# $K_6$-MINORS IN PROJECTIVE PLANAR GRAPHS

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# $K_6$-minors in projective planar graphs

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### Abstract

It is shown that every 5-connected graph embedded in the projective plane with face-width at least 3 contains the complete graph on 6 vertices as a minor.

## 1   Introduction

Let $G$ be a simple graph embedded in a surface $\Sigma$. For the most of the paper we will only concern the case where $\Sigma$ is the projective plane $\mathbb{N}_1$. We use the terminology from [1], basic facts about graph embeddings can be found in [2] or [5]. If $v$ is a vertex and $f$ is a face of $G$ then $\mathrm{d}(v)$ and $\mathrm{d}(f)$ stand for the *degree* of $v$ and *length* of $f$, respectively. By $\delta(G)$ we denote the minimum degree of vertices of $G$.

A *separation* of a graph $G$ is a pair of subgraphs $(H, K)$ such that $E(H) \cap E(K) = \emptyset$, $H \cup K = G$ and $|V(H)| > |V(H) \cap V(K)| < |V(K)|$. The *order* of the separation $(H, K)$ is $|V(H) \cap V(K)|$. A separation of order $k$ is a $k$-*separation*. A graph $G$ is $k$-*connected* if $|V(G)| > k$ and $G$ has no $k'$-separation for any $k' < k$. If $G$ is $k$-connected then $\delta(G) \geq k$.

Let $G$ be a graph embedded in $\Sigma$. We say that a vertex $v$ *sees* a vertex $u$ if $u$ and $v$ lie on a common face. Seeing is a symmetric relation among vertices of $G$ that contains adjacency.

Let $U, W$ be subsets of $V(G)$. A $U - W$ *path* is a path $P$ in $G$ such that one endvertex of $P$ is in $U$ the other endvertex is in $W$ and no intermediate vertex is in $U \cup W$. We say that $S \subseteq V(G)$ *separates* $U$ from $W$ if every $U - W$ path uses a vertex from $S$. We define similarly $u - W$, $U - w$, and $u - w$ paths where $u, w \in V(G)$.

Let $\Sigma$ denote any surface other than the sphere. Let $C$ be a contractible simple closed curve in $\Sigma$. In that case we can define the *interior* of $C$, $\text{Int}(C)$, as the component of $\Sigma \setminus C$ which is homeomorphic to an open disc. Similarly we define the *exterior* $\text{Ext}(C)$ of $C$. If a simple closed curve $C$ is not contractible it is called *essential*. A basic fact from algebraic topology states that any two essential closed curves in the projective plane are *homotopic*. The *face-width* or *representativity* of the graph $G$ in $\Sigma$ is the minimum number of intersecting points of $G$ with any essential simple closed curve and is denoted by $\text{fw}(G)$. See [8] or [5] for an introduction.

A graph $H$ is a *minor* of a graph $G$, $H \leq_m G$, if $H$ can be obtained by a series of contractions of edges from a subgraph of $G$. A minor $H$ of $G$ can be described in the following way: a vertex $v$ of $H$ corresponds to a connected subgraph $v_H$ of $G$, distinct vertices of $H$ correspond to disjoint subgraphs of $G$ and if two vertices $v$ and $u$ of $H$ are adjacent there is an edge of $G$ connecting a vertex of $v_H$ with a vertex of $u_H$. Every vertex of $G$ clearly belongs to at most one $v_H$. The phrase that a vertex of $G$ *is* a certain vertex of the minor $H$ will mean that it *belongs* to some $v_H$, $v \in V(H)$.

A $k$-*wheel* $(W_k)$ is a graph with vertices $v_0, v_1, \ldots, v_k, \ldots$ such that $v_0$ is adjacent exactly to vertices $v_1, \ldots, v_k$ and the vertices of $W_k$ distinct from $v_0$ induce a cycle; $v_0$ is called the *hub* of the wheel, the vertices $v_1, \ldots, v_k$ are called *spoke vertices*, the other vertices are called *rim* vertices and may or may not be present; edges incident with $v_0$ are called *spokes*, the remaining ones are called *rim* edges. Further on, we assume that spoke vertices appear along the cycle according to their indices.

Let $G$ be a graph embedded in $\Sigma$, and let $v \in V(G)$. The *surface neighborhood* of $v$ is the closure of the union of faces incident with $v$. If $G$ is 3-connected and $\text{fw}(G) \geq 3$ (or $G$ is plane and 3-connected), then the surface neigborhood of any vertex is a closed disc with induced embedding of a wheel. This implies that in this case no two faces of $G$ can both be incident with a pair of nonadjacent vertices of $G$.

