University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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SIMPLE EXPLICIT FORMULAS FOR THE FRAME-STEWART'S NUMBERS

Sandi Klavžar   Uroš Milutinović

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Simple explicit formulas for the Frame-Stewart's numbers

Sandi Klavžar*
Department of Mathematics, PEF, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
sandi.klavzar@uni-lj.si

Uroš Milutinović*
Department of Mathematics, PEF, University of Maribor
Koroška cesta 160, 2000 Maribor, Slovenia
uros.milutinovic@uni-mb.si

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Abstract

Several different approaches to the multi-peg Tower of Hanoi problem are equivalent. One of them is Stewart's recursive formula

$$S(n,p) = \min\{2S(n_1,p) + S(n-n_1,p-1) \mid n_1,\, n-n_1 \in \mathbb{Z}^+\}\,.$$

In the present paper we significantly simplify the explicit calculation of the Frame-Stewart's numbers $S(n,p)$ and give a short proof of the domain theorem that describes the set of all pairs (n,n_1), such that the above minima are achieved at n_1.

1 Introduction

The problem of finding the smallest number of moves in the Multi-peg Tower of Hanoi problem, that is, in the Tower of Hanoi problem with more than three pegs, has been posed by Dudeney [3] in 1908. (The classical Tower of Hanoi problem with three pegs goes back to Lucas [2].) In 1941 two solutions for the multi-peg version appeared, one due to Frame [5] and the other to Stewart [13]. However, already in the editorial note [4] following [13] it was pointed out that the solutions miss an argument that the proposed algorithms are indeed optimal. Proving that these "presumed optimal solutions" are optimal became a notorious open problem.

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Bode and Hinz [1] algorithmically verified that for four pegs and up to 17 disks the Frame-Stewart's approach agrees with the optimal solution. Recently Szegedy [14] proved that for k pegs at least $2^{C_k n^{1/(k-2)}}$ moves are needed. This bound is optimal up to a constant factor in the exponent for fixed k. For the definition of the problem as well as for more information on the history of it we refer to [6, 7]. Interestingly, a closer look to the Tower of Hanoi problem enabled Hinz and Schief [8] to compute the average distance of the Sierpiński gasket.

Stewart's presumed optimal solution is given by the recursive formula

$$S(n,p) = \min\{2S(n_1,p) + S(n-n_1,p-1) \mid n_1,\, n-n_1 \in \mathbb{Z}^+\}\,. \qquad (1)$$

In [9] it is proved that seven different approaches to the multi-peg Tower of Hanoi problem are equivalent, including Stewart's and Frame's approach. We thus call the numbers defined by recursion (1) *Frame-Stewart's numbers.* Our main result, stated and proved in Section 3, gives a simple explicit expression for these numbers. More precisely, let $h_p(k) = \binom{p-3+k}{p-2}$, and let $0 \le m \le h_p(k+1)$. Then

$$S(h_p(k)+m,p) = P_p(k)2^{k-1} + (-1)^p + m2^k\,,$$

where $P_p(k)$ is the following polynomial of degree $p-3$:

$$P_p(k) = 2(-1)^{p-1}\sum_{i=0}^{p-3}(-1)^i h_{i+2}(k)\,.$$

In the next section we give basic definitions and recall results needed for the proof of the above identity. The paper is concluded with a short proof of the domain theorem of the recursion (1).

2 Basic definitions and results

Definition 2.1 *For $p \ge 3$, let*

$$h_p(x) = \binom{p-3+x}{p-2},\ x \in \mathbb{R},\ x \ge 0.$$

On nonnegative reals these functions are strictly increasing and therefore they have inverses, which are strictly increasing on nonnegative reals as well.

