University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 38 (2000), 703

ON COMPLETE INTERSECTIONS

Franc Forstnerič

ISSN 1318-4865

May 24, 2000

Ljubljana, May 24, 2000

ON COMPLETE INTERSECTIONS

Franc Forstnerič

&1. The results.

A closed complex submanifold Y of codimension d in a complex manifold X is a *holomorphic complete intersection* if there exist d holomorphic functions $f_1, \ldots, f_d \in \mathcal{O}(X)$ such that

$$Y = \{x \in X : f_1(x) = \ldots = f_d(x) = 0\} \tag{1}$$

and the differentials $df_j(x)$ $(1 \le j \le d)$ are $\mathbf{C}$-linearly independent at each point $x \in Y$. These differential together induce a trivialization of the complex normal bundle $N_Y = TX|_Y/TY$ of Y in X. There is a partial converse when X is a Stein manifold: *If the normal bundle N_Y is trivial then Y is a holomorphic complete intersection in some open neighborhood of Y in X.* The reason is that a neighborhood of Y in X is biholomorphic to a neighborhood of the zero section in N_Y [GR, p. 256]. Similarly, a smooth real submanifold Y of real codimension d in a smooth manifold X is a *differentiable complete intersection* if there exist d smooth real functions on X satisfying (1) and such that their differentials are $\mathbf{R}$-linearly independent at each point of Y. In this case the triviality of the normal bundle N_Y is a necessary and sufficient condition for Y to be a complete intersection in some open neighborhood in X. For further results on complete intersections we refer the reader to the papers [Sch] and [BK] and the references therein.

We show that there is a difference between these notions already in low dimensions:

1.1 Theorem. *There exists a three dimensional closed complex submanifold in $\mathbf{C}^5$ which is a differentiable complete intersection but not a holomorphic complete intersection.*

More precisely, given any compact orientable two dimensional surface M of genus $g \ge 2$ we shall construct a three dimensional Stein manifold Y which is homotopically equivalent to M and whose tangent bundle TY is trivial as a real vector bundle but is non-trivial as a complex vector bundle. We then show that any proper holomorphic embedding $Y \hookrightarrow \mathbf{C}^5$ (or $Y \hookrightarrow \mathbf{C}^7$) satisfies the conclusion of theorem 1.1. In fact, we prove

Theorem 1.2. *Let Y be a Stein manifold of dimension m whose tangent bundle is trivial as a real vector bundle, but is non-trivial as a complex vector bundle. Choose integers m and d such that*

(a) $d = 2$ and $m \in \{2, 3\}$, or

(b) $d = 4$ and $2 \le m \le 7$.

Then the image of any proper holomorphic embedding $Y \hookrightarrow \mathbf{C}^{m+d}$ is a differentiable complete intersection but not a holomorphic complete intersection in $\mathbf{C}^{m+d}$.

Multiplying our $Y^3 \subset \mathbf{C}^5$ by $\mathbf{C}^k$ we obtain similar examples in higher dimensions. Submanifolds of this type don't exist in $\mathbf{C}^n$ for $n \le 3$, but we don't know the answer for two dimensional submanifolds in $\mathbf{C}^4$. Recall that every smooth holomorphic curve in $\mathbf{C}^n$ is a holomorphic complete intersection [FRa], and so is every complex hypersurface in $\mathbf{C}^n$ (since all divisors on $\mathbf{C}^n$ are principal).

Example 1. There exists a Stein manifold X of dimension four and a closed complex submanifold $Y \subset X$ of dimension two such that Y is a differentiable complete intersection but not a holomorphic complete intersection in X. We can choose Y to have the homotopy type of the real two-sphere. (See proposition 2.4 in section 2). ♠

We shall discuss a more general problem of *removing intersections* of maps from Stein manifolds into $\mathbf{C}^d$ with certain analytic subvarieties $\Sigma \subset \mathbf{C}^d$. To motivate our main result, theorem 1.3, we first look at the complete intersections problem in the more general context of complex spaces. A closed complex subvariety Y of a complex space X is a holomorphic complete intersection in X if there exist $d = \dim X - \dim Y$ global sections of the analytic sheaf of ideals $\mathcal{J}_Y$ which generate this sheaf at each point of X. Consider the short exact sequence

$$0 \to \mathcal{J}_Y^2 \to \mathcal{J}_Y \to \mathcal{J}_Y/\mathcal{J}_Y^2 \to 0.$$

When Y is a local complete intersection of codimension d, the quotient $N_Y^* = \mathcal{J}_Y/\mathcal{J}_Y^2$ is a locally trivial analytic sheaf of rank d with support on Y, that is, a holomorphic vector bundle of rank d over Y. The dual bundle N_Y of N_Y^* is by definition the normal bundle of Y in X; in the smooth case this coincides with the usual definition of N_Y.

Suppose now that X is Stein and $Y \subset X$ is a local complete intersection of codimension d in X with normal bundle N_Y. If Y is a complete intersection then N_Y is trivial (since its dual bundle $N_Y^* = \mathcal{J}_Y/\mathcal{J}_Y^2$ is generated by the images of the generators of $\mathcal{J}_Y$ and hence is trivial). The following partial converse was obtained in 1966 by Forster and Ramspott [FRa] by using the Oka-Grauert homotopy principle from [Gra], [Car]:

Let Y be a local complete intersection of codimension d with trivial normal bundle in a Stein space X. Suppose that $U \subset X$ is an open set containing Y and the functions $f = (f_1, \ldots, f_d) \in \mathcal{O}(U)^d$ generate $\mathcal{J}_Y$ on U. If there is a continuous map $\widetilde{f}: X \to \mathbf{C}^d$ such that $\widetilde{f} = f$ near Y and $\widetilde{f}^{-1}(0) = Y$, then Y is a holomorphic complete intersection in X. Such $\widetilde{f}$ always exists if $\dim Y < \dim X/2$, or if $X = \mathbf{C}^n$ and $\dim Y \le 2(n-1)/3$.

