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FROM BOUNDED STRUCTURAL RULES TO LINEAR LOGIC MODALITIES

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Abstract

In this paper, a baby-example of a modality free classical linear logic enriched with 2-bounded structural rules is presented. The corresponding embedding theorem into classical linear logic is proved yielding also the cut-elimination property of the system considered.

Key-words: bounded structural rules, linear logic, modality, embedding, cut-elimination.

MSC 2000: 03B47, 03F52, 03F05, 03F07.

1 Introduction

In this paper we present a modality-free axiomatic system of classical linear logic $\mathbf{CLL_a^2}$, enriched with 2-bounded structural rules. The system is based on two-fold sequents, the idea taken from Girard's LU (2), separating a structural part from a linear part. Solely in the structural part 2-bounded structural rules may act on an even number of copies of formulas while in the the linear part applications of logical rules specified below can be made. The two parts may also communicate with each other by the so called switching rules. It is easy to give a generalization of $\mathbf{CLL_a^2}$ corresponding to any given $n \geq 2$, and hence left to a reader. Moreover, as shown recently, the systems considered can be turned into equivalent usual sequent calculus formulation by applying the respective switching rule acting from a structural to a linear part. However, it is the so-called auxiliary system $\mathbf{CLL_a^2}$ that helped to remedy all the difficiencies of our previous attempts in formalizing logic with bounded structural rules, in particular the lack of the cut-elimination property. In what follows, we show that $\mathbf{CLL_a^2}$ can faithfully be embedded into classical linear logic ($\mathbf{CLL}$), as well as any of its generalized versions. The same holds true for the corresponding cut-free subsystems, and hence we may conclude that also the systems $\mathbf{CLL_a^n}$ enjoy the cut-elimination property. Moreover, relying on $\mathbf{CLL_a^n}$-derivable and provably equivalent formulas, displayed below for $n = 2$, we can specify the modified expressive power of the multiplicative connectives, i.e. tensor ($\otimes$) and, dually par ($\parr$). More precisely, given $n \geq 2$, then for any $2 \leq k < n$, k copies of a formula A linked by $\otimes$ (denoted by A^k) expresses availability of exactly k copies of A simultaneously, A^{nm} for $m \geq 1$ expresses availability of an arbitrary number of copies A (including zero), while A^{nm+l} with $1 \leq l < n$ expresses availability of at least l copies of A respectively. This means that up to a given n $\otimes$ behaves normally as in $\mathbf{CLL}$, A^n and its multiples mimic precisely $!A$ and A^{nm+l} act as $!A \otimes A^l$ in $\mathbf{CLL}$. Later on, it will be seen that this is the effect of n-bounded structural rules as well as properly modified right $\otimes$-rule and dually, left $\parr$-rule. To sum up, given $n \geq 2$ the expressive power of the additive connectives in $\mathbf{CLL_a^n}$ remains the same as in $\mathbf{CLL}$. On the other hand, by the multiplicative connectives we can express 'arbitrary many' and 'at least one', if $n = 2$, and exactly 2, ... , exactly $n - 1$, arbitrary many (including zero) as well as at least 1, ... , at least $n - 1$, if $n > 2$.

2 Auxiliary Axiomatic System with Bounded Structural Rules

The auxiliary axiomatic system $\mathbf{CLL}^{\mathbf{2}}_{\mathbf{a}}$ is based on the following concept of a two-fold sequent: $\Pi|\Gamma \vdash \Delta|\Sigma$, where Π and Σ may run over finite multisets of an even number of copies of formulas (including the empty set) and Γ, Δ denote arbitrary finite multisets of formulas in the language of classical linear logic without the modalities. We shall refer to Π and Σ as structural parts and to Γ and Δ as linear parts of a sequent considered. Following the standard notation, if any of the parts is uninhabited it will simply be denoted by a blank. Moreover, throughtout the below, we shall use the following abbreviations: given $n \geq 2$ and a formula A, let A^n and nA denote n copies of A linked by $\otimes$ and $⅋$ respectively for any ordering of brackets (for $n = 1$, A^1 and $1A$ being just A). Let further $A^{(n)}$ denote a multiset of n copies of A, and accordingly given a multiset Γ, let $\Gamma^n = \{A^n; A \in \Gamma\}$, $n\Gamma = \{nA; A \in \Gamma\}$ and $\Gamma^{(n)} := \{A^{(n)}; A \in \Gamma\}$.