Definition 2.2 *For $p \ge 3$, let $g_p = h_p^{-1}$, and let*

$$f_p(x) = \lceil g_p(x) \rceil.$$

Remark 2.3 *We shall use $h_2(x) = 1$ — this definition coincides with $\binom{2-3+x}{2-2}$. There are no g_2 or f_2.*

Proposition 2.4 *For any $p \geq 3$, $k \geq 1$,*

$$h_p(k) - h_p(k-1) = h_{p-1}(k) .$$

Proof. This property is just the basic property of binomial coefficients. ∎

A proof of the next theorem can be found in [9]. In fact, both formulas appeared already in [5], but have been treated rather heuristically, and have been presented as statements about the smallest number of moves in the Multi-peg Tower of Hanoi problem. The second formula will be crucial for our present work.

Theorem 2.5 *Let $n \geq 2$ and $p \geq 3$. Then $S(n,p)$ may be written as the following two sums:*

$$S(n,p) = \sum_{t=0}^{s-1} 2^t h_p(t) + 2^s(n - h_p(s)) , \tag{2}$$

where $s = f_p(n) - 1$, and

$$S(n,p) = \sum_{k=1}^{n} 2^{f_p(k)-1} . \tag{3}$$

We need to recall two more results.

Theorem 2.6 *Let $p \geq 4$ and $k \geq 2$. Then $n_1 = h_p(k-1)$ is the only value of n_1 for which*

$$S(h_p(k), p) = 2S(n_1, p) + S(h_p(k) - n_1, p-1)$$

holds true.

Proof. Is given in [9], in the proof of Theorem 5.1. Note that $h_p(k) - h_p(k-1) = h_{p-1}(k)$, by Proposition 2.4. ∎

Theorem 2.7 *Let $p \geq 4$. The set Σ of all pairs (n, n_1), for which $S(n,p) = 2S(n_1,p) + S(n-n_1, p-1)$ and $h_p(k) \leq n \leq h_p(k+1)$, $k \geq 2$, is the parallelogram Π in the (n, n_1)-plane, bounded by the lines $n_1 = h_p(k-1)$, $n_1 = h_p(k)$, $n_1 = n + h_p(k-1) - h_p(k)$, $n_1 = n + h_p(k) - h_p(k+1)$ (see Fig. 1).*

Theorem 2.7 was proved by a (double) induction on p and n in [10, 11]. In [10] Majumdar established the truth of the induction basis, that is, for $p = 4$, while in [11] he followed with a general argument. In fact, only one half of the proof is written down—for the “left” and the “bottom” part of the parallelogram. Thus, the whole argument along these lines would contain quite several pages. In Section 4 we give an alternative, complete, and short proof of the theorem.