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Let

$$C = v_1 f_1 \ldots v_k f_k v_1 \tag{1}$$

be a sequence such that $v_i$ $(i = 1, \ldots, k)$ are distinct vertices of $G$, $f_i$ $(i = 1, \ldots, k)$ are faces of $G$, and each face $f_i$ is incident with vertices $v_i$ and $v_{i+1}$. Then we say that $C$ is a *face chain* (FC). We call $v_i$ and $f_i$ $(i = 1, \ldots, k)$ the *vertices* and *faces* of $C$, respectively, and write $V(C) = \{v_1, \ldots, v_k\}$. We also call $v_i$ and $f_i$ the *beads* of $C$ and define $k$ to be the *length* of $C$. We define the length of a (contiguous) subsequence $C'$ of $C$ as one half $r$, where $r$ is the number of beads of $C'$ minus one. Note that the length of the subsequence is an integer if and only if the end beads are of the same type. The face chain $C$ determines a closed curve $\Gamma = \Gamma(C)$ which intersects $G$ in vertices $v_1, \ldots, v_k$ and runs through faces $f_1, \ldots, f_k$ (in this order). If each $v_i$ occurs on the boundary of $f_{i-i}$ and $f_i$ at most once, then $\Gamma(C)$ is determined up to homotopy. We may assume that $\Gamma$ is a simple closed curve unless there are indices $i < j$ such that $f_i = f_j$ and $v_i, v_j, v_{i+1}, v_{j+1}$ occur in this (interlaced) order along the boundary of $f_i$. If $\Gamma$ is essential, then we say that $C$ is an *essential face chain*. Suppose now that $\Gamma$ is contractible and simple. The graph $G_p := G \cap (\mathrm{Int}(\Gamma) \cup \Gamma)$ is clearly a plane graph and will be referred to as the *plane part* of $C$. The graph $G_x := G \cap (\mathrm{Ext}(\Gamma) \cup \Gamma)$ will be called the *exterior part* of $C$. If $|V(G_p)| > k$ then $C$ is called a *k-separating face chain* (*k*-SFC for short). Furthermore, if $|V(G_x)| > k$ then $(G_p, G_x)$ is a *k*-separation.

**Lemma 1** *Let $G$ be a 5-connected graph and let $C = v_1 f_1 \ldots v_5 f_5 v_1$ be a 5-SFC. Then either $V(G_p) = 6$ or $V(G_p) \geq 11$. In the latter case, there exists a vertex $v \in V(G_p)$ that sees no vertices of $C$.*

**Proof.** Let us call $v_1, \ldots, v_5$ the outer vertices of $G_p$ and call the remaining ones interior vertices. We have to show that there cannot be less than 6 interior vertices provided there are at least two of them. Without the loss of generality, we can assume that outer vertices induce a 5-cycle in the order of their indices. Suppose that $G_p$ is a counterexample with a minimum number of vertices and let $r = |V(G_p)|$.

Suppose first that $r = 7$. Let $u$ and $w$ be the interior vertices. Since $\mathrm{d}(u) \geq 5$, $u$ has at least four neighbors in $V(C)$. So has $w$. Then $u$ and $w$ have at least three common neighbors in $V(C)$ which is clearly not possible. Therefore $r \geq 8$.

If $u \in V(G_p)$ is the only interior vertex adjacent to $v_1$, the sequence $u, v_2, v_3, v_4, v_5$ would induce another 5-SFC, its plane part would have one vertex less than $G_p$ and at least 7 vertices which is a contradiction to the minimality of $G_p$. Therefore, every outer vertex of $G_p$ is adjacent to at least two interior vertices. Obviously, no interior vertex can be adjacent to three consecutive outer vertices and there can be at most one adjacent to three nonconsecutive ones. Since $\delta(G) \geq 5$, this implies that $G_p - V(C)$ is a plane graph with at most one vertex of degree $\leq 2$; hence it is not outerplanar. Let $v$ be an interior vertex of $G_p - V(C)$. Its neighborhood alone contains at least 6 vertices and these are distinct from vertices of $C$. This completes the proof. $\qquad\square$

We will use another technical lemma.