M. Schneider proved that for any local complete intersection $Y \subset X$ with trivial normal bundle the sheaf $\mathcal{J}_Y$ admits $d+1$ generators (theorem 2.5 in [Sch]).

Suppose now that Σ is a closed complex subvariety of $\mathbf{C}^d$ and $f: X \to \mathbf{C}^d$ is a holomorphic map from a Stein manifold X. Let $Y \subset X$ be a connected component (or a union of such) of $f^{-1}(\Sigma)$. *When is it possible to modify f to a holomorphic map $g: X \to \mathbf{C}^d$ such that $g^{-1}(\Sigma) = Y$ and $g - f$ vanishes to a given order on Y ?* (The problem on complete intersections discussed above corresponds to the case $\Sigma = \{0\} \subset \mathbf{C}^d$.) A necessary condition is that we can modify f to a *continuous* map with the required properties, and we are interested in the corresponding *homotopy principle*. Example 2 below shows that we must restrict the class of subvarieties Σ to obtain positive results. Denote by $\mathrm{Aut}\mathbf{C}^d$ the group of all holomorphic automorphisms of $\mathbf{C}^d$.

Definition 1. *A closed complex subvariety* $\Sigma \subset \mathbf{C}^d$ *is said to be tame if there is a* $\Phi \in \mathrm{Aut}\mathbf{C}^d$ *such that* $\Phi(\Sigma) \subset \Gamma = \{(z', z_d) \in \mathbf{C}^d : |z_d| \le 1 + |z'|\}$.

Every proper complex algebraic subvariety of $\mathbf{C}^d$ is tame. Conversely, a subset $\Sigma \subset \mathbf{C}^d$ of pure dimension $d-1$ contained in Γ is algebraic [Chi], and hence $\Sigma^{d-1} \subset \mathbf{C}^d$ is tame if and only if it is equivalent to an algebraic subset by an automorphism of $\mathbf{C}^d$. For discrete sets our notion of tameness coincides with that of Rosay and Rudin [RR].

1.3 Theorem. (Removal of intersections.) *Let* Σ *be a closed complex analytic subvariety of* $\mathbf{C}^d$ *satisfying one of the following conditions:*

(a) Σ *is tame and* $\dim \Sigma \le d-2$;

(b) a complex Lie group acts holomorphically and transitively on $\mathbf{C}^d \backslash \Sigma$.

Let X *be a Stein manifold,* $f: X \to \mathbf{C}^d$ *a holomorphic map and* $Y \subset X$ *a union of connected components of* $f^{-1}(\Sigma)$. *If there is a continuous map* $\widetilde{f}: X \to \mathbf{C}^d$ *which equals* f *in a neighborhood of* Y *and satisfies* $\widetilde{f}^{-1}(\Sigma) = Y$, *then for each* $r \in \mathbf{N}$ *there is a holomorphic map* $g: X \to \mathbf{C}^d$ *such that* $g^{-1}(\Sigma) = Y$ *and* $g - f$ *vanishes to order* r *along* Y. *Such* g *always exists if* $\dim X < 2(d - \dim \Sigma)$, *or if* X *is contractible and* $\dim Y \le 2(d - \dim \Sigma - 1)$.

Theorem 1.3 is proved in section 4. By the Oka-Grauert-Gromov homotopy principle from [Gro], [FP1], [FP2] we reduce it to an extension problem for continuous maps to which we apply the obstruction theory. The following example shows that theorem 1.3 fails for non-tame subvarieties, independently of their codimension.

Example 2. For each $d \ge 1$ there is a discrete set $\Sigma \subset \mathbf{C}^d$ such that every holomorphic map $g: \mathbf{C}^d \to \mathbf{C}^d \backslash \Sigma$ has rank at most $d-1$. For $d = 1$ this holds already if Σ contains two points (the complement is hyperbolic); for $d > 1$ such discrete sets were constructed by Rosay and Rudin [RR]. For such Σ the conclusion of theorem 1.3 fails for $Y = \{0\} \subset \mathbf{C}^d = X$. ♠

We observe that the complements of tame subvarieties of codimension at least two admit Fatou-Bieberbach domains (see sect. 4 for proof):

1.4 Proposition. *If* $\Sigma \subset \mathbf{C}^d$ *is a tame complex subset of codimension at least two, there exists an injective holomorphic map* $F: \mathbf{C}^d \to \mathbf{C}^d \backslash \Sigma$ *(a Fatou-Bieberbach map). If* $0 \notin \Sigma$, *we can choose* F *such that* $F(0) = 0$ *and* F *is tangent to the identity at* 0 *to arbitrary finite order. The same is true if* Σ *is a compact subset of* $\mathbf{C}^d$ *whose polynomial hull does not contain the origin.*

1.5 Corollary. *Let* $\Sigma \subset \mathbf{C}^d \backslash \{0\}$ *be as in proposition 1.4. If* $Y \subset X$ *is a complete intersection of codimension* d *in a complex space* X, *we can choose generators* $f_1, \ldots, f_d$ *of* $\mathcal{J}_Y$ *such that the map* $f = (f_1, \ldots, f_d): X \to \mathbf{C}^d$ *avoids* Σ.

Proof. If $g = (g_1, \ldots, g_d)$ is any set of generators for $\mathcal{J}_Y$ and F is as in proposition 1.4, the components of the map $f = F \circ g: X \to \mathbf{C}^d$ are generators of $\mathcal{J}_Y$, and we have $f(X) \subset \mathbf{C}^d \backslash \Sigma$. ♠

We conclude this introduction by mentioning two open problems.

Problem 1 (Murthy). Let $Y \subset \mathbf{C}^n$ be a local holomorphic complete intersection with trivial normal bundle. Is Y a complete intersection in $\mathbf{C}^n$? In particular, is every closed complex submanifold $Y \subset \mathbf{C}^n$ with trivial normal bundle a holomorphic complete intersection in $\mathbf{C}^n$? The first open case to consider is five dimensional submanifolds in $\mathbf{C}^8$ [Sch]. The answer is negative for differentiable complete intersections (example 1.1 in [BK]).

