reversing involution, and R the local rotation which cyclicaly permutes the darts in their natural order around vertices consistently with the global orientation. In studying the combinatorial properties of such a map we only need to consider the permutation group $\langle R, L \rangle$ on D (together with the generating set $\{R, L\}$), and consequently, its Schreier representation. See Jones and Singerman [25]. Equivalently, we only need to consider the action graph of the group $\langle R, L \rangle$ acting on D relative to the Cayley set $\{R, R^{-1}, L\}$. This graph, here denoted by $\mathrm{Map}(D; R, L)$ or sometimes by $\mathrm{Act}(M)$, is also known as the *truncation of the map* [40].

More generally, maps on all compact surfaces can be viewed combinatorially in terms of certain permutation groups and their generators, as shown by Bryant and Singerman [10]. For closed surfaces, the associated action graphs correspond to *graph encoded maps* of Lins [30]. ■

A walk W in the action graph $\mathrm{Act}(Z, G; S)$ defines a word $w(W) \in S^*$ over the alphabet S. Conversely, if $w \in S^*$ is a word, then $W(z, w)$ denotes the (set of "parallel") walk(s) starting at $z \in Z$ and determined by w. The fact that in the case of repeated generators there is no bijective correspondence between words over S and walks, rooted at a chosen vertex, is a minor technical difficulty which we usually (but not always) can ignore. The action graph is connected if and only if S generates a transitive subgroup of G. Without loss of generality we can in most cases assume that the action is transitive and that the Cayley (multi)set generates the group.

4 Morphisms arising from coverings of actions

A mapping $\phi + \theta : \tilde{Z} + \tilde{Z} \times \tilde{S} \rightarrow Z + Z \times S$ is a morphism of action graphs $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \rightarrow \mathrm{Act}(G, Z; S)$ if and only if

$$\theta(\tilde{z}, \tilde{s}) = (\phi(\tilde{z}), \theta_{\tilde{z}}(\tilde{s})), \tag{1}$$

where $\{\theta_{\tilde{z}} \mid \tilde{z} \in \tilde{Z}\}$ is a collection of mappings $\tilde{S} \rightarrow S$ satisfying

$$\phi(\tilde{z} \cdot \tilde{s}) = \phi(\tilde{z}) \cdot \theta_{\tilde{z}}(\tilde{s}) \quad \text{and} \quad (\theta_{\tilde{z}}(\tilde{s}))^{-1} = \theta_{\tilde{z} \cdot \tilde{s}}(\tilde{s}^{-1}). \tag{2}$$

This follows directly from the definition of a graph morphism. In particular, morphisms arising naturally from coverings of actions can be viewed as graph covering projections (by taking the Cayley (multi)sets to correspond bijectively in a natural way). We state this formally as a theorem, and list some of the well-known special cases and applications.

Theorem 4.1 *Let $(\phi, q) : (\tilde{Z}, \tilde{G}) \rightarrow (Z, G)$ be a covering of actions and let $\tilde{S} \subset \tilde{G}$ be a Cayley (multi)set. Consider $S = q(\tilde{S})$ as a multiset in a bijective correspondence with $\tilde{S}$. Then the induced graph morphism $p_{\phi,q} = \phi + \phi \times q|_{\tilde{S}} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \rightarrow \mathrm{Act}(Z, G; S)$ is a covering projection of action graphs.*

PROOF. The mapping $\theta = \phi \times q|_{\tilde{S}}$ satisfies (1) and (2). Hence $\phi + \phi \times q|_{\tilde{S}}$ is a graph morphism. Since it is onto with $q|_{\tilde{S}}$ a bijection, it is a covering projection. ■

Example 4.2 Let the action be transitive. Then the mapping $p_{c,\mathrm{id}} : \mathrm{Act}(Z, G; S) \rightarrow \mathrm{Sch}(G, G_b; S)$, where $c(b \cdot g) = G_b g$, is an equivariant isomorphism. ■

Example 4.3 Let $H \leq H' \leq G$. Then $p_{\phi,\mathrm{id}} : \mathrm{Sch}(G, H; S) \rightarrow \mathrm{Sch}(G, H'; S)$, where $\phi(Hg) = H'g$, is an equivariant covering projection. ■

Example 4.4 Let $K \triangleleft G$. Then $p_{\mathrm{id},q} : \mathrm{Sch}(G, K; S) \rightarrow \mathrm{Cay}(G/K; q(S))$ is an invariant 1-fold covering projection, and hence an invariant isomorphism. ■

Example 4.5 Let $(\mathrm{id}, q) : (Z, G) \rightarrow (Z, \bar{G})$ be an invariant covering of actions. Then $\mathrm{Act}(Z, G; S)$ is invariantly isomorphic to $\mathrm{Act}(Z, \bar{G}; q(S))$. Thus, in studying action graphs we may restrict to faithful actions by taking $q : G \rightarrow \bar{G} \cong G/G_Z$. A similar result dealing with isomorphisms of Cayley digraphs can be found in [22]. ■

Example 4.6 A homomorphism $\tilde{M} \rightarrow M$ of oriented maps is a morphism of the underlying graphs which extends to a mapping between the supporting surfaces. Topologically it corresponds to a branched covering with possible singularities in face-centres, edge-centres and vertices. Combinatorially we have a mapping of the respective dart-sets $\phi : \tilde{D} \to D$ such that $\phi(\tilde{x} \cdot \tilde{R}) = \phi(\tilde{x}) \cdot R$ and $\phi(\tilde{x} \cdot \tilde{L}) = \phi(\tilde{x}) \cdot L$. This together with $\tilde{L} \mapsto L$ and $\tilde{R} \mapsto R$ defines a covering of actions and consequently, a covering projection of action graphs $\mathrm{Map}(\tilde{D}; \tilde{R}, \tilde{L}) \to \mathrm{Map}(D; R, L)$. ■

Example 4.7 Recall from Example 2.3 that transitive and faithful actions of quotient groups of G can be "modeled by conjugacy classes of subgoups" in G. It follows that there exists a covering projection $\phi + \phi \times r : \mathrm{Act}(Z, Q; q(S)) \to \mathrm{Act}(Z', Q'; q'(S))$ arising from actions (where $rq = q'$) if and only if G_b^Q is contained in a conjugate subgroup of $G_b^{Q'}$. A special case is essentially considered in [50]. The situation as described above is encountered in the theory of maps and hypermaps. ■

Example 4.8 Oriented maps and their homomorphisms can be modeled by conjugacy classes within triangle groups, see Jones and Singerman [25]. The idea extends to all maps [10] and even hypermaps [27]. ■

5 Structure-preserving morphisms

Nonisomorphic actions can give rise to isomorphic graphs, as shown by Examples 4.4, 4.5 and by Example 5.1 below. Also, isomorphic actions can have isomorphic graphs with no graph isomorphism arising from an isomorphism of actions, see Example 5.2.

Example 5.1 The triangular prism is a Cayley graph for the groups S_3 and $\mathbb{Z}_6$. It is also an action graph for the group S_4, obtained by representing S_4 as the subgroup of S_6 generated by permutations (12)(45)(36), (23)(56)(14) and (13)(46)(25). See also Example 10.1. ■

Example 5.2 Take a Cayley graph $\mathrm{Cay}(G, S)$ where the generating set S is not a CI-set. Then there is a generating Cayley set T with $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ such that no automorphism of G maps S onto T. ■

In view of these remarks we note the following. When considering an action graph as having a certain structure arising from the action, we are actually considering the induced equivariant covering $\mathrm{Act}(Z, G; S) \to \mathrm{mnp}(S)$. A morphism $\phi + \theta : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ is *structure-preserving* if there exists a mapping of the monopoles $\mathrm{mnp}(\tilde{S}) \to \mathrm{mnp}(S)$ such that $\phi + \theta$, together with $\mathrm{mnp}(\tilde{S}) \to \mathrm{mnp}(S)$, is a morphism of covering projections. In other words, θ does not depend on the vertex: $\theta = \phi \times \psi$, where $\psi : \tilde{S} \to S$. For example, coverings arising from coverings of actions are structure-preserving, with $\mathrm{mnp}(\tilde{S}) \cong \mathrm{mnp}(S)$.

