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A counterexample to the uniform shortest path routing conjecture for vertex-transitive graphs

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Abstract

In this note we disprove the uniform shortest path routing conjecture for vertex-transitive graphs. An infinite family of counterexamples is given.

Let G be a connected graph on a vertex set V. A *routing* in G is a set $R = \{P_{uv}|\ (u,v) \in V \times V,\ u \neq v\}$ of $|V|(|V|-1)$ paths in G, where each individual path P_{uv} has initial vertex u and terminal vertex v. (We note that the paths P_{uv} and P_{vu} may be different.) We say that R is a *shortest path routing* if the length of each path $P_{uv} \in R$ is equal to the distance $d(u,v)$ of the vertices u and v in the graph G. Any subset R' of a routing R is a *partial routing*. For any vertex w of G the *load* of w in a partial routing $R' \subset R$, denoted by $\xi(R', w)$, is the number of paths in R' that *pass through* w (i.e., that contain w as an internal vertex). For a routing R let $\xi(R)$ be the maximum of the loads $\xi(R, w)$ over all vertices w of G. The *vertex-forwarding index* $\xi(G)$ of the graph G is the minimum of $\xi(R)$ over all routings R in G.

Routings and the associated forwarding indices have been studied extensively (see e.g. [1, 5, 6, 3]); for a most recent survey we recommend [4] from which the following basic facts can be extracted.

Proposition 1 *Let G be a connected graph of order n. Then*

$$\xi(G) \geq \frac{1}{n} \sum_{u} \sum_{v \neq u} (d(u,v) - 1) \ ,$$

with equality if and only if there exists a shortest path routing in G for which the load of all vertices is the same.

A routing for which equality holds in the above will be called a *uniform shortest path routing*. As observed by several authors (cf. [4] for details), each connected Cayley graph has a uniform shortest path routing. In this connection the following natural conjecture appeared in [5] (see also [4]):

Conjecture 2 *Every connected vertex-transitive graph admits a uniform shortest path routing.*

The result about uniform shortest path routings in Cayley graphs was extended in [3] to connected graphs having regular families of automorphisms, that is, *subsets* (not necessarily subgroups) of automorphisms that act regularly on the vertex sets. All such graphs are vertex-transitive, confirming thereby the above Conjecture.

A natural class of graphs to test the Conjecture are the *generalized Petersen graphs* $P(n,k)$ of order $2n$ with vertex set $\{u_i,\ v_i |\ i \in \mathbb{Z}_n\}$ and edge set $\{u_i v_i,\ u_i u_{i+1},\ v_i v_{i+k} |\ i \in \mathbb{Z}_n\}$. Note that $P(5,2)$ is the classical Petersen graph and $P(10,2)$ is the graph of the dodecahedron. It is well known [2] that $P(n,k)$ is vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod{n}$ or $(n,k) = (10,2)$. Also, it follows from [3] that all generalized Petersen graphs $P(n,k)$ with $k^2 \equiv \pm 1 \pmod{n}$ have a regular family of automorphisms, and thus admit uniform shortest path routings.

Curiously enough, the exceptional dodecahedron graph $P(10,2)$, which is well known to be vertex-transitive but not Cayley, turns out to be a counterexample to Conjecture 2!

Theorem 3 *The graph of the dodecahedron has no uniform shortest path routing.*

Proof. Throughout, let $G = P(10, 2)$ be the dodecahedron graph with vertex set V. We will refer to G as depicted in Figure 1; let $V = V_O \cup V_I$ where V_O is the set of the ten vertices on the outer rim in Figure 1 and V_I is the set of the remaining ten "inner" vertices of G.