Axioms

$$|A \vdash A| \quad |\bot \vdash \quad | \quad | \quad \vdash 1| \quad |\Gamma, 0 \vdash \Delta| \quad |\Gamma \vdash \top, \Delta|$$

Logical rules

$$\frac{\Pi|\Gamma \vdash \Delta|\Sigma}{\Pi|\Gamma, 1 \vdash \Delta|\Sigma} \qquad \frac{\Pi|\Gamma \vdash \Delta|\Sigma}{\Pi|\Gamma \vdash \Delta, \bot|\Sigma}$$

$$\frac{\Pi|\Gamma \vdash A, \Delta|\Sigma}{\Pi|\Gamma, \neg A \vdash \Delta|\Sigma} \qquad \frac{\Pi|\Gamma, A \vdash \Delta|\Sigma}{\Pi|\Gamma \vdash \neg A, \Delta|\Sigma}$$

$$\frac{\Pi|\Gamma, B \vdash \Delta|\Sigma}{\Pi|\Gamma, B\&C \vdash \Delta|\Sigma} \qquad \frac{\Pi|\Gamma, C \vdash \Delta|\Sigma}{\Pi|\Gamma, B\&C \vdash \Delta|\Sigma}$$

$$\frac{\Pi_1|\Gamma \vdash B, \Delta|\Sigma_1 \qquad \Pi_2|\Gamma \vdash C, \Delta|\Sigma_2}{\Pi_1, \Pi_2|\Gamma \vdash B\&C, \Delta|\Sigma_1, \Sigma_2}$$

$$\frac{\Pi|\Gamma \vdash B, \Delta|\Sigma}{\Pi|\Gamma \vdash B \oplus C, \Delta|\Sigma} \qquad \frac{\Pi|\Gamma \vdash C, \Delta|\Sigma}{\Pi|\Gamma \vdash B \oplus C, \Delta|\Sigma}$$

$$\frac{\Pi_1|\Gamma, B \vdash \Delta|\Sigma_1 \qquad \Pi_2|\Gamma, C \vdash \Delta|\Sigma_2}{\Pi_1, \Pi_2|\Gamma, B \oplus C \vdash \Delta|\Sigma_1, \Sigma_2}$$

Left $\otimes$-rule and right $⅋$-rule:

$$\frac{\Pi|\Gamma, B, C \vdash \Delta|\Sigma}{\Pi|\Gamma, B \otimes C \vdash \Delta|\Sigma} \qquad \frac{\Pi|\Gamma \vdash B, C, \Delta|\Sigma}{\Pi|\Gamma \vdash B \parr C, \Delta|\Sigma}$$

Restricted version of the right $\otimes$-rule:

$$\frac{\Pi_1|\Gamma_1 \vdash B, \Delta_1|\Sigma_1 \qquad \Pi_2|\Gamma_2 \vdash C, \Delta_2|\Sigma_2}{\Pi_1, \Pi_2|\Gamma_1, \Gamma_2 \vdash B \otimes C, \Delta_1, \Delta_2|\Sigma_1, \Sigma_2}$$

except when B is of the form A^{2k+1} and C is of the form A^{2m+1} for any $k, m \in \mathbf{N}$.

Promoted right $\otimes$-rule:

$$\frac{\Pi_1|\Gamma_1 \vdash A^{2k+1}, \Delta_1|\Sigma_1 \qquad \Pi_2|\Gamma_2 \vdash A^{2m+1}, \Delta_2|\Sigma_2}{\Pi_1, \Pi_2, \Gamma_1^{(2)}, \Gamma_2^{(2)}| \quad \vdash A^{2(k+m+1)}|\Delta_1^{(2)}, \Delta_2^{(2)}, \Sigma_1, \Sigma_2}$$

for any $k, m \in \mathbf{N}$.