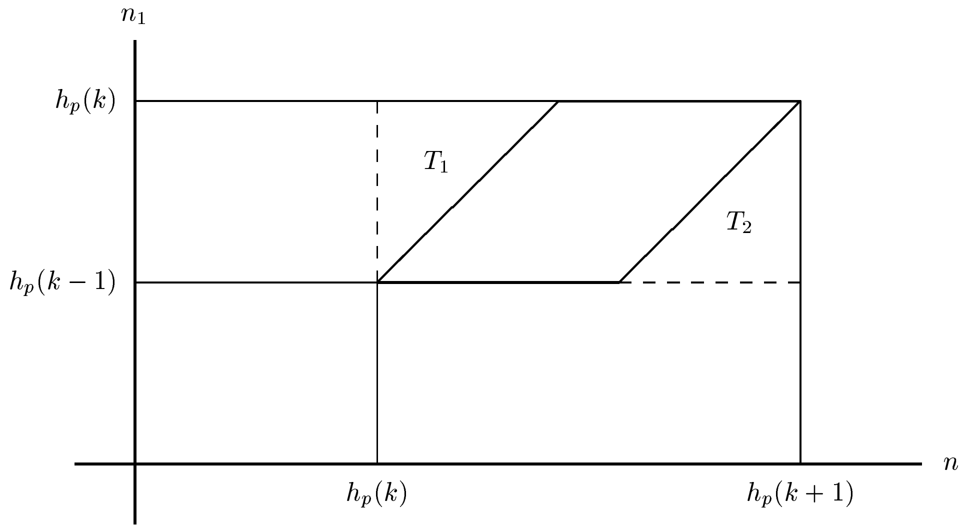

Figure 1: Set Σ

3 Explicit formulas

Since the points $(h_p(k+1), h_p(k))$ show special behavior—the uniqueness of the choice of n_1—we shall concentrate our attention on them. It will in turn enable us to give simple explicit expressions for all of the Frame-Stewart's numbers $S(n,p)$.

First, let us introduce a shorthand notation for $S(h_p(k),p)$.

Definition 3.1 *For $p \geq 3$, $k \geq 1$, let*

$$N(k,p) = S(h_p(k),p)\,.$$

Remark 3.2 *For $p = 3$, $k \geq 1$, $N(k,p) = S(k,p)$, since $h_3(k) = k$. Therefore, in this case no condensation occurs, but it is reasonable to treat it this way, since all the results obtained in this section, remain valid for $p = 3$ (with the above interpretation), too.*

The recursive formula of Theorem 2.6, written as

$$S(h_p(k+1),p) = 2S(h_p(k),p) + S(h_{p-1}(k+1),p-1)\,,$$

now becomes

$$N(k+1,p) = 2N(k,p) + N(k+1,p-1)\,.$$

It holds for $k \geq 1$, $p \geq 4$.

Taking into account $N(1,p) = S(1,p) = 1$ and $N(k,3) = S(k,3) = 2^k - 1$, and using standard methods of solving linear difference equations (obtained by fixing $p = 4, 5$), one easily gets the following explicit formulas, for $k \geq 1$:

$$\begin{aligned} N(k,3) &= 2^k - 1 \\ N(k,4) &= (k-1)2^k + 1 \\ N(k,5) &= (k(k-1)+2)2^{k-1} - 1\,. \end{aligned}$$

It suffices as a motivation for our main theorem.

Theorem 3.3 *For any* $p \geq 3$, $k \geq 1$

$$N(k,p) = P_p(k) \cdot 2^{k-1} + (-1)^p\,, \tag{4}$$

where P_p *is a polynomial of degree* $p-3$ *that can be written as*

$$P_p(k) = 2(-1)^{p-1} \sum_{i=0}^{p-3} (-1)^i h_{i+2}(k)\,. \tag{5}$$

Proof. We shall prove the claim by induction on p. By inductive assumption

$$N(k,p+1) = 2N(k-1,p+1) + N(k,p)$$

becomes

$$N(k,p+1) = 2N(k-1,p+1) + P_p(k) \cdot 2^{k-1} + (-1)^p\,,$$

for any $k \geq 2$. Multiplying

$$N(k+1-i,p+1) = 2N(k-i,p+1) + P_p(k+1-i) \cdot 2^{k-i} + (-1)^p\,,$$

by 2^{i-1}, and summing up the resulting identities for $i = 1, 2, \ldots, k-1$, one gets

$$\begin{aligned} &N(k,p+1) = \\ &2^{k-1}N(1,p+1) + 2^{k-1}(P_p(2) + P_p(3) + \cdots + P_p(k)) + (-1)^p(2^{k-1} - 1) = \\ &2^{k-1}(1 + (-1)^p + P_p(2) + P_p(3) + \cdots + P_p(k)) + (-1)^{p+1}\,, \end{aligned}$$

hence the statement of the theorem holds true for

$$P_{p+1}(k) = 1 + (-1)^p + P_p(2) + P_p(3) + \cdots + P_p(k)\,.$$

Using Euler-Maclaurin summation formula for Σm^i for $0 \leq i \leq p-3$, we see that there is such a polynomial and that its degree is $p-2$.