Problem 2. If the answer to problem 1 is negative in general, we may ask whether there exists a closed complex submanifold $Y \subset \mathbf{C}^n$ with the following properties:

(a) the complex normal bundle of Y in $\mathbf{C}^n$ is trivial,

(b) Y is a differentiable complete intersection in $\mathbf{C}^n$, but

(c) Y is not a holomorphic complete intersection in $\mathbf{C}^n$.

The paper is organized as follows. In section 2 we collect some preliminary material on vector bundles. In section 3 we prove theorems 1.1 and 1.2. In section 4 we prove theorem 1.3 and proposition 1.4.

&2. Preliminaries.

We begin by recalling some basic facts on real and complex vector bundles over CW-complexes; the proofs can be found in [Hus]. The results concerning complex vector bundles remain true for holomorphic vector bundles over Stein spaces in view of the Oka–Grauert principle [Gra], [Car] and the fact that any n-dimensional Stein space is homotopy equivalent to an n-dimensional CW-complex [Ham].

We denote by $\mathrm{Vect}^k_{\mathbf{R}}(X)$ (resp. $\mathrm{Vect}^k_{\mathbf{C}}(X)$) the topological isomorphism classes of real (respectively complex) vector bundles of rank k over a CW-complex X. If X is a Stein space then by Grauert's theorem [Gra], $\mathrm{Vect}^k_{\mathbf{C}}(X)$ coincides with the equivalence classes of holomorphic vector bundles of rank k over X. By $\mathcal{T}^k_{\mathbf{R}}$ (resp. $\mathcal{T}^k_{\mathbf{C}}$) we denote the trivial real (respectively complex) vector bundle of rank k over a given base (which will always be clear from the context).

2.1 Theorem. *Let X be an n-dimensional CW-complex. The map*

$$\mathrm{Vect}^k_{\mathbf{R}}(X) \to \mathrm{Vect}^{k+r}_{\mathbf{R}}(X), \quad E \to E \oplus \mathcal{T}^r_{\mathbf{R}} \qquad (k, r \geq 1)$$

is surjective if $k \geq n$ and is bijective if $k \geq n+1$.

2.2 Theorem. *Let X be an n-dimensional CW-complex. The map*

$$\mathrm{Vect}^k_{\mathbf{C}}(X) \to \mathrm{Vect}^{k+r}_{\mathbf{C}}(X), \quad E \to E \oplus \mathcal{T}^r_{\mathbf{C}} \qquad (k, r \geq 1)$$

is surjective when $k \geq [n/2]$ and is bijective when $k \geq [\frac{n+1}{2}]$. In particular, if $E \to X$ is a nontrivial complex vector bundle of rank $k \geq [\frac{n+1}{2}]$, the bundle $E \oplus \mathcal{T}^r_{\mathbf{C}}$ is nontrivial for each $r \in \mathbf{N}$.

Remark. Theorem 2.1 shows that any complete intersection submanifold Y in $\mathbf{C}^n$ is parallelizable, since $TY \oplus N_Y = T\mathbf{C}^n|_Y = \mathcal{T}^n_{\mathbf{C}}$ and N_Y trivial implies TY trivial. Likewise, any real submanifold $Y \subset \mathbf{R}^N$ which is a differentiable complete intersection is stably parallelizable, i.e., $TY \oplus \mathcal{T}^1_{\mathbf{R}}$ is trivial. ♠

We shall also need the following result which is proved in [BK].

2.3 Theorem. *Each smooth submanifold $Y \subset \mathbf{R}^n$ of codimension $d \in \{1,2,4,8\}$ and with trivial normal bundle is a differentiable complete intersection in $\mathbf{R}^n$.*

The proof in [BK] is somewhat hidden in an elaborate set of results; however, the main idea is very simple and we shall recall it here. The triviality of N_Y implies that there is an open set $U \subset \mathbf{R}^n$ containing Y and a smooth map $f = (f_1, \ldots, f_d): U \to \mathbf{R}^d$ which defines Y as complete intersection in U. Let $U^* = U \backslash Y$ and let $\phi: U^* \to S^{d-1}$ (the unit sphere in $\mathbf{R}^d$) be defined by $\phi(x) = f(x)/||f(x)||$. If $d \in \{2,4,8\}$, S^{d-1} admits $d-1$ linearly independent vector fields $v_2, \ldots, v_d$. Let $A(x)$ for $x \in U^*$ be the $d \times d$ matrix whose first columns is $\phi(x)$ and the remaining columns are $v_j \circ \phi(x)$, $2 \le j \le d$. Then $A: U^* \to GL(d, \mathbf{R})$ is a transition map of a smooth rank d vector bundle $E \to \mathbf{R}^n$ obtained by patching trivial bundles over the open covering (U, V) of $\mathbf{R}^n$, with $V = \mathbf{R}^n \backslash Y$. By construction of E, f extends to a section $\widetilde{f}: \mathbf{R}^n \to E$ which has no zeros outside of U (in fact it equals the constant section $(1, 0, \ldots, 0)$ in the vector bundle chart over V). But every vector bundle over $\mathbf{R}^n$ is trivial and hence $\widetilde{f}$ may be thought of as a map $\mathbf{R}^n \to \mathbf{R}^d$ which defines Y as a complete intersection in $\mathbf{R}^n$.

Remark. Theorem 2.3 holds (with the same proof) if we replace $\mathbf{R}^n$ by any contractible smooth manifold. However, the argument does not apply if $d \notin \{1,2,4,8\}$, and the authors of [BK] conjectured that the conclusion holds only for the indicated values of d. ♠

The following result justifies example 1 in the introduction.