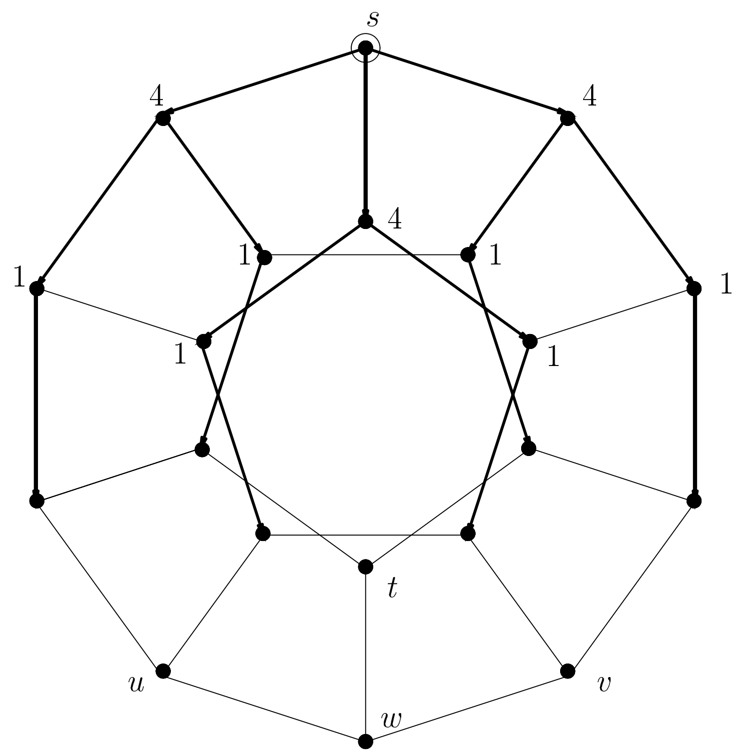

Figure 1: The local partial routing R_s^3 of paths of length at most 3.

Assume the contrary and let $R = \{P_{uv} | \ (u, v) \in V \times V, \ u \neq v\}$ be a uniform shortest path routing in G. For each vertex z of G let R_z be the induced "local" partial routing given by $R_z = \{P_{zv} | \ v \in V, \ v \neq z\}$. For any partial routing R' of R we introduce the concept of the *difference* $\Delta(R')$ defined by

$$\Delta(R') = \sum_{x \in V_O} \xi(R', x) - \sum_{y \in V_I} \xi(R', y) \ .$$

Of course, from the assumptions on our routing R it follows that $\Delta(R) = 0$. As $R = \cup_{z \in V} R_z$, we obviously have

$$\Delta(R) = \sum_{z \in V} \Delta(R_z) = \sum_{x \in V_O} \Delta(R_x) + \sum_{y \in V_I} \Delta(R_y) = 0. \qquad (1)$$

Let s be a fixed vertex in V_O and for $3 \leq i \leq 5$ let R_s^i denote the partial routing consisting of all paths in R_s of length at most i. The basic observation here is that because in our (partial) routings only shortest paths appear, the partial routing R_s^3 is uniquely determined – in the sense that it does not depend on the choice of the shortest path routing R we begin with. In Figure 1 the numbers at vertices are the loads induced by the partial routing R_s^3 (displayed in thick lines). For the corresponding difference we therefore obtain

$$\Delta(R_s^3) = \sum_{x \in V_O} \xi(R_s^3, x) - \sum_{y \in V_I} \xi(R_s^3, y) = 10 - 8 = 2.$$

Next, we focus on the three vertices t, u and v (Figure 1) at distance 4 from s. Looking at Figure 1 we see that for the path $P_{st} \in R_s^4 \subset R$ there are only two choices, and as each of them (when regarded as a partial routing consisting of a single path) has difference -1 it follows that $\Delta(P_{st}) = -1$. For the path $P_{su} \in R$ of length 4 we again have two possibilities but this time the difference of one of the paths is $+3$ while the difference of the other one is -3 (consult Figure 1); therefore $\Delta(P_{su}) = \pm 3$. Due to symmetry, the same holds for the vertex v . Thus, regardless of which of these paths of length 4 actually appear in $R_s^4 \subset R$, we have

$$\Delta(R_s^4) = \Delta(R_s^3) + \Delta(P_{st}) + \Delta(P_{su}) + \Delta(P_{sv}) = 2 - 1 \pm 3 \pm 3 \ ,$$

and therefore $\Delta(R_s^4) \in \{-5, 1, 7\}$.