Using this we may directly calculate all the values of $S(n,p)$, as follows. Describing this we shall simultaneously obtain a simpler recursive formula.

$$\begin{aligned} &S(h_p(k),p) = P_p(k)2^{k-1} + (-1)^p \\ &S(h_p(k)+1,p) = S(h_p(k),p) + 2^{f_p(h_p(k)+1)-1} = P_p(k)2^{k-1} + (-1)^p + 2^k \end{aligned}$$

$$\vdots$$

$$S(h_p(k)+m,p) = S(h_p(k)+m-1,p) + 2^{f_p(h_p(k)+m)-1} =$$
$$P_p(k)2^{k-1} + (-1)^p + m2^k$$

$$\vdots$$

$$S(h_p(k+1),p) = P_p(k)2^{k-1} + (-1)^p + (h_p(k+1) - h_p(k))2^k =$$
$$P_p(k)2^{k-1} + (-1)^p + h_{p-1}(k+1)2^k\,.$$

Since $S(h_p(k+1),p) = P_p(k+1)2^k + (-1)^p$, the last line of the above calculations gives the following recursive formula:

$$2P_p(k+1) = P_p(k) + 2h_{p-1}(k+1), \quad p \geq 4,\ k \geq 1\,. \tag{6}$$

Summing up the identities (6) $2P_p(i) = P_p(i-1) + 2h_{p-1}(i)$, multiplied by $1/2^{k-i}$, for $i = 2, 3, \ldots, k$, we get

$$P_p(k) = \frac{P_p(1)}{2^{k-1}} + \frac{h_{p-1}(2)}{2^{k-2}} + \cdots + \frac{h_{p-1}(k-2)}{2^2} + \frac{h_{p-1}(k-1)}{2} + h_{p-1}(k)\,. \tag{7}$$

Using $h_{p-1}(\ell) = h_p(\ell) - h_p(\ell-1)$ this identity can be rewritten as

$$P_p(k) = \frac{P_p(1)}{2^{k-1}} - \frac{h_p(1)}{2^{k-2}} - \cdots - \frac{h_p(k-2)}{2^2} - \frac{h_p(k-1)}{2} + h_p(k)\,.$$

Equation (7) for subscript $p+1$ reads as

$$P_{p+1}(k) = \frac{P_{p+1}(1)}{2^{k-1}} + \frac{h_p(2)}{2^{k-2}} + \cdots + \frac{h_p(k-2)}{2^2} + \frac{h_p(k-1)}{2} + h_p(k)\,.$$

Summing the last two equations and taking into account $P_p(1) = 1 - (-1)^p$, $P_{p+1}(1) = 1 + (-1)^p$, and $h_p(1) = 1$, we get the following recursion—this time in p:

$$P_{p+1}(k) + P_p(k) = 2h_p(k), \quad p \geq 3,\ k \geq 1\,. \tag{8}$$

Summation of identities (8) $P_j(k) = -P_{j-1}(k) + 2h_{j-1}(k)$, multiplied by $(-1)^{p-j}$, for $j = 4, \ldots, p$, yields

$$P_p(k) = 2(-1)^{p-1}\sum_{i=1}^{p-3}(-1)^i h_{i+2}(k) - (-1)^{p-2}P_3(k)\,.$$

Taking into account $P_3(k) = 2$ and $h_2(k) = 1$ we finally obtain

$$P_p(k) = 2(-1)^{p-1}\sum_{i=0}^{p-3}(-1)^i h_{i+2}(k)\,.$$

∎

Formula (5) is in a way the most natural explicit presentation of the polynomials P_p, since the polynomials h_2, h_3, $\ldots$, h_{p-1} form a standard base for the space of all integer-valued polynomials of degree $\leq p-3$ (see [12], p. 129).