**Lemma 2** *Let $G$ be either a plane or projective plane graph with minimum degree 5. Then $G$ contains a vertex $v$ of degree 5 which is incident with at least four triangular faces and the fifth face $f$ incident with $v$ has length at most 5. Moreover, if $\mathrm{d}(f) = 4$, at least three vertices of $f$ have degree 5. If $\mathrm{d}(f) = 5$, all vertices of $f$ have degree 5.*

**Proof.** We will prove this fact using discharging method. Let $\mathrm{d}(v) - 6$ and $2\mathrm{d}(f) - 6$ be the initial charges of every vertex $v$ and face $f$, respectively. Plugging

$$\sum_{v \in V(G)} \mathrm{d}(v) = 2|E(G)| \quad \text{and} \quad \sum_{f \in F(G)} \mathrm{d}(f) = 2|E(G)|$$

into Euler's formula we obtain

$$\sum_{v \in V(G)} (\mathrm{d}(v) - 6) + \sum_{f \in F(G)} (2\mathrm{d}(f) - 6) < 0.$$

The total initial charge is negative. Now we redistribute the charge from positively charged faces of $G$ according to the following rule:

R. Every face $f$ of length at least 4 distributes its positive charge uniformly among vertices of degree 5 incident with it.

After charge distribution, only vertices of degree five can be negatively charged. Since the total charge of $G$ remains negative, there exists a negatively charged

vertex $v$ which can be incident with at most one nontriangular face. More-over, if $v$ is incident with a face $f$ of length $\geq 4$, then more than $2\mathrm{d}(f) - 6$ vertices of $f$ have degree 5. This completes the proof. $\qquad\square$

Throughout the paper we will use the following two theorems. The first one can be proved using Menger's theorem (cf. [6] or [9]).

**Theorem 3** *Let $G$ be a plane graph and let $s_1, \ldots, s_l, t_l, \ldots, t_1$ be vertices along the outer face $f_0$ in that order. Then $G$ contains disjoint $s_i - t_i$ paths $P_i$ ($i = 1, \ldots, l$) if and only if there is no $k$-separating face chain $C$ such that $f_0$ is a bead of $C$ and vertices of $C$ separate more than $k$ pairs $s_i, t_i$.*

The second result has been proven by Robertson and Seymour in [6].

**Theorem 4** *Let $G$ be a projective plane graph and $f$ a face bounded by a facial cycle $s_1 s_2 s_3 t_1 t_2 t_3$ of length 6. Then $G$ contains disjoint $s_i - t_i$ paths $P_i$ ($i = 1, 2, 3$) if and only if there is no $k$-SFC with $k \leq 5$ containing $f$ in its interior.*

## 2   Main theorem

Throughout this section, let $G$ be a 5-connected graph embedded in the projective plane $\mathbb{N}_1$ with face-width at least 3. We will show that $G$ contains $K_6$ as a minor.

**(1)** *There exist no $k$-SFC for $k \leq 4$.*

**Proof.** Let $C$ be a $k$-SFC and $k \leq 4$. Let $G_p$ and $G_x$ be the plane part and the exterior part with respect to $C$. If $|V(G_x)| = k \leq 4$ we can find an essential curve $\Gamma$ with $|\Gamma \cap G| \leq 2$. Otherwise $|V(G_x)| > k$ and $(G_p, G_x)$ is a $k$-separation. A contradiction. $\qquad\square$

**(2)** *Suppose $C$ is a 5-SFC with vertices $V(C) = \{v_1, v_2, v_3, v_4, v_5\}$. If $V(G_x) = V(C)$ then $v_1 v_3 v_5 v_2 v_4$ is a 5-cycle of $G$ and is contained in $G_x$.*

**Proof.** With local deformation of $C$ we may assume that $G_p$ contains all the edges of $G$ of the form $v_i v_{i+1}$. It is then enough to show that $\delta(G_x) = 2$. Suppose this is not the case. Then $G_x$ is a proper subgraph of the 5-cycle $v_1 v_3 v_5 v_2 v_4$. We may assume that the edge $v_1 v_3$ is not present in $G_x$. In this case, vertices $v_4$ and $v_5$ cover all edges of $G_x$. Now the induced embedding of $G - v_4 - v_5$ is in the plane. A contradiction to $\mathrm{fw}(G) \geq 3$. $\qquad\square$

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## Choosing a 5-wheel minor $W$

Choose an arbitrary vertex $v$ of degree 5 which satisfies the conclusion of Lemma 2. Let $v_1, v_2, v_3, v_4$ and $v_5$ be its neighbors and let $g_i$ $(i = 1, \ldots, 5)$ denote the face incident with $v$, $v_i$, and $v_{i+1}$ (indices modulo 5). If $v$ is incident with a nontriangular face, let $g_5$ be the one. Vertices $v_i$ $(i = 1, \ldots, 5)$ are spoke vertices of the 5-wheel induced by the surface neighborhood of $v$.