2.4 Proposition. *There exists a Stein manifold X of dimension four and a closed complex submanifold $Y \subset X$ of dimension two which is homotopy equivalent to the two-sphere such that Y is a differentiable complete intersection but not a holomorphic complete intersection in X.*

Proof. We take X to be the total space of a rank two holomorphic vector bundle over a two dimensional Stein manifold Y such that the bundle is trivial as a real vector bundle but non-trivial as a complex vector bundle over Y. Its zero section, which we identify with Y, is then a differentiable complete intersection but not a holomorphic complete intersection in X. To obtain such a bundle we let S be the Riemann sphere and set $E = TS \oplus \mathcal{T}^1_{\mathbf{C}} \to S$, where TS is the holomorphic tangent bundle of S. Since TS is non-trivial and the base has dimension two, theorem 2.2 shows that E is non-trivial as a complex vector bundle. However, as a real bundle we have $E = (TS \oplus \mathcal{T}^1_{\mathbf{R}}) \oplus \mathcal{T}^1_{\mathbf{R}}$ which is trivial. We now take Y to be a Stein complexification of S, containing S as a maximal real submanifold, and we extend E to a holomorphic vector bundle $X \to Y$. ♠

It is instructive to carry out the above procedure explicitly by defining a non-trivial complex structure on the trivial rank four bundle $\mathcal{T}^4_{\mathbf{R}} \to S$. The last part of the argument below is essentially the same one which can be used to prove theorem 2.2.

Explicit construction of a non-trivial complex structure on $\mathcal{T}^4_{\mathbf{R}}$ over the 2-sphere. Let $x = (x_1, x_2, x_3, x_4)$ be real coordinates on $\mathbf{R}^4$ and let $\{\mathbf{e}_j : 1 \le j \le 4\}$ be the corresponding standard basis of $T_x\mathbf{R}^4$. Let $S \subset \{0\} \times \mathbf{R}^3 \subset \mathbf{R}^4$ be the unit hypersurface sphere in the hyperplane $x_1 = 0$, and let $V = T\mathbf{R}^4|_S \cong S \times \mathbf{R}^4$. We can equip V with the structure of a rank 2 complex vector bundle over S by choosing a map $J: S \to GL(4,\mathbf{R})$ satisfying $J_x^2 = -Id$ for each $x \in S$. One such choice is $J_x^0\mathbf{e}_1 = \mathbf{e}_2$, $J_x^0\mathbf{e}_3 = \mathbf{e}_4$; in this structure $V \to S$ is a trivial $\mathbf{C}$-vector bundle over S. Another choice is obtained by starting with

$$J_x\mathbf{e}_1 = x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4, \qquad x = (0, x_2, x_3, x_4) \in S.$$

Let $V_x^1 \subset V_x$ be the real 2-plane spanned by $\mathbf{e}_1$ and $J_x\mathbf{e}_1$, and let $V_x^2 \subset V_x$ denote the orthogonal complement to V_x^1. Notice that $V^2 = TS$ and hence it is nontrivial. Since V^2 is an oriented plane bundle, we can choose an orientation preserving $J_x: V_x^2 \to V_x^2$, depending continuously on $x \in S$ and such that $J_x^2 = -Id$ on V_x^2. (The choice is unique if we require that J_x be orthogonal.) We then extend J_x by linearity to V_x.

We claim that the $\mathbf{C}$-bundle (V, J) over S is not equivalent to the trivial $\mathbf{C}$-bundle (V, J^0). Suppose on the contrary that there exists an equivalence $A: S \to GL(4,\mathbf{R})$ between the two bundles, meaning that $AJ^0A^{-1} = J$. The group preserving J^0 is precisely $GL(2,\mathbf{C})$, and hence for any map $B: S \to GL(2,\mathbf{C})$ we have

$$J = AJ^0A^{-1} = ABJ^0B^{-1}A^{-1} = (AB)J^0(AB)^{-1}.$$

We claim that we can choose B such that $B^{-1}A^{-1}\mathbf{e}_1 = \mathbf{e}_1$ on S. Since $\mathbf{R}^4\backslash\{0\} \simeq S^3$, every map $S = S^2 \to \mathbf{R}^4\backslash\{0\}$ is homotopic to a constant map. Thus there is a homotopy $v_t: S \to \mathbf{R}^4\backslash\{0\}$ ($t \in [0,1]$) from $v_0 = \mathbf{e}_1$ to $v_1 = A^{-1}\mathbf{e}_1$. Denote by $\tau: GL(2,\mathbf{C}) \to \mathbf{R}^4\backslash\{0\}$ the map $\tau(B) = B\mathbf{e}_1$. Clearly this map is a Serre fibration, i.e., it has the homotopy lifting property. Thus there is a homotopy $B_t: S \to GL(2,\mathbf{C})$ ($t \in [0,1]$), with $B_0 = Id$, satisfying $B_t\mathbf{e}_1 = v_t$ for each $t \in [0,1]$. At $t = 1$ we get the desired map $B = B_1$ satisfying $B\mathbf{e}_1 = A^{-1}\mathbf{e}_1$.

Write $C = AB$; hence $J = CJ^0C^{-1}$. By construction we have $C\mathbf{e}_1 = \mathbf{e}_1$ and $C\mathbf{e}_2 = CJ^0\mathbf{e}_1 = J\mathbf{e}_1$. Thus C maps the trivial subbundle $U = \mathbf{R}^2 \times \{0\}^2 \subset V$ onto the subbundle $V^1 \subset V$, and hence it induces an isomorphism of quotient bundles $V/U \cong V/V^1 \cong V^2 \cong TS$. This is a contradiction since the first bundle is trivial while the second is not.

&3. Proof of theorems 1.1 and 1.2.