It remains to examine the unique vertex w at distance 5 from s. The path $P_{sw} \in R_s^5 \subset R$ passes either through the vertex t or (without loss of generality) through u. In the first case we have a choice of two paths, both of difference -2. In the second case we again have two possible paths, this time of difference 4 and -2 (which can be seen in Figure 1). Therefore $\Delta(P_{sw}) \in \{-2, 4\}$, and so

$$\Delta(R_s) = \Delta(R_s^5) = \Delta(R_s^4) + \Delta(P_{sw}) \in \{-7, -1, 5, 11\} \ .$$

In particular, it follows that for each $x \in V_O$ we have

$$\Delta(R_x) \equiv -1 \pmod 6 \quad \text{for each } x \in V_O \ . \tag{2}$$

In a similar manner the reader may check that for vertices in V_I the difference values of the local partial routings are even more restricted; namely,

$$\Delta(R_y) = -1 \quad \text{for each } y \in V_I \ . \tag{3}$$

The rest is straightforward: By the equations (1), (2) and (3) we obtain

$$0 = \Delta(R) = \sum_{x \in V_O} \Delta(R_x) + \sum_{y \in V_I} \Delta(R_y) \equiv -20 \pmod 6 \ ,$$

a contradiction. □

More examples can be found from the dodecahedron graph, for example by forming graph products with other graphs. The *strong product* $G \boxtimes H$ of graphs G and H has as vertices the pairs (g, h) where $g \in V(G)$ and $h \in V(H)$. Vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $\{g_1, g_2\}$ is an edge of G and $h_1 = h_2$ or if $g_1 = g_2$ and $\{h_1, h_2\}$ is an edge of H or if $\{g_1, g_2\}$ is an edge of G and $\{h_1, h_2\}$ is an edge of H. It is well known and easy to see that the strong product of vertex transitive factors is a vertex transitive graph. For more details of strong products we recommend [7]. Let K_q be the complete graph on q vertices.

Theorem 4 *Let $q \not\equiv 0 \pmod 3$. The graphs $K_q \boxtimes P(10, 2)$ have no uniform shortest path routing.*

Proof. Let (g_1, h_1) and (g_2, h_2) be arbitrary two vertices of $K_q \boxtimes P(10, 2)$. Observe that the distance between them is the same as the distance between h_1 and h_2 in $P(10, 2)$ and furthermore that any shortest path projects to a shortest path of the same length in the factor $P(10, 2)$.

One can now argue along the same lines as in the proof of Theorem 3, considering all q vertices that project to the same vertex of $P(10, 2)$ at the same time.

Take any vertex that projects to s, say $\hat{s} = (g, s)$. First consider the vertices at distance at most 3. Recalling above observation, each shortest path projects to a shortest path in the factor. Since exactly q vertices project to each vertex of $P(10, 2)$, the difference is $10q - 8q = 2q$. The vertices at distance 4 from $\hat{s}$ project to t, u and v. The corresponding differences are -1, ± 3 and ± 3, and hence the sum of differences can be any number from the set $\{-7q, -7q+6, \ldots, -q, \ldots, 5q\}$, in each case however it is $-q$ mod 6. The vertices at distance 5 contribute q times 4 or -2, which gives $4q$ mod 6 in each case.

Thus the difference

$$\Delta(R_{\hat{s}}) \equiv (2q - q + 4q) = -q \pmod 6.$$

Therefore all vertices that project to the outer rim contribute $10q(-q) = -10q^2$ mod 6. Similarly, one could see that the $10q$ vertices that project to the inner rim give difference $-q$ each. Hence

$$\Delta(R) \equiv -20q^2 \pmod 6.$$

which is not congruent to 0 unless q is divisible by 3. □

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References

[1] F. Chung, E. Coffman, M. Reiman and B. Simon, On the capacity and vertex-forwarding index of communication networks, *IEEE Trans. Inform. Theory* **33** (1987), 224-232.

[2] R. Frucht, J.E. Graver and M.E. Watkins, The groups of generalized Petersen graphs, *Proc. Cambridge Philos. Soc.* **70** (1971), 211-218.

[3] G. Gauyacq, On quasi-Cayley graphs, *Discrete Applied Math.* **77** (1997) No. 1, 43-58.

[4] M.-C. Heydemann, Cayley graphs and interconnection networks, in: *"Graph symmetry"* (G.Hahn and G.Sabidussi, eds.), Kluwer Academic Publishers, Netherlands, 1997, 167-224.

[5] M.-C. Heydemann, J.-C. Meyer and D. Sotteau, On forwarding indices of networks, *Discrete Applied Math.* **23** (1989), 103-123.

[6] M.-C. Heydemann, J.-C. Meyer, J. Opatrný and D. Sotteau, Forwarding indices of consistent routings and their complexity, *Networks* **24** (1994), 75-82.

[7] W. Imrich and S. Klavžar, "Product Graphs, structure and recognition", Wiley-Interscience, 2000.