Let $C = u_1 f_1 u_2 f_2 u_3 f_3 u_4 f_4 u_5 f_5 u_1$ be a maximal (in the sense of containment of edges in its interior) 5-SFC that contains $v$ in its interior. Clearly, $C$ exists, because $v_1 f_1 v_2 f_2 \ldots v_5 f_5 v_1$ is a 5-SFC which contains $v$ in its interior. We define $G_p$ and $G_x$ with respect to $C$.

First suppose that $v$ is the only vertex in $\mathrm{Int}(C)$. In this case $v_i = u_i$ for $i = 1, \ldots, 5$. Also observe that $f_i \neq g_i$ for $i = 1, \ldots, 4$ and $f_5 = g_5$ if and only if $g_5$ is not a triangle. We set $W$ as the 5-wheel induced by the neighborhood of $v$.

Now, suppose $\mathrm{Int}(C)$ contains at least two vertices. By Lemma 1 there is an interior vertex $u_0$ of $G_p$ which does not see any of the vertices $u_i$ $(i = 1, \ldots, 5)$. Since $G$ is 5-connected, there are internally disjoint paths $P_i$ joining $u_0$ with $u_i$, $i = 1, \ldots, 5$. It is easy to see that $P_i \subseteq G_p$ for all $i = 1, \ldots, 5$. The neighorhood of $u_0$ is strictly contained in $\mathrm{Int}(C)$. It is easy to observe that it contains internally disjoint $P_i - P_{i+1}$ paths $Q_i$ for $i = 1, \ldots, 5$. Let $W$ be a graph induced by the union $\bigcup_{i=1}^{5}(P_i \cup Q_i)$. Clearly $W$ is contractible to a 5-wheel with hub $u_0$ and $u_1, \ldots, u_5$ are contained in the spoke vertices of the 5-wheel minor.

In both cases the following is true.

**(3)** *Let $C$ be a maximal 5-SFC in $G$ containing $v$ in its interior and let $u_1, \ldots, u_5$ be the vertices of $C$. Then there is a 5-wheel minor $W$ having vertices of $C$ as its spoke vertices. Moreover, $W$ is entirely in $\mathrm{Int}(C) \cup V(C)$ except in the case when $\mathrm{Int}(C)$ contains exactly one vertex of degree five which is not incident with triangles only. In this case, only the subdivided rim edge $u_1 u_5$ is realized in the exterior of $C$.*

## Finding a suitable cycle minor $U$ in $G_x$

In this part we will mostly work with the outer part $G_x$ of the 5-SFC $C = u_1 f_1 u_2 f_2 u_3 f_3 u_4 f_4 u_5 f_5 u_1$ chosen above. Let $U \subseteq G_x$ denote a graph which contracts to a 5-cycle with vertices $u_1, u_3, u_5, u_2, u_4$ in that order. Our goal is to show that $U$ exists. Observe that if $W$ is the 5-wheel minor from (3) and

$W$ is contained in $\text{Int}(C) \cup V(C)$, then $W \cup U$ is contractible to $K_6$. If the subdivided rim edge $u_1 - u_5$ of $W$ is in the exterior part of $C$, then we shall argue that $U$ can be chosen so that $W \cup U$ still contains $K_6$ minor. More precisely, let $U_i$ be the connected subgraph of $U$ which is contracted to $u_i$ in order to get the 5-cycle. Then $W \cup U$ is contractible to $K_6$ if $U_3 \cap W = \{u_3\}$, i.e. $U_3$ does not contain a vertex of the $u_1 - u_5$ segment of $W$.

First we prove a statement concerning 6-SFCs in $G$.

**(4)** *At most one of the vertices $u_i$ $(i = 1, \ldots, 5)$ can be a vertex of a 6-SFC $D$ such that $\text{Int}(C) \cup f_i \cup f_{i-1} \subseteq \text{Int}(D)$. If such a 6-SFC exists, then $D$ and $C$ have no beads in common apart from $u_i$.*

**Proof.** First we show that apart from $u_i$ the 6-SFC $D$ which satisfies the assumptions contains no other bead of $C$. Suppose this is not the case. Let $x$ be another common bead of $C$ and $D$. Then $u_i$ and $x$ divide $C$ and $D$ into subsequences of lengths $c_1$, $c_2$ and $d_1$, $d_2$, respectively, see Figure 1. By the
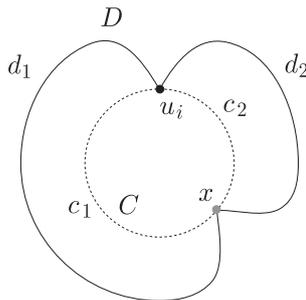


Figure 1: A 6-separating face chain $D$.

maximality of $C$ we have $c_1 + d_2 \geq 6$ and $c_2 + d_1 \geq 6$. On the other hand, $c_1 + c_2 + d_1 + d_2 = 5 + 6 = 11$ which is a contradiction.