Proposition 5.3 *Let* $\phi + \phi \times \psi : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ *be a structure-preserving covering, where* $\tilde{S}$ *and* S *are generating (multi)sets and* G *is faithful. Then this covering arises from a covering of actions.*

PROOF. By induction we have $\phi(\tilde{z} \cdot \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_n) = \phi(\tilde{z}) \cdot \psi(\tilde{s}_1)\psi(\tilde{s}_2) \dots \psi(\tilde{s}_n)$ for each $\tilde{z} \in \tilde{Z}$ and any choice of generators $\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n \in \tilde{S}$. Let $\tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_n = 1$. As ϕ is onto and G faithful, we have $\psi(\tilde{s}_1)\psi(\tilde{s}_2) \dots \psi(\tilde{s}_n) = 1$. Hence ψ extends to a homomorphism, as required. ■

Example 5.4 A covering projection of action graphs $\mathrm{Map}(\tilde{D}; \tilde{R}, \tilde{L}) \to \mathrm{Map}(D; R, L)$ such that $\tilde{R} \mapsto R$ and $\tilde{L} \mapsto L$ arises from a covering of actions $\langle \tilde{R}, \tilde{L} \rangle \to \langle R, L \rangle$, and hence represents a homomorphism of the respective maps. ■

Although coverings of action graphs, even isomorphisms, in general do not arise from actions nor are at least structure-preserving, we may still ask the following. Let $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to X$ and $\tilde{X} \to \mathrm{Act}(Z, G; S)$ be covering projections. Is there an action-structure for the graph X (respectively, $\tilde{X}$) such that these projections arise from coverings of actions?

Theorem 5.5 *Let* $\phi + \theta : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to X$ *be a covering projection. Then ther exists an action graph structure for* X *such that* $\phi + \theta$ *is equivalent to a covering arising from actions if and only if there exists a covering projection* $X \to \mathrm{mnp}(\tilde{S})$ *which makes the following diagram*

$$\begin{array}{ccc} \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) & \xrightarrow{\phi+\theta} & X \\ \downarrow & & \downarrow \\ \mathrm{mnp}(\tilde{S}) & \xrightarrow{\mathrm{id}} & \mathrm{mnp}(\tilde{S}) \end{array}$$

commutative. In this case the action-structure for X *can be chosen in such a way that the respective covering is equivariant.*

PROOF(SKETCH). If $\phi + \theta$ is equivalent to a projection arising from a covering of actions, then such a decomposition clearly exists. Conversely, let Z be the vertex-set of X. One can show easily that $z \cdot \tilde{g} = \phi(\tilde{z} \cdot \tilde{g})$, $\tilde{z} \in \phi^{-1}(z)$, is a well defined action of $\tilde{G}$ on Z, with $(\phi, \mathrm{id}) : (\tilde{Z}, \tilde{G}) \to (Z, \tilde{G})$ an equivariant covering of actions. The projection $p_{\phi,\mathrm{id}} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, \tilde{G}; \tilde{S})$ is equivalent to $\phi + \theta$. ■

We can interpret Theorem 5.5 by saying that a quotient of a Schreier coset graph decomposing the natural projection onto a monopole is again a Schreier coset graph of the same group. Stated in this form, the result is due to Širan and Škoviera [50].

We now turn to the second question above. We assume that (Z, G) is transitive and that S is a generating Cayley (multi)set. We may also assume that the covering projection $p = p_\xi : \tilde{X} = \mathrm{Cov}(F, \Gamma; \xi) \to \mathrm{Act}(Z, G; S)$ is given by means of a voltage space $(F, \Gamma; \xi)$ on $\mathrm{Act}(Z, G; S)$.

Theorem 5.6 *With the notation and assumptions above there exists an action graph structure for* $\mathrm{Cov}(F, \Gamma; \xi)$ *such that the natural projection* $\mathrm{Cov}(F, \Gamma; \xi) \to \mathrm{mnp}(\tilde{S})$ *completes the diagram*

$$\begin{array}{ccc} \mathrm{Cov}(F,\Gamma;\xi) & \xrightarrow{p_\xi} & \mathrm{Act}(Z,G;S) \\ \downarrow & & \downarrow \\ \mathrm{mnp}(\tilde{S}) & \xrightarrow{\approx} & \mathrm{mnp}(S). \end{array}$$

Moreover, if G is faithful then p_ξ is equivalent to a covering arising from actions.

Proof. The derived graph has $Z \times F$ as the vertex-set and $Z \times S \times F$ as the dart-set, with the incidence function and the switching involution being, respectively, $\mathrm{beg}(z, s, i) = (z, i)$ and $\mathrm{inv}(z, s, i) = (z \cdot s, s^{-1},\ i \cdot \xi_{(z,s)})$. By the unique walk lifting, the collection of closed walks in $\mathrm{Act}(Z, G; S)$ representing the orbits of $s \in S$ lift to a collection of closed walks representing a permutation of $Z \times F$ which we denote by α_s. This permutation is defined by $(z, i) \cdot \alpha_s = (z \cdot s,\ i \cdot \xi_{(z,s)})$. Note that $s \mapsto \alpha_s$ is a bijection and that $\alpha_{s^{-1}} = \alpha_s^{-1}$.

Let $\tilde{G} = \langle \alpha_s \mid s \in S \rangle$ and $\tilde{S} = \{\alpha_s \mid s \in S\}$. The action graph $\mathrm{Act}(Z \times F, \tilde{G}, \tilde{S})$ has $Z \times S$ as the vertex-set and $Z \times F \times \tilde{S}$ as the dart-set. The incidence is given by $\mathrm{beg}(z, i, \alpha_s) = (z, i)$ and the switching involution is $\mathrm{inv}(z, i, \alpha_s) = (z \cdot s, i \cdot \xi_{(z,s)}, \alpha_s^{-1})$. The mappings id on vertices and $(z, s, i) \mapsto (z, i, \alpha_s)$ on darts define an isomorphism $\mathrm{Cov}(F, \Gamma; \xi) \to \mathrm{Act}(Z \times F, \tilde{G}; \tilde{S})$. This induces a projection $\mathrm{Act}(Z \times F, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ equivalent to p_ξ, and the natural projection $\mathrm{Act}(Z \times F, \tilde{G}; \tilde{S}) \to \mathrm{mnp}(\tilde{S})$ induces an equivalent natural projection $\mathrm{Cov}(F, \Gamma; \xi) \to \mathrm{mnp}(\tilde{S})$ making the required diagram commutative.

The last statement in the theorem follows by Proposition 5.3. ■

Example 5.7 Let a connected graph $\mathrm{Cov}(F, \Gamma; \xi)$ be a covering of the action graph $\mathrm{Map}(D; R, L)$ associated with a map M, and let $\mathrm{Cov}(F, \Gamma; \xi)$ inherit the action-structure as in Theorem 5.6. Since α_L is an involution, $\mathrm{Cov}(F, \Gamma; \xi)$ is the action graph $\mathrm{Map}(D \times F; \alpha_R, \alpha_L)$ of some map $\tilde{M}$, and the covering projection essentially arises from the map homomorphism $\tilde{M} \to M$, by Example 5.4.

This shows that homomorphisms of oriented maps can be studied just by considering coverings of associated action graphs. Compare with the discussion on Schreier representations of maps in [41]. We define a map homomorphism $\tilde{M} \to M$ to be *regular* if $\mathrm{Map}(\tilde{D}; \tilde{R}, \tilde{L}) \to \mathrm{Map}(D; R, L)$ is a regular covering. See also [36]. ■

Finally, let us briefly consider the following question. What is the necessary and sufficient condition for a connected graph $\mathrm{Cov}(F,\Gamma;\xi)$ as in Theorem 5.6 to be the Cayley graph of the group $\tilde{G} = \langle \alpha_s \mid s \in S \rangle$ relative to $\tilde{S} = \{\alpha_s \mid s \in S\}$?