Proof of theorem 1.2. Let $F: Y \to \mathbf{C}^n$, $n = m+d$, be any proper holomorphic embedding. We identify Y with the submanifold $F(Y) \subset \mathbf{C}^n$ and denote by N_Y its holomorphic normal bundle. By the Oka-Cartan theory we have a holomorphic splitting $T\mathbf{C}^n|_Y \cong TY \oplus \mathbf{N}_Y$ [GR]. Since TY is a trivial real bundle, theorem 2.1 shows that its complement N_Y is also real trivial. Since the real codimension of Y is $2d$ which is either 4 or 8, theorem 2.3 implies that Y is a differentiable complete intersection in $\mathbf{C}^n$. On the other hand, since TY is a non-trivial as a complex bundle over Y, theorem 2.2 implies that N_Y is also a non-trivial complex vector bundle. Thus Y is not a holomorphic complete intersection in any open set $U \subset \mathbf{C}^n$ containing Y. This proves theorem 1.2. ♠

In the proof of theorem 1.1 we shall need the following:

3.1 Proposition. *For any compact orientable two dimensional surface M of genus $g \geq 2$ there exists a Stein manifold Y of dimension three which is homotopically equivalent to M and whose tangent bundle TY is trivial as a real vector bundle, but non-trivial as a complex vector bundle.*

Proof. Let M be any surface as in the proposition; such M is the connected sum of $g \geq 2$ tori. Observe that its tangent bundle TM is non-trivial, but $TM \oplus \mathcal{T}^1_{\mathbf{R}}$ is trivial since M embeds as a real hypersurface in $\mathbf{R}^3$.

By theorem 1.8 in [For1] there exists a smooth (even real-analytic) embedding $M \hookrightarrow \mathbf{C}^2$ which is totally real except at finitely many points which are hyperbolic in the sense of Bishop, and the embedded submanifold $M \subset \mathbf{C}^2$ has small Stein neighborhoods $\Omega \subset \mathbf{C}^2$ which can be retracted onto M. We endow the tangent bundle TM with a structure of a complex vector bundle and extend this bundle to a complex line bundle on Ω by using a retraction $\pi: \Omega \to M$.

There exists a holomorphic line bundle $E \to \Omega$ which is equivalent to the given complex line bundle constructed above (and hence is non-trivial). We can see this by appealing to the Oka-Grauert theorem [Gra], [Car]; however, in the present situation we can construct such bundle quite explicitly as follows. Assume (as we may) that the embedding $M \subset \mathbf{C}^2$ is real-analytic. We can represent the tangent bundle TM by a 1-cocycle defined by real-analytic functions $c_{ij}: U_{ij} \to \mathbf{C}\backslash\{0\}$ on a (finite) open covering $\mathcal{U} = \{U_i\}$ of M such that the closure of each of the sets $U_{ij} = U_i \cap U_j$ for $i \neq j$ is contained in the totally real part of M (we only need to avoid the finitely many complex tangents in M). The complexifications of c_{ij} now determine a holomorphic line bundle in an open neighborhood of M in $\mathbf{C}^2$.

We claim that the total space $Y = E$ of this bundle satisfies proposition 3.1. Since the base Ω is Stein, Y is also Stein, and it is homotopy equivalent to Ω and hence to M. Its tangent bundle along the zero section $\Omega \subset Y$ equals $TY|_\Omega = T\Omega \oplus E = \mathcal{T}^2_{\mathbf{C}} \oplus E$. Denote this bundle by H. Since E is nontrivial and the base Y has the homotopy type of a real two dimensional CW-complex, it follows from theorem 2.2 that H is non-trivial as a $\mathbf{C}$-vector bundle over Ω. Now $TY = p^*H$ where $p: Y \to \Omega$ is the base projection, whence TY is also non-trivial.

On the other hand, considering H as a real vector bundle we have $H = \mathcal{T}^4_{\mathbf{R}} \oplus E = \mathcal{T}^3_{\mathbf{R}} \oplus (\mathcal{T}^1_{\mathbf{R}} \oplus E)$. We have already observed that the second summand is trivial and hence H is trivial. Therefore $TY = p^*H \to Y$ is also a trivial real bundle. ♠

Proof of theorem 1.1. Let Y be as in proposition 3.1. By the embedding theorem of Eliashberg and Gromov [EGr] and Schürmann [Schur] there exists a proper holomorphic embedding $Y \hookrightarrow \mathbf{C}^5$. By theorem 1.2 Y is then a differentiable complete intersection but not a holomorphic complete intersection in $\mathbf{C}^5$. The same argument applies to any embedding $Y \hookrightarrow \mathbf{C}^7$. ♠

&4. Removal of intersections.

Proof of proposition 1.4. Consider first the case when $\Sigma \subset \mathbf{C}^d$ is a tame subvariety of dimension at most $d-2$. For $1 \leq j \leq d$ we denote by $\pi_j: \mathbf{C}^d \to \mathbf{C}^{d-1}_j$ the projection onto the coordinate hyperplane $\{z_j = 0\}$. Tameness of Σ implies that, after a biholomorphic change of coordinates on $\mathbf{C}^d$, the restriction of π_j to Σ is proper for each j, and hence $\Sigma_j = \pi_j(\Sigma)$ is a proper closed analytic subset of $\mathbf{C}^{d-1}_j$. By translation we may also assume that Σ_j does not contain the origin for any j. Choose a holomorphic function g_j on $\mathbf{C}^{d-1}_j$ such that $g_j(0) = -\log 2$ and $g_j = 0$ on Σ_j, and set

$$\Phi_j(z) = \left(z_1, \ldots, z_{j-1}, e^{g_j(\hat{z}_j)} z_j, z_{j+1}, \ldots, z_d\right),$$

where $\hat{z}_j = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$. Then $\Phi = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_d \in \mathrm{Aut}\mathbf{C}^d$, Φ restricts to the identity on Σ, $\Phi(0) = 0$, and $D\Phi(0) = \frac{1}{2}I$. Thus $0 \in \mathbf{C}^d$ is an attracting fixed point of Φ whose basin of attraction is a Fatou-Bieberbach domain $\Omega \subset \mathbf{C}^d \backslash \Sigma$. We obtain the corresponding Fatou-Bieberbach map $F\colon \mathbf{C}^d \to \Omega$ as in [RR].