Let $i \neq j$ and suppose that there exist $D_i$ and $D_j$, each satisfying the conditions of the proposition, containing vertices $u_i$ and $u_j$, respectively. By the preceeding paragraph, $u_i$ lies in the interior of $D_j$, and $u_j$ lies in the interior of $D_i$. Therefore $D_i$ and $D_j$ share at least two beads. Denote them by $x$ and $y$ and label the lengths of segments according to Figure 2. We may assume that, apart from $x$ and $y$, the segments labeled with $e_1$ and $e_3$ contain no bead of $D_i$. If $e_1 + e_3 \leq d_2$, we could replace the segment of $D_i$ of length $d_2$ by the two segments of $D_j$. This would either contradict maximality of $C$ or give a 6-SFC intersecting $V(C)$ in two vertices. Hence,
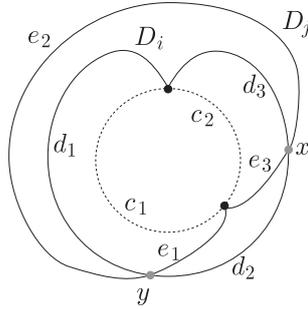
Figure 2: 6-separating face chains $D_i$ and $D_j$.

$d_2 \leq e_1 + e_3 - 1$. This easily implies that $d_2 + e_2 \leq 5$. The corresponding segments of $D_i$ and $D_j$ thus form a 5-SFC $C'$ (contradicting maximality of $C$) unless they have a vertex $z$ in common. In that case, $C'$ contains a face chain of length $\leq 2$. Since $G$ is 3-connected, this face chain is essential. This implies that $\mathrm{fw}(G) \leq 2$, a contradiction. □

We call a vertex $u_i$ *nice* if it is not contained as a bead in a 6-SFC which satisfies the conditions of (4). Since at least one of $u_2$ and $u_4$ is nice we may assume that $u_2$ is a nice vertex. We say that the face $f_i$ ($i = 1, \ldots, 5$) is *essential* if $f_i \cup W$ contains an essential cycle. This is true if $f_i$ contains the vertex $v_{i+3}$ (index modulo 5). There is another possibility for $f_2$ and $f_3$ to be essential without containing vertices $u_5$ and $u_1$, respectively. This is possible if $W$ is not contained in $G_p$ and $f_2$ (or $f_3$) intersects the $u_1 - u_5$ segment which is in $G_x$.

Suppose that $f_i$ is an essential face. Then $f_i$ or $f_5$ is a face of every 6-SFC $D$ which contains $\mathrm{Int}(C)$ in its interior. Now, (4) implies:

(5) *If some face $f_i$ ($i = 1, \ldots, 5$) is essential, then all vertices $u_1, \ldots, u_5$ are nice.*

We now have the necessary tools to construct the suitable cycle minor.

(6) *Suppose that neither $f_1$ nor $f_2$ are essential faces and that $u_2$ is a nice vertex. Then $G_x$ contains a subgraph $U$ which contracts to a 5-cycle through vertices $u_1, u_3, u_5, u_2, u_4$, respectively, and such that $U_3 \cap W = \{u_3\}$.*

**Proof.** Let $x_1$ and $x_3$ be the vertices of $G_x$ incident with $f_1$ and $f_2$, respectively, and adjacent to $u_2$. Observe that $x_1 \neq u_1$ and $x_3 \neq u_3$ by the
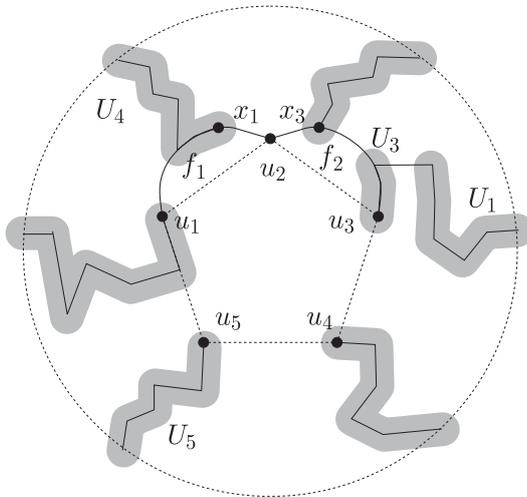
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Figure 3: Contracting paths in $U$.