First of all, $\alpha_{s_1}\alpha_{s_2}\dots\alpha_{s_n} \in \tilde{G}_{(z,i)}$ if and only if $s_1 s_2 \dots s_n \in G_z$ and $\xi_{W(z,s_1 s_2\dots s_n)} \in \Gamma_i$, and hence $\alpha_{s_1}\alpha_{s_2}\dots\alpha_{s_n} = \mathrm{id}$ if and only if $s_1 s_2 \dots s_n \in G_Z$ and $\xi_{W(z,s_1 s_2 \dots s_n)} \in \Gamma_F$ for each $z \in Z$. Thus, assuming that G is faithful and the covering is regular, the answer to the above question is the following: any closed walk with trivial voltage must correspond to a relation $s_1 s_2 \dots s_n = 1$, and in this case, all closed walks corresponding to this word must have trivial voltage. In particular, a regular covering of a Cayley graph as in Theorem 5.6 is the required Cayley graph if and only if the following holds: whenever a closed walk has trivial voltage then all closed walks corresponding to the respective word must have trivial voltage. This condition is equivalent to saying that the equivariant group lifts, see Section 8 and [35].

Example 5.8 Let $\tilde{M} \to M$ be a regular homomorphism of oriented maps, where M is a regular map (see Section 7). Since the action graph $\mathrm{Act}(\tilde{M})$ can be reconstructed from the action graph $\mathrm{Act}(M)$ as in Theorem 5.6 (see Theorem 6.1), it follows that $\tilde{M}$ is a regular map if and only if $\mathrm{Aut}\, M$ lifts (see Section 7 and Example 8.9, and also [36]). ■

6 Reconstruction

Let $(\phi, q) : (\tilde{Z}, \tilde{G}) \to (Z, G)$ be a covering of transitive actions, let $\tilde{S} \subset \tilde{G}$ be a generating Cayley (multi)set and $S = q(\tilde{S})$ a (multi)set in bijective correspondence with $\tilde{S}$. We would like to reconstruct the action graph $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ from $\mathrm{Act}(Z, G; S)$ in terms of voltages.

This can be done by means of a *canonical* voltage space $(\tilde{Z}, \tilde{G}; \xi)$ (relative to (ϕ, q)) on $\mathrm{Act}(Z, G; S)$, with voltages on darts defined by the rule $\xi_{zs} = \tilde{s} = q^{-1}(s)$. The derived covering graph $\mathrm{Cov}(\tilde{Z}, \tilde{G}; \xi)$ has vertex-set $\tilde{V} = Z \times \tilde{Z}$ and the dart-set $\tilde{D} = Z \times S \times \tilde{Z}$. The incidence function $\mathrm{beg} : \tilde{D} \to \tilde{V}$ is given by the projection $\mathrm{beg}(z, s, \tilde{z}) = (z, \tilde{z})$, and the switching involution is $\mathrm{inv}(z, s, \tilde{u}) = (z \cdot s, s^{-1}, \tilde{u} \cdot \tilde{s})$, where $\tilde{s} = q^{-1}(s)$. The corresponding local group is $\mathrm{Loc}^b = q^{-1}(G_b)$. Its action on $\tilde{Z}$ is, modulo relabeling, the same as the action of $\mathcal{W}^b$ on the vertex fibre over b in $\mathrm{Cov}(\tilde{Z}, \tilde{G}; \xi)$. But $\mathcal{W}^b$ and $q^{-1}(G_b)$ also act on the fibre fib_b in $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$. In fact, $(\mathrm{id}, \xi) : (\mathrm{fib}_b, \mathcal{W}^b) \to (\mathrm{fib}_b, q^{-1}(G_b))$ is an invariant covering of actions.

Theorem 6.1 *With the notation above, the component* $\mathrm{C}(b, \tilde{b})$ *of* $\mathrm{Cov}(\tilde{Z}, \tilde{G}; \xi)$ *containing the vertex* $(b, \tilde{b})$, $b = \phi(\tilde{b})$, *consists exactly of all vertices of the form* $(z, \tilde{z})$, $z = \phi(\tilde{z})$, *and the restriction* $p_\xi : \mathrm{C}(b, \tilde{b}) \to \mathrm{Act}(Z, G; S)$ *is equivalent to* $p_{\phi,q}$:

$\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$. *(If $\tilde{G}$ is a permutation group, then the action graph structure imposed on* $\mathrm{C}(b, \tilde{b})$ *as in Theorem 5.6 coincides with* $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$.)

Next, all restrictions of $p_\xi : \mathrm{Cov}(\tilde{Z}, \tilde{G}; \xi) \to \mathrm{Act}(Z, G; S)$ *to its connected components are equivalent to* $p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ *if and only if the restrictions of the action of* $q^{-1}(G_b)$ *on all its orbits have the same conjugacy class of stabilizers. In particular, let the action of* $\tilde{G}$ *be such that no stabilizer is properly contained in another stabilizer (say, the group is finite). Then all restrictions of* p_ξ *to its connected components are equivalent to* $p_{\phi,q}$ *if and only if* $q(\tilde{G}_{\tilde{b}}) \leq G_Z$.

PROOF. Let $(z, \tilde{u})$ be in the same component as $(b, \tilde{b})$, $b = \phi(\tilde{b})$. Then $\tilde{u} = \tilde{b} \cdot (\tilde{s}_1 \tilde{s}_2 \ldots \tilde{s}_n)$ and $\phi(\tilde{u}) = b \cdot (s_1 s_2 \ldots s_n) = z$. So all vertices in this component are of the required form. If $(z, \tilde{z})$ and $(u, \tilde{u})$ have the same label $\tilde{z} = \tilde{u}$, then $z = \phi(\tilde{z}) = \phi(\tilde{u}) = u$. Hence no two vertices are labeled by the same label. Moreover, all labels from $\tilde{Z}$ actually appear since $\tilde{S}$ generates $\tilde{G}$ and $\tilde{G}$ is transitive on $\tilde{Z}$. The fibre over b in $\mathrm{C}(b, \tilde{b})$ is labelled by the orbit of Loc^b on $\tilde{Z}$ which is precisely $\mathrm{fib}_b = \phi^{-1}(b)$. It follows that the actions of $\mathcal{W}^b$ on fibres over b in $\mathrm{C}(b, \tilde{b})$ and $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ are essentially the same. Hence $p_\xi : \mathrm{C}(b, \tilde{b}) \to \mathrm{Act}(Z, G; S)$ and $p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ are equivalent. The explicit graph isomorphism which establishes this equivalence is $(z, \tilde{z}) \mapsto \tilde{z}$ on vertices and $(z, s, \tilde{z}) \mapsto (\tilde{z}, \tilde{s})$ on darts. If $\tilde{G}$ is faithful, then the action graph structure imposed on $\mathrm{C}(b, \tilde{b})$ as in Theorem 5.6 obviously coincides with $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$.

Clearly, all restrictions of p_ξ to the components of $\mathrm{Cov}(\tilde{Z}, \tilde{G}; \xi)$ are equivalent if and only if the induced actions of $\mathrm{Loc}^b = q^{-1}(G_b)$ on its orbits in $\tilde{Z}$ are equivariantly isomorphic, that is, all of these actions must have the same conjugacy class of stabilizers in Loc^b.