Theorem 3.3 is formulated for the values $S(h_p(k),p)$. Nevertheless, its proof shows that we have the following explicit formula for all the values $S(n,p)$:

Corollary 3.4 *Let $p \geq 3$, $k \geq 1$, and let $0 \leq m \leq h_{p-1}(k+1)$. Then*

$$S(h_p(k)+m,p) = P_p(k)2^{k-1} + (-1)^p + m2^k\,.$$

In particular,

$$S(h_p(k+1),p) = P_p(k)2^{k-1} + (-1)^p + h_{p-1}(k+1)2^k\,.$$

4 Proof of Theorem 2.7

Recall that Σ denotes the set of all pairs (n, n_1), for which $S(n,p) = 2S(n_1,p) + S(n-n_1,p-1)$ and $h_p(k) \leq n \leq h_p(k+1)$, $k \geq 2$, and that Π is the parallelogram in the (n, n_1)-plane as depicted on Fig. 1.

We shall prove $\Sigma \subseteq \Pi$ by induction on n (from the above range).

By Theorem 2.6 we know that $(h_p(k), h_p(k-1))$ is the only point of Σ for which the first coordinate equals $h_p(k)$. It obviously belongs to Π, being one of its vertices.

Let $S(n,p) = 2S(n_1,p)+S(n-n_1,p-1)$, $h_p(k) < n \leq h_p(k+1)$, and assume that all points from Σ with the first coordinate less than n belong to Π.

First note that $h_p(k) < n \leq h_p(k+1)$ is equivalent to $k < g_p(n) \leq k+1$, and this is equivalent to $f_p(n) = k+1$.

If $n_1 > h_p(k)$, then $g_p(n_1) > k$, and finally $f_p(n_1) > k$. Hence

$$\begin{aligned}
&2S(n_1,p) + S(n-n_1,p-1) = \\
&2(S(n_1-1,p) + 2^{f_p(n_1)-1}) + S(n-n_1,p-1) > \\
&2S(n_1-1,p) + S(n-n_1,p-1) + 2^k \geq S(n-1,p) + 2^k \geq \\
&S(n-1,p) + 2^{f_p(n)-1} = S(n,p)\,.
\end{aligned}$$

By this, it is proved that $n_1 \leq h_p(k)$.

If $n_1 \leq h_p(k-1)$, then $n-n_1 \geq n-h_p(k-1) > h_p(k)-h_p(k-1) = h_{p-1}(k)$. It follows that $g_{p-1}(n-n_1) > k$, and hence $f_{p-1}(n-n_1) \geq k+1$. Therefore

$$\begin{aligned}
&2S(n_1,p) + S(n-n_1,p-1) = \\
&2S(n_1,p) + S(n-n_1-1,p-1) + 2^{f_{p-1}(n-n_1)-1} \geq \\
&S(n-1,p) + 2^k = S(n-1,p) + 2^{f_p(n)-1} = S(n,p)\,.
\end{aligned}$$

From $S(n,p) = 2S(n_1,p) + S(n-n_1,p-1)$ it follows

$$S(n-1,p) = 2S(n_1,p) + S(n-1-n_1,p-1)\,.$$

Since $h_p(k) \leq n-1 < h_p(k+1)$, by the inductive assumption we get $n_1 \geq h_p(k-1)$.

It follows that $n_1 \leq h_p(k-1)$ is possible only when $n_1 = h_p(k-1)$. This proves that $n_1 \geq h_p(k-1)$.

Now, we know that $h_p(k) < n \leq h_p(k+1)$, $h_p(k-1) \leq n_1 \leq h_p(k)$, and $S(n,p) = 2S(n_1,p)+S(n-n_1,p-1)$. Denote by T_1 the triangle consisting of all pairs (n, n_1) satisfying $n > h_p(k)$, $n_1 \leq h_p(k)$, and $n_1 > n + h_p(k-1) - h_p(k)$; also denote by T_2 the triangle consisting of all pairs (n, n_1) satisfying $n \leq h_p(k+1)$, $h_p(k-1) \leq n_1$, and $n_1 < n + h_p(k) - h_p(k+1)$ (see Figure 1).