maximality of $C$. Delete $u_2$ from $G_x$ and add an edge joining vertices $x_1$ and $x_3$. We add edges $x_1u_1$, $x_3u_3$ and $u_1u_5$ as well. Denote the resulting graph by $G'_x$. We want to find disjoint $x_1 - u_4$, $x_3 - u_5$ and $u_1 - u_3$ paths in $G'_x$. According to Theorem 4 the only obstruction for existence of such paths is a 5-SFC $C'$ containing the endpoints of required paths in $\text{Int}(C') \cup C'$. By the maximality of $C$, $C'$ is not a 5-SFC in $G$. This is only possible if one of the beads of $C'$ is a face incident with the edge $x_1x_3$. Such a $C'$ would then induce a 6-SFC containing $u_2$ and satisfying the assumptions of (4). This is a contradiction since $u_2$ is a nice vertex.

Contracting appropriately the obtained paths as shown in Figure 3, the noncontracted edges induce a required cycle. We may assume that the $u_1 - u_3$ path has connected intersection with $f_2$. Then we may achieve $U_3 \subseteq f_2$, so that $U_3 \cap W = \{u_3\}$. $\qquad\square$

The following proposition settles another case.

**(7)** *Suppose that $f_1$ is an essential face but $f_2$ is not essential. Then we can find a subgraph $U$ in $G_x$ as stated in (6).*

**Proof.** The situation is roughly depicted in Figure 4. Since $f_1$ is an essential face, it contains $u_4$. We cut the projective plane along the corresponding essential curve (shown as the dotted line), split the vertex $u_4$ and obtain a 'planar' drawing of $G_x$. This drawing is represented in Figure 4 as the square
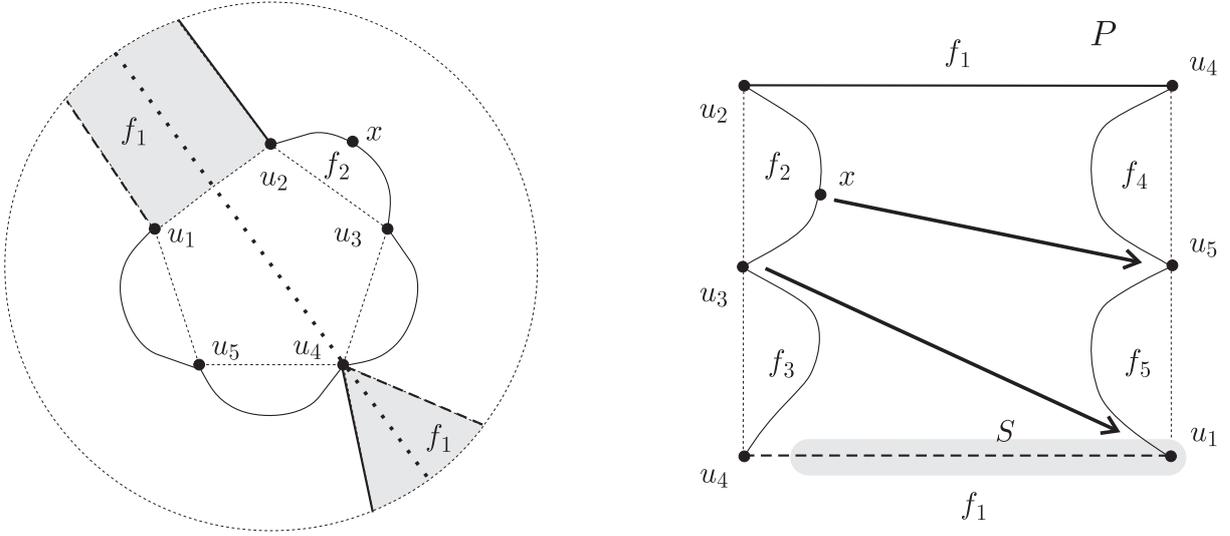
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Figure 4: Cutting when $f_1$ is an essential face.

$P$. Let $S$ denote the set of vertices of the $(u_4, u_1]$ segment along $f_1$ and let $x \in V(G_x)$ be the neighbor of $u_3$ incident with $f_2$. In order to find $U$ it is enough to find pairwise disjoint paths $Q_1$ linking $x$ with $u_5$, and $Q_2$ linking $u_3$ with $u_1$ (respectively) that are disjoint from vertices on the top segment of $P$ and the left copy of $u_4$.

Suppose that $f$ is a face lying inside the square $P$, where $f \neq f_i$ ($i = 1, \ldots, 5$). The following observations are true and they are easy to argue:

1. $f$ is not incident with the base and the top of $P$ at the same time.