Observe that $\mathrm{Loc}^b_{\tilde{z}} = \tilde{G}_{\tilde{z}}$ for $\tilde{z} \in \phi^{-1}(b)$, and $\mathrm{Loc}^b_{\tilde{u}} = \tilde{G}_{\tilde{u}} \cap \mathrm{Loc}^b$ for $\tilde{u} \notin \phi^{-1}(b)$. Now, if all stabilizers of $\tilde{G}$ are contained in Loc^b, then all the above actions of Loc^b do have the same conjugacy class of stabilizers. Conversely, the fact that each $\mathrm{Loc}^b_{\tilde{u}} = \tilde{G}_{\tilde{u}} \cap \mathrm{Loc}^b$ is also the stabilizer of some point $\tilde{z} \in \phi^{-1}(b)$ implies $\tilde{G}_{\tilde{z}} \leq \tilde{G}_{\tilde{u}}$. Since the action of $\tilde{G}$ is not pathological, we have equality, and hence each stabilizer of $\tilde{G}$ must be contained in $\mathrm{Loc}^b = q^{-1}(G_b)$. This condition can be further rephrased as follows. Since q is onto and $\tilde{G}_{\tilde{z}}$, $\tilde{z} \in \tilde{Z}$ are conjugate subgroups, we have $g^{-1} q(\tilde{G}_{\tilde{b}}) g \leq G_b$ for all $g \in G$. Thus, $q(\tilde{G}_{\tilde{b}}) \leq G_Z$. Conversely, if $q(\tilde{G}_{\tilde{b}}) \leq G_Z$, then $q(\tilde{G}_{\tilde{z}}) = g^{-1} q(\tilde{G}_{\tilde{b}}) g \leq g^{-1} G_Z g = G_Z \leq G_b$. This completes the proof. ■

Example 6.2 The action graph $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ can be reconstructed from $\mathrm{Act}(Z, G; S)$ by taking just any connected component of the derived covering, for instance, when $\tilde{G}$ (or G) acts with a normal stabilizer. A special case is encountered when reconstructing Cayley graphs form Cayley graphs of quotient groups [19]. ■

Example 6.3 Consider the dihedral group $G = D_{18} = \langle R, L \mid R^9 = L^2 = (RL)^2 = 1\rangle$ with subgroups $H = \langle 1, L\rangle \cong \mathbb{Z}_2$ and $H' = \langle R^3, L\rangle \cong S_3$, and let $S = \{R, R^{-1}, L\}$. Then $\mathrm{Sch}(G, H; S)$ equivariantly covers $\mathrm{Sch}(G, H'; S)$. Since not all conjugates of H' contain H, the graph $\mathrm{Sch}(G, H; S)$ cannot be reconstructed by taking just any connected component of the derived covering $\mathrm{Cov}(H|G, G; \xi)$.

In this particular case we can also apply the Burnside-Frobenius counting lemma. Namely, the action of H' on the cosets of H has two orbits, whereas the derived covering is 9-fold. ■

7 Automorphisms

We henceforth assume that actions are transitive and that Cayley (multi)sets generate the groups in question. A bijective self-mapping $\tau + \theta$ of $Z + Z \times S$ is an automorphism of $\mathrm{Act}(Z, G; S)$ if and only if $\theta(z, s) = (\tau(z), \theta_z(s))$, where $\{\theta_z \mid z \in Z\}$ is a collection of bijective self-mappings of S satisfying $\tau(z \cdot s) = \tau(z) \cdot \theta_z(s)$ and $(\theta_z(s))^{-1} = \theta_{z \cdot s}(s^{-1})$. Studying general automorphisms, even the subgroup of structure-preserving ones, is difficult. Next in the line is the subgroup $\mathrm{Aut}^S(Z, G)$ of *action-automorphisms*, $\tau + \tau \times \psi$, where ψ extends to (or just is) an automorphism of G. Proposition 5.3 implies Proposition 7.1.

Proposition 7.1 *Let a transitive action (Z, G) be faithful, and let $S \subset G$ be a generating Cayley (multi)set. Then each automorphism of $\mathrm{Act}(Z, G; S)$ of the form $\tau + \tau \times \psi$ is an action-automorphism.* ■

Let $\mathrm{Aut}^S G = \{\psi \in \mathrm{Aut}\, G \mid \psi(S) = S\}$ denote the subgroup of *S-automorphisms* of G, and let $\mathrm{Adm}_Z^S G = \mathrm{Aut}^S G \cap \mathrm{Adm}_Z G$ denote its *S-admissible subgroup* formed by automomorphisms which preserve S as well as the conjugacy class of stabilizers of G (as a set). Of course, $\tau + \tau \times \psi$ is an action-automorphism of $\mathrm{Act}(Z, G; S)$ if and only if $(\tau, \psi) \in \mathrm{Aut}(Z, G)$ with $\psi \in \mathrm{Adm}_Z^S G$. In particular, we can identify $\mathrm{Eq}(Z)_G$, the *equivariant group of automorphisms* of $\mathrm{Act}(Z, G; S)$, and the equivariant group of the action, $\mathrm{Aut}(Z)_G$. It is easy to see that the projection $\mathrm{Aut}^S(Z, G) \to \mathrm{Adm}_Z^S G$, $(\tau, \psi) \mapsto \psi$ is a group epimorphism with kernel $\mathrm{Aut}(Z)_G$. We state this observation formally.

Theorem 7.2 *The group $\mathrm{Aut}^S(Z, G)$ of action-automorphisms of $\mathrm{Act}(Z, G; S)$ is isomorphic to an extension of the equivariant group $\mathrm{Aut}(Z)_G$ by $\mathrm{Adm}_Z^S G$.* ■

While $\mathrm{Aut}(Z)_G$ is isomorphic to $\mathrm{N}_G(G_b)/G_b$, the identification of $\mathrm{Adm}_Z^S G$ and the extension itself might not be easy. Here is a simple and well-known example.

Example 7.3 In view of Proposition 7.1 there are two kinds of automorphism of Cayley graphs. Those for which the mapping of darts is not constant at all points, and those which are action-automorphisms. Let $\tau + \tau \times \psi$ be an action-automorphism. Then $\tau(g) = \tau_{a,\psi}(g) = a\psi(g)$, for some $a \in G$ and $\psi \in \mathrm{Adm}_G^S G = \mathrm{Aut}^S G$. Moreover, the assignment $(a, \psi) \mapsto (\tau_{a,\psi}, \psi)$ is an isomorphism $G \rtimes_{\mathrm{id}} \mathrm{Aut}^S G \to \mathrm{Aut}^S(G, G)_r$. Hence the group of action-automorphisms $\mathrm{Aut}^S(G, G)_r$ of $\mathrm{Cay}(G; S)$ is isomorphic to a subgroup in the holomorph of G.

The group $\mathrm{Aut}^S(G, G)_r$ can as well be characterized as the normalizer of the left regular representation $\{\tau_{a,\mathrm{id}} \mid a \in G\}$ of G within the full automorphism group of $\mathrm{Cay}(G; S)$ [18]. ■

Example 7.4 From the very definition it follows that the automorphism group $\mathrm{Aut}\, M$ of an oriented map M can be identified with the equivariant group $\mathrm{Aut}(D)_{\langle R,L \rangle}$ [25]. ■

Example 7.5 An oriented map is called *regular* if its automorphism group acts transitively (and hence regularly) on the dart-set of the underlying graph. Recall that the equivariant group acts regularly if and only if the group itself is regular. Therefore, a map is regular if and only if $\mathrm{Map}(D; R, L)$ is a Cayley graph for the group $\langle R, L \rangle$ [25, 40]. Since any $\mathrm{Map}(D; R, L)$ is equivariantly covered by $\mathrm{Cay}(\langle R, L \rangle; \{R, R^{-1}, L\})$, every map is a (branched) regular quotient of some regular map [25, 26, 46]. The idea extends to maps on bordered surfaces [5]. ■

As a last remark, let us ask what are the necessary and sufficient conditions for a transitive faithful action (Z, G) to extend to a group of automorphisms of $\mathrm{Act}(Z, G; S)$. This problem is of interest [45] (however, action graphs in [45] differ from ours), and difficult in general. But let us assume that the induced automorphism group preserves the natural structure as a covering of $\mathrm{mnp}(S)$. Then the answer is trivial.

Proposition 7.6 *Let the action (Z, G) be transitive and faithful, and let $S \subset G$ be a generating Cayley (multi)set. Then $T(G) = \{\tau_g + \tau_g \times \psi_g \mid g \in G\}$, where $\tau_g(z) = z \cdot g$, is a group of (action) automorphisms of $\mathrm{Act}(Z, G; S)$ if and only if S is a union of conjugacy classes. In particular, $T(G)$ acts as a subgroup of the equivariant group if and only if the action graph is a Cayley graph $\mathrm{Cay}(G; S)$ and G is abelian.* ■

Other types of symmetries of action graphs will not be discussed here. Arc-transitivity of Cayley digraphs is considered, for instance, in [2, 22].