If K is a compact subset in $\mathbf{C}^d$ whose polynomial hull does not contain the origin, we can construct a Fatou-Bieberbach map $F\colon \mathbf{C}^d \to \mathbf{C}^d \backslash K$ by the push-out method of Dixon and Esterle [DE] (see also [For2]). Here is the outline. Replacing K by $\widehat{K}$ we may assume that K is polynomially convex and $0 \notin K$. Denote by B_r the closed ball of radius r in $\mathbf{C}^d$. Proposition 2.1 in [For2] (or the results in [FRo]) gives an automorphism $G_0 \in \mathrm{Aut}\mathbf{C}^d$ which is tangent to identity to a given order r at 0 and satisfies $G_0(K) \cap B_1 = \emptyset$. Set $K_1 = G_0(K)$. Next we choose $\Phi_1 \in \mathrm{Aut}\mathbf{C}^d$ which approximates the identity map on B_1, is tangent to the identity at 0 and satisfies $\Phi_1(K_1) \cap B_2 = \emptyset$, and we let $G_1 = \Phi_1 \circ G_0$. Continuing inductively we obtain a sequence $G_j \in \mathrm{Aut}\mathbf{C}^d$ ($j \in \mathbf{Z}_+$) which converges on some domain $\Omega \subset \mathbf{C}^d$ to a biholomorphic map $G\colon \Omega \to \mathbf{C}^d$ of G *onto* $\mathbf{C}^d$ (proposition 5.1 in [For2]). By construction G is tangent to the identity to order r at 0 and $K \cap \Omega = \emptyset$. The map $F = G^{-1}\colon \mathbf{C}^d \to \Omega$ satisfies proposition 1.4. ♠

Proof of theorem 1.3. We use the same notation as in the statement of the theorem. Let $\mathcal{J}_\Sigma$ be the sheaf of ideals of $\Sigma \subset \mathbf{C}^d$ and let $\mathcal{J}_Y$ be the sheaf of ideals of $Y \subset X$. We define an analytic sheaf of ideals $\mathcal{S}$ on X as follows: at points $x \in Y$ we take $\mathcal{S}_x$ to be the pull-back of $\mathcal{J}_{\Sigma, f(x)}$ by f, and for $x \in X \backslash Y$ we take $\mathcal{S}_x = \mathcal{O}_{X,x}$. More precisely, if $x \in Y$ and if $\mathcal{J}_\Sigma$ is generated by functions $g_1, \ldots, g_k$ in some neighborhood of $f(x)$ in $\mathbf{C}^d$, we take the functions $g_j \circ f$ $(1 \le j \le k)$ as the generators of $\mathcal{S}$ in a neighborhood of x. Clearly $\mathcal{S}$ is a coherent analytic sheaf of ideals on X and $\mathcal{O}_X/\mathcal{S}$ is supported on Y.

Choose $r \in \mathbf{N}$ and let $\mathcal{R} = \mathcal{S}\mathcal{J}_Y^r$; this is also a coherent sheaf of ideals on X which coincides with $\mathcal{O}_X$ on $X \backslash Y$. By the Oka-Cartan theory we can choose finitely many sections $\xi_1, \ldots, \xi_k$ of $\mathcal{R}$ such that $Y = \{x \in X \colon \xi_j(x) = 0,\ 1 \le j \le k\}$. (We do not require that the ξ_j's generate $\mathcal{R}$!) We seek a map $g\colon X \to \mathbf{C}^d$ satisfying theorem 1.3 in the form

$$g(x) = f(x) + \sum_{j=1}^{k} \xi_j(x) g_j(x) = f(x) + G(x)\xi(x), \tag{2}$$

where $G(x) = (g_1(x), \ldots, g_k(x))$ is a holomorphic $d \times k$ matrix-valued function and $\xi = (\xi_1, \ldots, \xi_k)^t$. For any choice of G the map $g = f + G\xi$ agrees with f to order $r + 1$

along Y. Our goal is to choose G such that $g^{-1}(\Sigma) = Y$. Define a holomorphic map $\Phi: X \times \mathbf{C}^{dk} \to \mathbf{C}^d$ by

$$\Phi(x; v_1, \ldots, v_k) = f(x) + \sum_{j=1}^{k} \xi_j(x) v_j$$

and let

$$\widetilde{\Sigma} = \Phi^{-1}(\Sigma) \backslash (Y \times \mathbf{C}^{dk}).$$

Then $g = f + G\xi$ satisfies theorem 1.3 if and only if G is holomorphic and its graph in $X \times \mathbf{C}^{dk}$ avoids $\widetilde{\Sigma}$.

Observe that for each fixed $x \in X \backslash Y$ the map $\Phi(x, \cdot\,): \mathbf{C}^{dk} \to \mathbf{C}^d$ is an affine surjection, while for $x \in Y$ we have $\Phi(x, \cdot\,) = f(x)$ (hence $Y \times \mathbf{C}^{dk} \subset \Phi^{-1}(\Sigma)$). Let $p: X \times \mathbf{C}^{dk} \to X$ denote the base projection. We shall need the following

4.1 Lemma. *$\widetilde{\Sigma}$ is a closed complex subvariety of $X \times \mathbf{C}^{dk}$. Moreover, for each point $a \in X \backslash Y$ there is a neighborhood $U \subset X \backslash Y$ of a and a biholomorphic self-map Ψ of $\widetilde{U} = U \times \mathbf{C}^{dk}$, with $p \circ \Psi = p$, such that $\Psi(x, \cdot\,)$ is affine linear for each $x \in U$ and $\Psi(\widetilde{\Sigma} \cap \widetilde{U}) = U \times (\Sigma \times \mathbf{C}^{d(k-1)})$.*

Proof. By definition $\widetilde{\Sigma}$ is a closed complex subvariety in $(X \backslash Y) \times \mathbf{C}^{dk}$. The second statement follows immediatelly from the observation that $\Phi(x, \cdot\,): \mathbf{C}^{dk} \to \mathbf{C}^d$ is an affine surjection for any $x \in X \backslash Y$ and hence is locally (with respect to the base) equivalent to the projection of $\mathbf{C}^{dk}$ onto $\mathbf{C}^d \times \{0\}^{d(k-1)}$.