2. $f$ is not incident with $f_4$ and the left copy of $u_4$ at the same time.

3. If $f$ is incident with the left copy of $u_4$ and $f'$ is a face in $P$ incident with the right copy of $u_4$, then $f$ and $f'$ do not share a vertex (distinct from $u_4$).

Concerning the above observations and using Theorem 3 (applied to $P$ after removal of the left copy of $u_4$ and the top segment of $P$) the only possible obstruction for the required paths $Q_1$ and $Q_2$ is a pair of faces $g$ and $g' \neq f_4$ in $P$ sharing a common vertex $y$ such that $g$ is incident with the left copy of $u_4$ and $g'$ is incident with a vertex $z \neq u_4$ (by observation 3) on
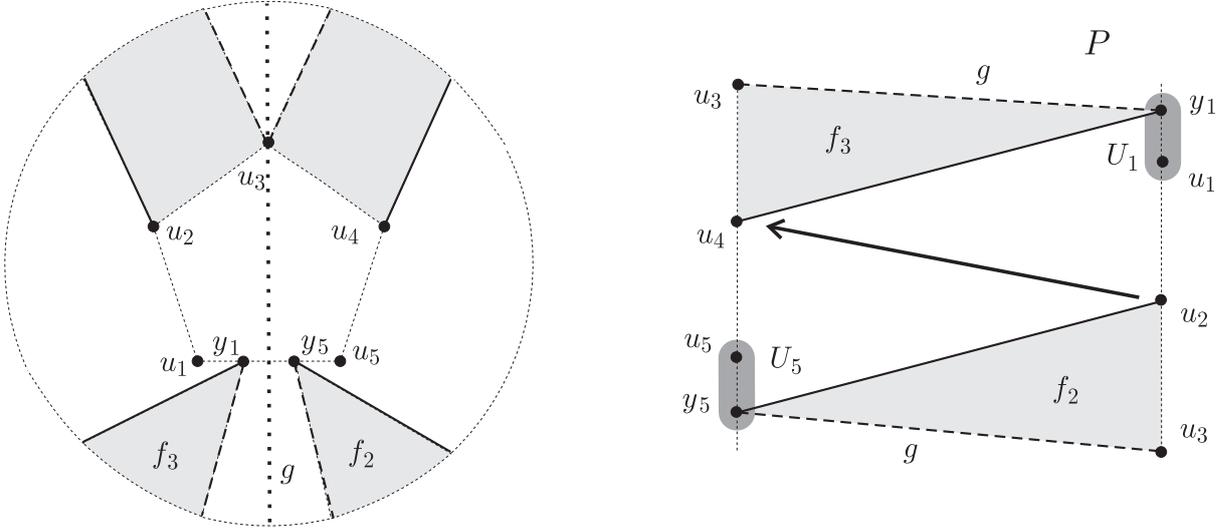
Figure 5: Cutting when both $f_2$ and $f_3$ are essential.

the segment $[u_2, u_4)$ in $P$. In this case the sequence $u_4 g y g' z f_1 u_1 f_5 u_5 f_4 u_4$ is a 5-SFC contradicting the maximality of $C$.

Now it is easy to observe, that we can choose $U_3$ to consist of the vertex $u_3$ only. $\qquad \square$

By (6) and (7) we may assume that $f_2$ is essential. By (5), all vertices $u_i$ $(i = 1, \ldots, 5)$ are nice. Hence, we may apply (6) to $u_4$, $f_3$, and $f_4$ if neither $f_3$ nor $f_4$ is essential. If $f_4$ is essential and $f_3$ is not essential, we apply the arguments of (7) to $u_4$, $f_3$ and $f_4$, by symmetry. Therefore we may assume, that both, $f_2$ and $f_3$ are essential. This case is considered in the next proposition.

**(8)** *Suppose that $f_2$ and $f_3$ are essential faces. Then $G_x$ contains $U$ as stated in (6).*

**Proof.** Since $f_2$ and $f_3$ are essential, they both contain at least one vertex from the $u_1 - u_5$ segment of $W$. Denote them by $y_5$ and $y_1$, respectively. The situation is depicted in Figure 5.

Note that if $W \subseteq G_p$, then $y_1 = u_1$ and $y_5 = u_5$. In the case $W \not\subseteq G_p$, $f_2$ and $f_3$ do not share a vertex apart from $u_3$, since $d(u_3) \geq 5$. In both cases $y_5 \neq y_1$.