8 Lifting and projecting automorphisms

Let $p_{\phi,q}: \mathrm{Act}(\tilde{Z},\tilde{G};\tilde{S}) \to \mathrm{Act}(Z,G;S)$ be a covering arising from transitive actions, where $\tilde{S}$ and $S = q(\tilde{S})$ are generating Cayley (multi)sets. The problem of lifting automorphisms has recently obtained considerable attention [4, 5, 6, 9, 15, 23, 24, 29, 33, 35, 36, 47, 49, 51] in various contexts. From the general theory we infer that the lifting condition, expressed in terms of the canonical voltages valued in $\tilde{G}$, reads: An automorphism $\tau + \theta$ lifts along $p_{\phi,q}$ if and only if there exists $\tilde{z} \in \mathrm{fib}_{\tau b}$ such that, for each $W \in \mathcal{W}^b$,

$$\xi_W \in \tilde{G}_{\tilde{b}} \text{ if and only if } \xi_{\theta W} \in \tilde{G}_{\tilde{z}}. \tag{3}$$

However, we here explore the possibility that the lifting condition be expressed without the usual explicit reference to mappings of closed walks and their voltages, but, rather, expressed in terms of certain subgroups of G and $\tilde{G}$. To this end we have, in spite of the fact that coverings arising from actions are somewhat peculiar, restrict our considerations either to a very special class of action-automorphisms (the equivariant group), or else to action-automorphisms with addtional requirements imposed on the covering of actions (Example 8.2). We also note that the lifts of action-automorphisms, although structure-preserving, need not be action-automorphisms.

Example 8.1 Let $\tilde{G}$ be faithful. Then, by Proposition 7.1, a lift of an action-automorphism of $\mathrm{Act}(Z,G;S)$ is an action-automorphism of $\mathrm{Act}(\tilde{Z},\tilde{G};\tilde{S})$,. ∎

Example 8.2 Suppose that a covering of actions is equivariant. Then the conclusion as in the previous example holds, too. Moreover, from (3) we easily derive that an action-automorphism $\tau + \tau \times \psi$ lifts if and only if there exists $g \in G$ such that $\tau(b) = b \cdot g$ and $\psi(G_{\tilde{b}}) = g^{-1} G_{\tilde{b}} g$. (An alternative direct proof avoiding (3) is readily at hand and is left to the reader.) In particular, action-automorphisms do lift along the equivariant covering $\mathrm{Cay}(G;S) \to \mathrm{Act}(Z,G;S)$. ∎

We now focus our attention on the equivariant group $\mathrm{Eq}(Z)_G$ of $\mathrm{Aut}(Z,G;S)$. That one can derive a reasonable lifting condition in terms of subgroups of $\tilde{G}$ and G is not surprising because with equivariant automorphisms we could as well consider just coverings of actions.

Theorem 8.3 *The equivariant group* $\mathrm{Eq}(Z)_G$ *lifts along* $p_{\phi,q} : \mathrm{Act}(\tilde{Z},\tilde{G};\tilde{S}) \to \mathrm{Act}(Z,G;S)$ *if and only if* $q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}))$ *intersects every coset of* G_b *within* $\mathrm{N}_G(G_b)$*, and the lifted group is then a subgroup of the equivariant group* $\mathrm{Eq}(\tilde{Z})_{\tilde{G}}$*. In particular, if the covering projection is regular, then* $\mathrm{Eq}(Z)_G$ *lifts if and only if* $\mathrm{N}_G(G_b) \leq q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}))$ *(equivalently,* $q^{-1}(\mathrm{N}_G(G_b)) \leq \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$*).*

PROOF. We present a proof which avoids the reference to (3). Clearly, a lift of an equivariant automorphism is equivariant. Consider a pair of equivariant automorphisms of the covering graph and of the base graph, respectively. Their action on vertices is given, relative to $\tilde{b}$ and $b = \phi(\tilde{b})$, by $\tilde{\tau}_{\tilde{c}}(\tilde{b}\cdot\tilde{g}) = \tilde{b}\cdot\tilde{c}\tilde{g}$ where $\tilde{c} \in \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}) \bmod \tilde{G}_{\tilde{b}}$, and $\tau_c(b\cdot g) = b\cdot cg$, where $c \in \mathrm{N}_G(G_b) \bmod G_b$. Let $\tilde{\tau}_{\tilde{c}} + \tilde{\tau}_{\tilde{c}} \times \mathrm{id}$ be a lift of $\tau_c + \tau_c \times \mathrm{id}$, that is, let $\phi\tilde{\tau}_{\tilde{c}} = \tau_c\phi$. We easily get $q(\tilde{c}) \in G_b c$. Conversely, if c and $\tilde{c}$ satisfy this condition, then $\tilde{\tau}_{\tilde{c}} + \tilde{\tau}_{\tilde{c}} \times \mathrm{id}$ is a lift of $\tau_c + \tau_c \times \mathrm{id}$. The lifting condition can now be expressed as: for each $c \in N_G(G_b)$ there exists $\tilde{c} \in \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$ such that $q(\tilde{c}) \in G_b c$. The claim follows.

The covering projection $p_{\phi,q}$ is regular if and only $q^{-1}(G_b) \leq \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$, by Theorem 9.1 below. This implies $G_b \leq q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}))$, and hence the lifting condition now obviously reduces to $\mathrm{N}_G(G_b) \leq q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}))$. The alternative form follows because $\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$ contains $\mathrm{Ker}\, q$. ■

Example 8.4 Let the group $\tilde{G}$ act with a normal stabilizer. Then the equivariant group lifts. In particular, $\mathrm{Eq}(Z)_G$ lifts along $p_{\phi,q} : \mathrm{Cay}(\tilde{G};\tilde{S}) \to \mathrm{Act}(Z,G;S)$. ■

Example 8.5 Let $\tilde{M} \to M$ be a homomorphism of oriented maps. If $\tilde{M}$ is a regular map, then $\mathrm{Aut}\, M$ lifts [36]. ■

Example 8.6 The lift of the equivariant group $\mathrm{Eq}(Z)_G$ along a regular covering projection $p_{\phi,q}$ is isomorphic to $q^{-1}(\mathrm{N}_G(G_b))/\tilde{G}_{\tilde{b}}$. ■

Proposition 8.7 *Let the covering* $p_{\phi,q} : \mathrm{Act}(\tilde{Z},\tilde{G};\tilde{S}) \to \mathrm{Cay}(G;S)$ *be regular. If the equivariant group of* $\mathrm{Cay}(G;S)$ *lifts, then* $\mathrm{Act}(\tilde{Z},\tilde{G};\tilde{S})$ *is isomorphic to the Cayley graph* $\mathrm{Cay}(\tilde{G}/\tilde{G}_{\tilde{Z}};\tilde{S})$.

PROOF. By Theorem 8.3 we have $q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})) = G$. Thus $\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$ intersects every coset of $\mathrm{Ker}\, q$. Since $\mathrm{Ker}\, q \leq q^{-1}(G_b)$ and $q^{-1}(G_b) \leq \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$ (see Theorem 9.1 below), $\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$ contains $\mathrm{Ker}\, q$. Hence $\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}) = \tilde{G}$, and the proof follows. ■

Example 8.8 Consider a regular covering $p_{\phi,q} : \mathrm{Act}(\tilde{Z},\tilde{G};\tilde{S}) \to \mathrm{Cay}(G;S)$, where $\tilde{G}$ is faithful. Then $\mathrm{Act}(\tilde{Z},\tilde{G};\tilde{S})$ is isomorphic to the Cayley graph $\mathrm{Cay}(\tilde{G};\tilde{S})$ if and only if the equivariant group of $\mathrm{Cay}(G;S)$ lifts, by Example 8.4 and Proposition 8.7. In view of the lifting condition (3) we may rephrase this as in Section 5. ■

Example 8.9 Let $\tilde{M} \to M$ be a regular homomorphism, where M is a regular map. If $\mathrm{Aut}\, M$ lifts, then $\tilde{M}$ is also a regular map. In view of Example 8.5 we obtain the if and only if statement of Example 5.8. See also [20, 36]. ■

Let us now consider projecting automorphisms of $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ along $p_{\phi,q}$. An automorphism $\tilde{\tau}+\tilde{\theta}$ projects whenever each vertex-fibre is mapped onto some vertex-fibre and each dart-fibre is mapped onto some dart-fibre. If the covering is regular, then an automorphism projects if and only if it normalizes the group of covering transformations. This is actually a theorem of Macbeath [37] which holds for general topological coverings as well as in our combinatorial context. A similar result for digraphs is proved in [39].