It remains to show that $\widetilde{\Sigma}$ is closed in $X \times \mathbf{C}^{dk}$. We need to show that, as $x \in X \backslash Y$ approaches a point $x_0 \in Y$, the fibers $\widetilde{\Sigma}_x$ leave any compact subset of $\mathbf{C}^{dk}$. Choose a neighborhood $V \subset \mathbf{C}^d$ of the point $f(x_0)$ and holomorphic functions $h = (h_1, \ldots, h_m)$ on V which generate the ideal sheaf $\mathcal{J}_\Sigma$ on V. Also choose a neighborhood $U \subset X$ of x_0 with $f(U) \subset V$. Let ξ_j $(1 \le j \le m)$ be sections of the sheaf $\mathcal{R}$ as above. By Taylor expansion of h at the point $f(x)$ for $x \in U$ we get

$$\begin{aligned} h \circ \Phi(x, v) &= h\left(f(x) + \sum \xi_j(x) v_j\right) \\ &= h(f(x)) + \sum_{|\alpha| \ge 1} c_\alpha D^\alpha h(f(x)) \left(\sum \xi_j(x) v_j\right)^\alpha \\ &= h(f(x)) + A(x, v)\xi(x), \end{aligned}$$

where A is a holomorphic $d \times k$ matrix function. Denoting by $||\cdot||$ the Euclidean norm on $\mathbf{C}^d$ (and the corresponding matrix norm), we have

$$||h(\Phi(x, v))|| \ge ||h(f(x))|| - ||A(x, v)|| \cdot ||\xi(x)||.$$

The components of $h(f(x))$ generate the sheaf $\mathcal{S}$ at each point of U. Hence, as $x \to x_0 \in Y$, the term $||\xi(x)||$ is of size $o(||h(f(x))||)$ by the definition of the sheaf $\mathcal{R} = \mathcal{S}\mathcal{J}^r$. Hence for each $C > 0$ there is a neighborhood $U_C \subset U$ of x_0 such that for all $x \in U_C$ and $v \in \mathbf{C}^{dk}$

with $||v|| \leq C$ we have $||h \circ \Phi(x,v)|| \geq ||h(f(x))||/2$, and hence $\Phi(x,v) \in \Sigma$ if and only if $x \in Y$. Thus for $x \in U_C$ the fiber $\widetilde{\Sigma}_x$ does not intersect the ball of radius C in $\mathbf{C}^{dk}$. This proves that $\widetilde{\Sigma}$ is closed in $X \times \mathbf{C}^{dk}$. ♠

We continue with the proof of theorem 1.3. The assumptions on Σ imply that the complement $\mathbf{C}^d \backslash \Sigma$ admits a spray in the sense of Gromov (see [FP1] and lemma 7.1 in [FP2]). From this and the second statement in lemma 4.1 it follows that the holomorphic submersion $h\colon Z = (X \times \mathbf{C}^{dk}) \backslash \widetilde{\Sigma} \to X$ admits a fiber dominating spray in a small neighborhood of any point $x \in X \backslash Y$ ([Gro] or definition 1.1 in [FP2]). By theorem 1.2 in [FP2] (see also [Gro], 4.5 Main Theorem) the homotopy principle holds for sections of Z, meaning that any continuous section $\widetilde{G}\colon X \to Z$ can be deformed to a holomorphic section.

A continuous extension $\widetilde{f}\colon X \to \mathbf{C}^d$ of f as in theorem 1.3 can be lifted to a continuous section $\widetilde{G}\colon X \to Z$ which is holomorphic near Y (see lemma 8.1 in [FP2]). The homotopy principle gives a holomorphic section $G\colon X \to Z$ such that the corresponding map $g\colon X \to \mathbf{C}^d$ (2) satisfies theorem 1.3.

In the remainder we investigate the existence of a continuous extension $\widetilde{f}$ using the obstruction theory (see e.g. section V.5 in [Whi]). By [Ham] the subvariety Y has a closed neighborhood $A \subset X$ such that the pair (X, A) is homotopy equivalent to a relative CW-complex of dimension $n = \dim X$ and Y is a deformation retraction of A. Moreover, we may choose A so small that $\{x \in A\colon f(x) \in \Sigma\} = Y$. Hence f maps $A^* = A \backslash Y$ to $\Omega = \mathbf{C}^d \backslash \Sigma$, and we wish to find an extension of f to a map from $X^* = X \backslash Y$ to Ω.

The pair (X^*, A^*) can be represented by the same relative CW-complex as (X, A). Denote by X_q its q-dimensional skeleton, so our goal is to extend f to a map $X_n \to \Omega$. We begin by extending f to the zero-skeleton X_0 by arbitrarily prescribing the values at the points of X_0. Since Ω is connected, we can further extend to a map $f_1\colon X_1 \to \Omega$. Suppose inductively that f has already been extended to $f_q\colon X_q \to \Omega$ for some $q \geq 1$. The next skeleton X_{q+1} is obtained by attaching $(q+1)$-cells e_{q+1} to X_q by maps $\partial e_{q+1} \to X_q$. Composing this attaching map with $f_q\colon X_q \to \Omega$ we obtain for each such cell e_{q+1} a map $\partial e_{q+1} \to \Omega$ which defines an element of the fundamental group $\pi_q(\Omega)$. In this way we obtain a singular cochain $c^{q+1} \in \Gamma^{q+1}(X^*, A^*; \pi_q(\Omega))$ (which is in fact a $(q+1)$-cocycle, called the *obstruction cocycle*), and f_q extends to a map $f_{q+1}\colon X_{q+1} \to \Omega$ if and only if $c^{q+1} = 0$.