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Now, consider the 3-FC $u_3 f_3 y_1 f_5 y_5 f_2 u_3$. Since it is not a separating face-chain, it bounds a triangular face. We denote the latter by $g$. This implies, that $u_3$ is adjacent to both $y_1$ and $y_5$ and we can set $U_3$, the connected subgraph which contracts to the third vertex of $U$, to be $\{u_3\}$.

Furthermore, we set $U_i := \{y_i, u_i\}$ $(i = 1, 5)$. We cut the projective plane along the dotted line splitting the vertex $u_3$ in order to obtain a 'planar' drawing $P$ of $G_x$. Now, it suffices to find a path $Q$, linking $u_2$ with $u_4$, which is disjoint from both $U_1$ and $U_5$.

Applying Theorem 3 to the disk obtained from $P$ by deleting $u_3$, $U_1$, and $U_5$, yields that the only possible obstruction for $Q$ would be a face $h \neq f_5$ containing vertices from both, $U_1$ and $U_5$. However, such a face $h$ does not exist since it would give rise to an essential curve intersecting $G$ only in two vertices, contradicting $\mathrm{fw}(G) \geq 3$. $\qquad\square$

Now we state the main theorem which is a direct consequence of (3), (6)–(8), and the remarks preceeding (8).

**Theorem 5** *If $G$ is a 5-connected graph embedded in the projective plane with $\mathrm{fw}(G) \geq 3$ then it contains a $K_6$ minor.*

# 3    Final remarks

Theorem 5 is in a sense best possible. We cannot relax any of the assumptions. Observe that $K_6$ triangulates the projective plane. Therefore every embedding of $K_6$ in $\mathbb{N}_1$ has face-width at least 3. Since the face-width is minor-monotone, no projective graph of face-width less than 3 contains a $K_6$-minor. Neither can we relax the connectivity assumption.

**Proposition 6** *There is an infinite family of 4-connected projective planar graphs with minimum degree 5 and face-width 3 which do not contain a $K_6$-minor.*

**Proof.**  Let $G$ be a graph shown on Figure 6. The frame of $G$ is a graph $H$ depicted in black which consists of 7 vertices and 13 edges. $H_1$ and $H_2$ are plane graphs attached to the frame such that $G$ is 4-connected, has minimum degree 5 and face-width 3. Such graphs $H_1$ and $H_2$ obviously exist; any 4-connected planar graph with minimum degree 5 and with a face of length 4 is a candidate.
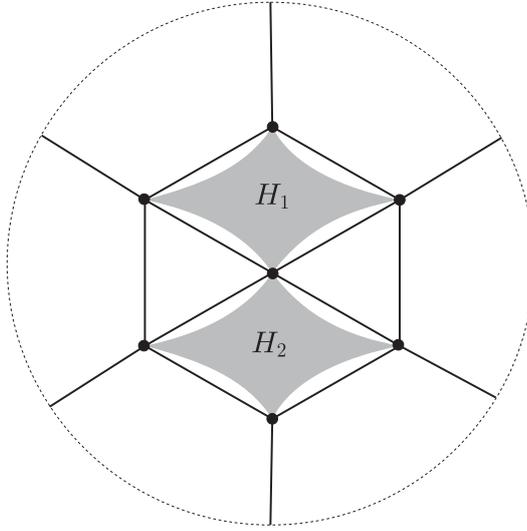
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Figure 6: A graph with no $K_6$ minor.

Suppose $G$ contains a $K_6$ minor. Since $G$ is connected we can obtain a $K_6$ using contractions and deletions of edges only. First, contract and delete as many edges of $E(G) \setminus E(H)$ as possible such that the resulting graph $G'$ still contains a $K_6$ minor. Denote by $H'_1$ and $H'_2$ the resulting minors of $H_1$ and $H_2$, respectively. Observe that the subset of vertices of $G'$ that corresponds to a vertex of $K_6$ is either a connected set of frame vertices or a single vertex.

Let $U'_i = V(H'_i) \setminus V(H)$, $i = 1, 2$. If $U'_1 = U'_2 = \emptyset$, then $H'_1$ and $H'_2$ contribute to the $K_6$ minor at most one diagonal edge each. Hence $G'$ contains 7 vertices and at most 15 edges. Since $K_6$ has 15 edges as well, it cannot be a minor of $G'$.

Suppose now that $r := |U'_1| > 0$. The vertices of $U'_1$ induce the complete graph $K_r$, hence $r \leq 4$. Each of these vertices is joined to the vertices of the $K_6$ minor outside $U'_1$. This implies that $r \neq 4$. If $1 \leq r \leq 3$, each vertex $u \in U'_1$ has $6 - r$ edges joining $u$ with the frame. This is easily seen to be impossible. The proof is complete. $\qquad\qquad\square$

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