Note that projections of action-automorphisms are structure-preserving but need not be action-automorphisms. One particular instance when such projections are indeed action-automorphisms is when the covering is equivariant. The case when G acts faithfully is another. In general, the following holds.

Proposition 8.10 *Suppose that an action-automorphism* $\tilde{\tau}+\tilde{\tau}\times\tilde{\psi}$ *of* $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ *projects along* $p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$. *Then the projected automorphism is an action-automorphism of* $\mathrm{Act}(Z, G; S)$ *if and only* $\mathrm{Ker}\, q$ *is invariant for* $\tilde{\psi}$.

Proof. Let $\tau+\tau\times\psi$ be the projected automorphism. We know that ψ is defined as $\psi = q\tilde{\psi}q^{-1}$ on S. Now ψ extends to an automorphism of G if and only if $\psi(s_1)\ldots\psi(s_n) = 1$ whenever $s_1 \ldots s_n = 1$. Equivalently, we must have $q\tilde{\psi}(\tilde{s}_1 \ldots \tilde{s}_n) = 1$ whenever $(\tilde{s}_1 \ldots \tilde{s}_n) \in \mathrm{Ker}\, q$, that is, $\tilde{\psi}(\tilde{s}_1 \ldots \tilde{s}_n) \in \mathrm{Ker}\, q$ whenever $(\tilde{s}_1 \ldots \tilde{s}_n) \in \mathrm{Ker}\, q$. ■

Theorem 8.11 *An action-automorphism* $\tilde{\tau}+\tilde{\tau}\times\tilde{\psi}$ *of* $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ *projects along* $p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ *if and only if there exists* $\tilde{g} \in \tilde{G}$ *such that*

$$\tilde{\tau}\tilde{b} = \tilde{b}\cdot\tilde{g} \quad \text{and} \quad \tilde{\psi}(q^{-1}(G_b)) = \tilde{g}^{-1}(q^{-1}(G_b))\tilde{g}.$$

Proof. First of all, if an action-automorphism is vertex-fibre preserving, then it is also dart-fibre preserving. Moreover, it is enough to require that just one vertex-fibre is mapped to a fibre. The proof of this fact is left to the reader. It follows that $\tilde{\tau}+\tilde{\tau}\times\tilde{\psi}$ projects if and only if $\tilde{\psi}$ maps $q^{-1}(G_b)$ onto $q^{-1}(G_{\phi\tilde{\tau}\tilde{b}})$, where $\tilde{b} \in \mathrm{fib}_b$. Writing $\tilde{\tau}\tilde{b} = \tilde{b}\cdot\tilde{g}$ and taking into account that the stabilizers are conjugate subgroups, we obtain the desired result. ■

Example 8.12 An action-automorphism $\tilde{\tau}+\tilde{\tau}\times\tilde{\psi}$ of $\mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ projects along $p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Cay}(G; S)$ if and only if $\tilde{\psi}(\mathrm{Ker}\, q) = \mathrm{Ker}\, q$, and the projection is necessarily an action-automorphism. ■

Theorem 8.13 *The group* $\mathrm{Eq}(\tilde{Z})_{\tilde{G}}$ *projects along* $p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ *if and only if* $\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}) \leq \mathrm{N}_{\tilde{G}}(q^{-1}(G_b))$ *(equivalently,* $q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})) \leq \mathrm{N}_G(G_b)$*). The projected group is a subgroup of* $\mathrm{Eq}(Z)_G$. *For regular coverings, the condition simplifies to* $q^{-1}(G_b) \lhd \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$.

PROOF. Clearly, the projection of an equivariant automorphism is equivariant. By Theorem 8.11, an automorphism $\tilde{\tau}_{\tilde{c}} + \tilde{\tau}_{\tilde{c}} \times \mathrm{id}$, where $\tilde{\tau}_{\tilde{c}}(\tilde{b} \cdot \tilde{g}) = \tilde{b} \cdot \tilde{c}\tilde{g}$ and $\tilde{c} \in \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}) \bmod \tilde{G}_{\tilde{b}}$, projects if and only if there exists $\tilde{g} \in \tilde{G}$ such that $\tilde{b} \cdot \tilde{c} = \tilde{b} \cdot \tilde{g}$ and $\tilde{g} \in \mathrm{N}_{\tilde{G}}(q^{-1}(G_b))$. This is equivalent to saying that $\mathrm{N}_{\tilde{G}}(q^{-1}(G_b))$ intersects the coset $\tilde{G}_{\tilde{b}}\tilde{c}$. Since $\tilde{c} \in \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$ was arbitrary, $\mathrm{N}_{\tilde{G}}(q^{-1}(G_b))$ should intersect every coset of $\tilde{G}_{\tilde{b}}$ within $\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$. But $\tilde{G}_{\tilde{b}} \leq q^{-1}(G_b) \leq \mathrm{N}_{\tilde{G}}(q^{-1}(G_b))$, implying that $\mathrm{N}_{\tilde{G}}(q^{-1}(G_b))$ should contain $\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$, as claimed. The alternative form is also evident. (A direct proof similar to the proof of Theorem 8.3 is left to the reader.) The rest follows by Theorem 9.1. ■

Example 8.14 Let G act with a normal stabilizer. Then the group $\mathrm{Eq}(\tilde{Z})_{\tilde{G}}$ projects. In particular, it projects along $p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Cay}(G; S)$. ■

Example 8.15 Let $\tilde{M} \to M$ be a homomorphism of oriented maps, where M is a regular map. Then $\mathrm{Aut}\,\tilde{M}$ projects [36]. ■

Corollary 8.16 *Let the group* $\mathrm{Eq}(\tilde{G})_{\tilde{G}}$ *project along* $p_{\phi,q} : \mathrm{Cay}(\tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$. *Then* $\mathrm{Act}(Z, G; S)$ *is isomorphic to the Cayley graph* $\mathrm{Cay}(G/G_Z; S)$.

PROOF. By Theorem 8.13 we have $q^{-1}(G_b) \triangleleft \tilde{G}$. Hence G_b is normal in G, and the proof follows. ■

Example 8.17 Let $\tilde{M} \to M$ be a homomorphism of oriented maps, where $\tilde{M}$ is a regular map. In view of Example 8.15 and Corollary 8.16 the group $\mathrm{Aut}\,\tilde{M}$ projects if and only if M is also a regular map. (The homomorphism itself must then be regular, see Example 9.3.) ■

Corollary 8.18 *Let the covering projection* $p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ *be regular. Then the equivariant group* $\mathrm{Eq}(Z)_G$ *lifts and the equivariant group* $\mathrm{Eq}(\tilde{Z})_{\tilde{G}}$ *projects if and only if* $q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})) = \mathrm{N}_G(G_b)$. *In this case,* $\mathrm{Eq}(Z)_G$ *lifts to* $\mathrm{Eq}(\tilde{Z})_{\tilde{G}}$ *and* $\mathrm{Eq}(\tilde{Z})_{\tilde{G}}$ *projects onto* $\mathrm{Eq}(Z)_G$.

PROOF. By Theorems 8.3 and 8.13 we must have $\mathrm{N}_G(G_b) \leq q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}))$ and $q(\mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})) \leq \mathrm{N}_G(G_b)$, and the claim follows. The last statement is evident as well. ■

Example 8.19 The projection $p_{\phi,q} : \mathrm{Cay}(\tilde{G}; \tilde{S}) \to \mathrm{Cay}(G; S)$ is regular (see Example 9.3). The left regular representation of G lifts to the left regular representation of $\tilde{G}$, and hence the latter projects onto the former. In particular, if $\tilde{M} \to M$ is a homomorphism of regular maps, then $\mathrm{Aut}\,M$ lifts to $\mathrm{Aut}\,\tilde{M}$ and $\mathrm{Aut}\,\tilde{M}$ projects onto $\mathrm{Aut}\,M$ [36]. ■

9 The structure of lifted groups

Theorem 9.1 *Let a covering projection* $p = p_{\phi,q} : \mathrm{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \mathrm{Act}(Z, G; S)$ *arise from a covering of transitive actions, where* $\tilde{S}$ *and* $S = q(\tilde{S})$ *are generating Cayley (multi)sets. Choose* $\tilde{b} \in \tilde{Z}$ *and* $b = \phi(\tilde{b})$ *as base-points. Then:*

(a) $\mathrm{CT}(p) = \{\tilde{\tau} + \tilde{\tau} \times \mathrm{id} \mid \tilde{\tau} \in \mathrm{Aut}(\tilde{Z})_{\tilde{G}},\ \phi\tilde{\tau} = \phi\}$ *is a subgroup of* $\mathrm{Eq}(\tilde{Z})_{\tilde{G}}$.