If $n_1 > h_p(k-1)$, i.e. $f_p(n_1) = k$ (while $f_p(n) = k+1$), then $S(n,p) = 2S(n_1,p) + S(n-n_1,p-1)$ implies $S(n-1,p) + 2^{f_p(n)-1} = 2S(n_1-1,p) + 2 \cdot 2^{f_p(n_1)-1} + S(n-n_1,p-1)$. It follows that $S(n-1,p) = 2S(n_1-1,p) + S(n-n_1,p-1)$; it means that $(n-1,n_1-1) \in \Sigma$. Since by finitely many repetitions of this transformation, any point from T_1 is transformed to a point strictly above $(h_p(k), h_p(k-1))$, not belonging to Σ, it follows that $T_1 \cap \Sigma = \emptyset$.

If $n_1 < h_p(k)$ (hence $f_p(n_1+1) = k$) and $n < h_p(k+1)$ (hence $f_p(n+1) = k+1$), then from $S(n,p) = 2S(n_1,p) + S(n-n_1,p-1)$ it follows $S(n,p) + 2^{f_p(n+1)-1} = 2(S(n_1,p) + \cdot 2^{f_p(n_1+1)-1}) + S(n-n_1,p-1)$, i.e. $S(n+1,p) = 2S(n_1+1,p) + S(n-n_1,p-1)$. It means that $(n+1,n_1+1) \in \Sigma$. Since by finitely many repetitions of this transformation, any point from T_2 is transformed to a point strictly below $(h_p(k+1), h_p(k))$, not belonging to Σ, it follows that $T_2 \cap \Sigma = \emptyset$.

By all these we have proved that Σ is the subset of the parallelogram Π. The other inclusion $\Pi \subseteq \Sigma$ remains to be proved.

First, note that $n_1 = h_p(k-1)$ and $n = h_p(k)$ satisfy $n_1 < h_p(k)$ and $n < h_p(k+1)$. Therefore, since $(h_p(k), h_p(k-1)) \in \Sigma$, it follows that $(h_p(k)+1, h_p(k-1)+1), (h_p(k)+2, h_p(k-1)+2), \ldots \in \Sigma$, until we get that all the points of the parallelogram, lying on the line $n_1 = n + h_p(k-1) - h_p(k)$, belong to Σ.

Next, note that all points of the parallelogram, except those from the line $n_1 = n + h_p(k) - h_p(k+1)$, satisfy $n_1 > n + h_p(k) - h_p(k+1)$, i.e. $n - n_1 > h_p(k+1) - h_p(k) = h_{p-1}(k+1)$, as well as $n_1 \leq n + h_p(k-1) - h_p(k)$, i.e. $n - n_1 \leq h_p(k) - h_p(k-1) = h_{p-1}(k)$. That means that $f_{p-1}(n-n_1+1) = k+1$. If $h_p(k) \leq n < h_p(k+1)$, then also $f_p(n+1) = k+1$.

Therefore, for such an $(n, n_1) \in \Sigma$, for which n and n_1 satisfy these additional conditions, $S(n,p) = 2S(n_1,p) + S(n-n_1,p-1)$ implies

$$\begin{aligned}
&S(n+1,p) = 2S(n_1,p) + S(n-n_1,p-1) + 2^{f_p(n+1)-1} = \\
&2S(n_1,p) + S(n-n_1,p-1) + 2^{f_{p-1}(n-n_1+1)-1} = \\
&2S(n_1,p) + S(n+1-n_1,p-1)\,,
\end{aligned}$$

i.e. it follows that $(n+1, n_1) \in \Sigma$.

Thus, we are able to complete our proof, by repeatedly translating points from the left edge of the parallelogram, for which we already know that they belong to Σ, to the right for vector $(1,0)$, keeping them in Σ, until we cover all integral points from the parallelogram.

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