In our case we have $\pi_q(\Omega) = \pi_q(\mathbf{C}^d \backslash \Sigma) = 0$ for $1 \leq q \leq 2s-2$, where $s = d - \dim \Sigma$. This implies that f can be extended to the skeleton X_{2s-1}. Hence, if $\dim X < 2s = 2(d - \dim \Sigma)$, we have an extension $\widetilde{f}\colon X \backslash Y \to \mathbf{C}^d \backslash \Sigma$ as required.

Assume now that X is contractible (e.g., $X = \mathbf{C}^n$). We shall use the following more precise result from obstruction theory ([Whi], theorem V.5.14):

Let $f_q\colon X_q \to \Omega$ for some $q \geq 1$. Then $f_q | X_{q-1}$ can be extended to a map $f_{q+1}\colon X_{q+1} \to \Omega$ if and only if $\gamma^{q+1}(f) = [c^{q+1}] = 0 \in H^{q+1}(X^*, A^*; \pi_q(\Omega))$, i.e., the cohomology class of the obstruction cocycle c^{q+1} equals zero.

By excision we have $H^q(X^*, A^*; G) = H^q(X, A; G)$ for any abelian coefficient group G. Since X is contractible, the long exact sequence for the cohomology of the pair $A \hookrightarrow X$

gives $H^{q+1}(X, A; G) = H^q(A; G)$ for $q \geq 1$. Furthermore, since Y is a deformation retract of A we have $H^q(A; G) = H^q(Y; G)$. Together we obtain

$$H^{q+1}(X^*, A^*; \pi_q(\Omega)) = H^q(Y; \pi_q(\Omega)), \quad q \geq 1.$$

Since Y is a Stein manifold of dimension m, it is homotopy equivalent to an m-dimensional CW-complex and hence $H^q(Y; \pi_q(\Omega)) = 0$ for $q > m$. Thus, if $f\colon A^* \to \Omega$ admits an extension to the skeleton X_{m+1}, it also admits an extension to all higher dimensional skeleta and hence to X^*. Earlier we have seen that there is an extension to X_{2s-1} with $s = d - \dim \Sigma$. If we assume $m + 1 \leq 2s - 1$, we thus obtain a desired continuous extension of f to X^*. This completes the proof of theorem 1.3. ♠

Acknowledgements. I wish to thank E. L. Stout for having brought to my attention the question answered by theorem 1.1 and for stimulating discussions. This research was supported in part by an NSF grant, by the Vilas foundation at the University of Wisconsin–Madison, and by the Ministry of Science of the Republic of Slovenia.

References.

[BK] J. Bochnak and W. Kucharz: Complete intersections in differential topology and analytic geometry. Boll. Un. Mat. Ital. B (7) **10** (1996), no. 4, 1019–1041.

[Car] H. Cartan: Espaces fibrés analytiques. Symposium Internat. de topologia algebraica, Mexico, 97–121 (1958). (Also in Oeuvres, vol. 2, Springer, New York, 1979.)

[Chi] E. M. Chirka: Complex analytic sets. Kluwer Academic Publishers Group, Dordrecht, 1989.

[DE] P. G. Dixon and J. Esterle: Michael's problem and the Poincaré–Fatou–Bieberbach phenomenon. Bull. Amer. Math. Soc. (N.S.) **15** (1986), 127–187.

[FRa] O. Forster and K. J. Ramspott: Analytische Modulgarben und Endromisbündel. Invent. Math. **2** (1966), 145–170.

[For1] F. Forstnerič: Complex tangents of real surfaces in complex surfaces. Duke Math. J. **67** (1992), 353–376.

[For2] F. Forstnerič: Interpolation by holomorphic automorphisms and embeddings in $\mathbf{C}^n$. J. Geom. Anal. **9**, no.1, (1999) 93-118.

[FP1] F. Forstnerič and J. Prezelj: Oka's principle for holomorphic fiber bundles with sprays. Math. Ann., to appear.

[FP2] F. Forstnerič and J. Prezelj: Oka's principle for holomorphic submersions with sprays. Math. Ann., to appear.

[FRo] F. Forstnerič and J.-P. Rosay: Approximation of biholomorphic mappings by automorphisms of $\mathbf{C}^n$. Invent. Math. **112** (1993), 323–349.

[Gra] H. Grauert: Analytische Faserungen über holomorph-vollständigen Räumen. Math. Ann. **135** (1958), 263–273.

[GRe] H. Grauert and R. Remmert: Theory of Stein Spaces. Grundl. Math. Wiss. **227**, Springer, New York, 1977.

[Gro] M. Gromov: Oka's principle for holomorphic sections of elliptic bundles. J. Amer. Math. Soc. **2** (1989), 851-897.

[GR] C. Gunning and H. Rossi: Analytic functions of several complex variables. Prentice–Hall, Englewood Cliffs, 1965.

[Ham] H. A. Hamm: Zur Homotopietyp Steinscher Räume. J. Reine Angew. Math. **338** (1983), 121–135.

[Hus] D. Husemoller: Fibre bundles. Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994.

[RRu] J.-P. Rosay and W. Rudin: Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. Trans. Amer. Math. Soc. **310**, 47–86 (1988)

[Sch] M. Schneider: On the number of equations needed to describe a variety. Complex analysis of several variables (Madison, Wis., 1982), 163–180, Proc. Sympos. Pure Math., 41, Amer. Math. Soc., Providence, R.I., 1984.

[Whi] G. W. Whitehead: Elements of homotopy theory. Graduate Texts in Mathematics, 61. Springer, New York–Berlin, 1978.

IMFM, University of Ljubljana, Jadranska 19, 1000–Ljubljana, Slovenia