(b) $\mathrm{CT}(p) = \{\tilde{\tau} + \tilde{\tau} \times \mathrm{id} \mid \tilde{\tau}(\tilde{b} \cdot \tilde{g}) = \tilde{b} \cdot \tilde{a}\tilde{g},\ \tilde{a} \in (q^{-1}(G_b) \cap \mathrm{N}(\tilde{G}_{\tilde{b}})) \bmod \tilde{G}_{\tilde{b}}\}$ *is isomorphic to* $(q^{-1}(G_b) \cap \mathrm{N}(\tilde{G}_{\tilde{b}}))/\tilde{G}_{\tilde{b}}$.

(c) *The covering projection* p *is* $[q^{-1}(G_b) : \tilde{G}_{\tilde{b}}]$*-fold, and is regular if and only if* $\tilde{G}_{\tilde{b}} \triangleleft q^{-1}(G_b)$ *(equivalently,* $q^{-1}(G_b) \le \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}})$*).*

PROOF. The statement (a) is obvious. Since $\tilde{G}$ is transitive on $\tilde{Z}$ we can explicitly calculate the elements of $\mathrm{Aut}(\tilde{Z})_{\tilde{G}}$ relative to $\tilde{b}$ as $\tilde{\tau}(\tilde{b} \cdot \tilde{g}) = \tilde{b} \cdot \tilde{c}\tilde{g}$, where $\tilde{c} \in \mathrm{N}_{\tilde{G}}(\tilde{G}_{\tilde{b}}) \bmod \tilde{G}_{\tilde{b}}$. Now $\phi\tilde{\tau} = \phi$ implies $b \cdot q(\tilde{c}) = \phi(\tilde{b} \cdot \tilde{c}) = \phi(\tilde{\tau}(\tilde{b})) = \phi(\tilde{b}) = b$. Consequently, $\tilde{c} \in q^{-1}(G_b)$, giving (b).

The first statement of (c) follows from the fact that the covering is connected. Indeed, the group $q^{-1}(G_b)$ acts transitively on the fibre $\phi^{-1}(b)$ and has $\tilde{G}_{\tilde{b}}$ as its stabilizer. The covering is regular, by definition, if $\mathrm{CT}(p)$ is transitive on $\phi^{-1}(b) = \{\tilde{b} \cdot \tilde{a} \mid \tilde{a} \in q^{-1}(G_b)\}$. But this holds if and only if $q^{-1}(G_b) \le q^{-1}(G_b) \cap \mathrm{N}(\tilde{G}_{\tilde{b}})$, and part (c) follows. ■

Example 9.2 Let $H \le H' \le G$, and let $p = p_{\phi,\mathrm{id}} : \mathrm{Sch}(G, H; S) \to \mathrm{Sch}(G, H'; S)$ be the corresponding covering projection, where $\phi(Hg) = H'g$. Then $\mathrm{CT}(p) = \{\tilde{\tau} + \tilde{\tau} \times \mathrm{id} \mid \tilde{\tau}(Hg) = Hag,\ a \in (H' \cap \mathrm{N}(H)) \bmod H\}$ is isomorphic to $(H' \cap \mathrm{N}(H))/H$. The covering projection is $[H' : H]$-fold, and is regular if and only if $H \triangleleft H'$. ■

Example 9.3 A covering $p_{\phi,q} : \mathrm{Cay}(\tilde{G}; \tilde{S}) \to \mathrm{Act}(Z; G; S)$ is always regular. Hence a homomorphism $\tilde{M} \to M$ of oriented maps, where $\tilde{M}$ is a regular map, must be a regular homomorphism. In particular, homomorphisms between regular maps are regular [36]. ■

A lifted group of automorphisms is an extension of the group of covering transformations by the respective group of the base graph. This extension is difficult to analyze in general [9, 20, 33, 35, 36, 51]. We end this section by considering the case when the equivariant group lifts as a split extension along a covering projection arising from actions. As the equivariant group acts without fixed points, each orbit of an arbitrary complement to $\mathrm{CT}(p)$ within the lifted group intersects each fibre in at most one point (thus forming an *invariant transversal*). This is equivalent with the requirement that the covering projection be reconstructed by means of a voltage space for which the distribution of voltages is well behaved relative to the action of the equivariant group. The claim takes a particularly nice form whenever the covering projection is regular [33, 35]. A straightforward application of these considerations to regular homomorphisms of oriented maps gives Theorem 9.4. A direct proof in terms of voltages associated with angles of the map can be found in [36]. By $\mathcal{W}^\Omega$ we denote the set of all walks with endvertices in a subset of vertices Ω.

Theorem 9.4 *Let $\tilde{M} \to M$ be a regular homomorphism of oriented maps, and let Ω be an orbit (or a union of orbits) of a dart in M relative to* $\mathrm{Aut}\, M$. *Then* $\mathrm{Aut}\, M$ *lifts as a spit extension of* CT *(the lift of the identity automorphism) if and only if the covering projection of action graphs* $\mathrm{Act}(\tilde{M}) \to \mathrm{Act}(M)$ *can be reconstructed by means of a Cayley voltage space* $(\mathrm{CT}, \mathrm{CT}; \xi)$ *such that the set of walks* $\{W \in \mathcal{W}^\Omega \mid \xi_W = 1\}$ *in* $\mathrm{Act}(M)$ *is invariant for the action of* $\mathrm{Aut}\, M$. *Moreover, the extension is a direct product if and only if* $\mathrm{Act}(\tilde{M}) \to \mathrm{Act}(M)$ *can be reconstructed by a Cayley voltage space* $(\mathrm{CT}, \mathrm{CT}, \xi)$ *such that each of the sets* $\{W \in \mathcal{W}^\Omega \mid \xi_W = \nu\}$ $(\nu \in \mathrm{CT})$ *is invariant for the action of* $\mathrm{Aut}\, M$.

PROOF. The map automorphism group corresponds to the equivariant group in the associated action graph, and the lift of the identity corresponds to the group of covering transformations for the regular covering projection of the respective action graphs. The theorem follows by applying Theorems 9.1 and 9.3, and Corollaries 9.7 and 9.8 of [35]. ■

10 Generators and relations

Let $\overrightarrow{S} \subset G$ denote a nonempty antisymmetric subset, that is, $\overrightarrow{S} \cap \overrightarrow{S}^{-1}$ is either empty or else all of its elements are of order 2. With (Z, G) and $\overrightarrow{S}$ we associate the *action digraph* $\mathrm{Act}(Z, G; \overrightarrow{S})$ with the vertex-set Z and the arc-set $Z \times \overrightarrow{S}$, where $\mathrm{beg}(z, s) = z$ and $\mathrm{end}(z, s) = z \cdot s$. Like in the case of action graphs it is sometimes necessary to consider the set $\overrightarrow{S}$ as a multiset. The *underlying graph* of $\mathrm{Act}(Z, G; \overrightarrow{S})$ is the action graph $\underline{\mathrm{act}}(Z, G; \vec{S}) = \mathrm{Act}(Z, G; \overrightarrow{S} \cup \overrightarrow{S}^{-1})$. (Note that involutory loops collapse to semiedges.) Omitting formal basic definitions we only mention that epimorphisms of actions give rise to covering projections of action digraphs.

The study of group presentations involves a variety of techniques, see [13, 31, 32, 48] and the references therein. Action digraphs can often provide much insight to a formal algorithmic approach, and were (in disguise) at least partially present in the original works of Reidemeister and Schreier.

Choose a spanning tree in $\mathrm{Act}(Z, G; \vec{S})$, where (Z, G) is transitive and $\vec{S}$ generates G. Each cotree arc gives rise to a unique fundamental closed walk based at $z \in Z$, and the set of all such walks generates the set of all closed walks at z, up to reduction. Thus, if $\vec{\mathcal{C}}$ is a set of labels bijectively associated with all the cotree arcs, which evaluate to words in $(\vec{S} \cup \vec{S}^{-1})^*$ defined by the fundamental closed walks rooted at $z \in Z$, then each element of G_z, expressed as a word in $(\vec{S} \cup \vec{S}^{-1})^*$, can be written as a word in $(\vec{\mathcal{C}} \cup \vec{\mathcal{C}}^{-1})^*$. This is done by trailing the closed walk associated with a given word in $(\vec{S} \cup \vec{S}^{-1})^*$, and simultaneously keeping track of the labels in $\vec{\mathcal{C}} \cup \vec{\mathcal{C}}^{-1}$ when traversing a cotree arc. The process is known as the *rewriting process* relative to $z \in Z$. A variant of the Schreier-Reidemeister theorem now states the following. Let $G = \langle \vec{S}; \mathcal{R} \rangle$ be a presentation of G, and let $\mathrm{Rew}\mathcal{R}$ be the set of (reduced) words in $(\vec{\mathcal{C}} \cup \vec{\mathcal{C}}^{-1})^*$, obtained from all the relators in $\mathcal{R}$ by a rewriting process relative to all $z \in Z$. Then the stabilizer G_b has the presentation $\langle \vec{\mathcal{C}}; \mathrm{Rew}\mathcal{R} \rangle$. Many of the generators and relators obtained by this method can be redundant. However, sophisticated techniques for simplifying the presentation do exist in certain cases [12, 13, 32, 48, 52].

An action digraph $\mathrm{Act}(Z, G; \vec{S})$, where $\vec{S}$ generates G, obviously determines the group G/G_Z up to isomorphism (also if G is not transitive). Suppose that the action digraph is finite. Denote by $\vec{S_1}$ the generators of the stabilizer G_{b_1}, expressed as words in $(\overrightarrow{S} \cup \overrightarrow{S}^{-1})^*$ associated with fundamental closed walks at b_1 relative to a spanning tree in the appropriate component of $\mathrm{Act}(Z, G; \vec{S})$. By repeating this process on $\mathrm{Act}(Z \backslash \{b_1\}, G_{b_1}; \vec{S_1})$, $\mathrm{Act}(Z \backslash \{b_1, b_2\}, G_{b_1,b_2}; \vec{S_2})$ and so on, we can recursively construct a generating set for the pointwise stabilizer G_Z. Hence if G is faithful, we can find a presentation of G. If $|\vec{S}| = n$ and $|Z| = m$, then the number of generators of G_Z obtained in this way can amount up to $(n-1)m! + 1$. Thus, the method is not practical unless one can detect enough many redundant generators at each step, or has sufficient control over the recursive construction of the generators. As for the improvements which allow effective computer implementation we refer to [11, 12, 48] and the references therein.

Example 10.1 We leave to the reader to check that the following three permutations $a = (12)(45)(36)$, $b = (23)(56)(14)$ and $c = (13)(46)(25)$ in the symmetric group S_6 generate a subgroup which is isomorphic to S_4. ■

Despite the remarks above, $\mathrm{Act}(Z, G; \vec{S})$ proves useful in gathering at least partial information about the defining relations, particularly when its underlying graph is highly asymmetric with special structure. The idea is to use graph-theoretical properties of $\mathrm{Act}(Z, G; \vec{S})$ to derive such information. The following example is taken from [34].

Example 10.2 Consider the alternating group A_n, where $n \geq 11$ is odd, and the generators $a = (1, 2, 3, 4, 5, 6, 7, 8, \ldots, n)$ and $b = (3, 6, 1, 4, 5, 7, 8, \ldots, n)$. A careful analysis of the action digraph $\mathrm{Act}(\{1, 2, \ldots, n\}, A_n; \{a, b\})$ shows that in the Cayley graph $\mathrm{Cay}(A_n; \{a, a^{-1}, b, b^{-1}\})$, the cycles of girth-length, which is 6, arise essentially from the relation $(ab^{-1})^3 = 1$, and that cycles of length n arise essentially from the obvious relations $a^n = b^n = 1$. This information is crucial in proving that the above Cayley graph is 1/2-transitive. ■

What we have discussed so far can be applied to a problem encountered with lifts of automorphisms. Let $p : \mathrm{Cov}(F, \Gamma; \xi) \to X$ be a covering projection of connected graphs (or even more general topological spaces, see [33]), given by means of a voltage space $(F, \Gamma; \xi)$, where Γ acts faithfully on F. A necessary condition (also sufficient if the covering is regular) for an automorphism to lift is that the set of all closed paths with trivial voltage be invariant under its action [33, 35]. In order to test this effectively (assuming of course, that the covering has finite number of folds and that the fundamental group of X is finitely generated) we need the generators of the kernel of $\xi : \pi^b \to \Gamma$, expressed in terms of a generating set $\vec{S}$ of π^b, save for those cases where ad-hoc techniques apply. One possibility is to consider the auxiliary regular covering $\mathrm{Cov} = \mathrm{Cov}(\Gamma, \Gamma; \xi) \to X$. The required generators of $\mathrm{Ker}\,\xi$ are then obtained by projecting the generators of $\pi(\tilde{b}, \mathrm{Cov})$, where $\tilde{b} \in \mathrm{fib}_b$. However, this requires the construction of $\mathrm{Cov}(\Gamma, \Gamma; \xi)$, which is not always appropriate.

A better alternative is to consider the fundamental group π^b acting on Γ by right multiplication $\nu \cdot \alpha = \nu\xi(\alpha)$. The stabilizer of this action is $\mathrm{Ker}\,\xi$, and so the required generators can be found by means of a spanning tree in $\mathrm{Act}(\Gamma, \pi^b; \vec{S}) \cong \mathrm{Cay}(\Gamma; \xi(\vec{S}))$. It is close to first constructing the coset representatives of $\mathrm{Ker}\,\xi$, that is, finding a closed path (rooted at the based point) with voltage α, for all $\alpha \in \Gamma$, and then applying the Schreier method. Another possibility is to consider the action of π^b on the abstract fibre F given by $i \cdot \alpha = i \cdot \xi(\alpha)$. The kernel $\mathrm{Ker}\,\xi$ is then equal to the pointwise stabilizer of this action. Thus, the required generators can be found recursively by considering the action digraph $\mathrm{Act}(F, \pi^b; \vec{S})$.

A similar problem is to construct a generating set for the trivial voltage paths with endpoints in an orbit of a given group A of automorphisms of X. Namely, a necessary and sufficient condition for A to lift along a regular covering projection as a special kind of split extension of $\mathrm{CT}(p)$ (one with an invariant transversal [33, 35]) is that the set of paths as above be invariant for the action of A (recall Theorem 9.4). Suppose that X is a graph and Ω a vertex orbit. Introduce a new vertex B not in X, and connect B with all the vertices in Ω. Moreover, extend the voltage assignment so that these new edges carry the trivial voltage. The required generating set of walks is obtained in the same way as before by considering the extended graph with B as the base point. Note that the group A has the required type of lift if and only if, viewed as the stabilizer of B in the extended graph, lifts along the extended covering projection. (The idea clearly extends to finite CW-complexes.)

The preceding discussion is summarized in the following theorem.

Theorem 10.3 *With notation and assumptions above, the problem whether a given group A of automorphisms of a graph X (or a more general topological space) lifts along a regular covering projection given by a faithful voltage space $(F, \Gamma; \xi)$ can be tested in time, proportional to the number of generators of A, multiplied by the time required for the construction of* $\mathrm{Cay}(\Gamma; \xi(\vec{S}))$ *and its spanning tree (or multiplied by the time required for the construction of* $\mathrm{Act}(F, \pi^b; \vec{S})$ *and finding the generators of the pointwise stabilizer).*

If X is a graph (or a finite CW-complex), a similar statement holds for the problem whether a given group of automorphisms of X lifts as a split extension of $\mathrm{CT}(p)$ *with an invariant transversal.* ∎

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