Restricted version of the left $\parr$-rule:

$$\frac{\Pi_1|\Gamma_1, B \vdash \Delta_1|\Sigma_1 \qquad \Pi_2|\Gamma_2, C \vdash \Delta_2|\Sigma_2}{\Pi_1, \Pi_2|B \parr C, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2|\Sigma_1, \Sigma_2}$$

except when B is of the form $(2k+1)A$ and C is of the form $(2m+1)A$ for any $k, m \in \mathbf{N}$.

Promoted left $\parr$-rule:

$$\frac{\Pi_1|\Gamma_1, (2k+1)A \vdash \Delta_1|\Sigma_1 \qquad \Pi_2|\Gamma_2, (2m+1)A \vdash \Delta_2|\Sigma_2}{\Pi_1, \Pi_2, \Gamma_1^{(2)}, \Gamma_2^{(2)}|2(k+m+1)A \vdash \quad |\Delta_1^{(2)}, \Delta_2^{(2)}, \Sigma_1, \Sigma_2}$$

for any $k, m \in \mathbf{N}$.

Stuctural rules

Left and right contraction rules:

$$\frac{\Pi, A^{(2n)}|\Gamma \rightarrow \Delta|\Sigma}{\Pi, A^{(2k)}|\Gamma \rightarrow \Delta|\Sigma} \qquad \frac{\Pi|\Gamma \vdash \Delta|A^{(2n)}, \Sigma}{\Pi|\Gamma \vdash \Delta|A^{(2k)}, \Sigma}$$

for any $n > k \geq 1$.

Left and right weakening rules:

$$\frac{\Pi|\Gamma \vdash \Delta|\Sigma}{\Pi, A^{(2n)}|\Gamma \vdash \Delta|\Sigma} \qquad \frac{\Pi|\Gamma \vdash \Delta|\Sigma}{\Pi|\Gamma \vdash \Delta|A^{(2n)}, \Sigma}$$

for any $n \geq 1$.

Left detensoring rule and right deparing rule:

$$\frac{\Pi, (A^n)^{(2)}|\Gamma \vdash \Delta|\Sigma}{\Pi, A^{(2n)}|\Gamma \vdash \Delta|\Sigma} \qquad \frac{\Pi|\Gamma \vdash \Delta|(nA)^{(2)}, \Sigma}{\Pi|\Gamma \vdash \Delta|A^{(2n)}, \Sigma}$$

for any $n > 1$.

Switching rules

Switching from a linear part to a structural part:

$$\frac{\Pi|\Gamma, A \vdash \Delta|\Sigma}{\Pi, A^{(2)}|\Gamma \vdash \Delta|\Sigma} \qquad \frac{\Pi|\Gamma \vdash \Delta, A|\Sigma}{\Pi|\Gamma \vdash \Delta|A^{(2)}, \Sigma}$$

Switching from a structural part to a linear part:

$$\frac{\Pi, A^{(2n)}|\Gamma \vdash \Delta|\Sigma}{\Pi|A^{2n}, \Gamma \vdash \Delta|\Sigma} \qquad \frac{\Pi|\Gamma \vdash \Delta|A^{(2n)}, \Sigma}{\Pi|\Gamma \vdash 2nA, \Delta|\Sigma}$$

Cut rule:

$$\frac{\Pi_1|\Gamma_1 \vdash A, \Delta_1|\Sigma_1 \qquad \Pi_2|\Gamma_2, A \vdash \Delta_2|\Sigma_2}{\Pi_1, \Pi_2|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2|\Sigma_1, \Sigma_2}$$

To continue with, within $\mathbf{CLL_a^2}$ the following are provably equivalent for any $n \geq 1$ and any formula A:

$$A^2 \vdash\dashv A^{2n} \text{ and dually, } 2A \vdash\dashv 2nA$$
$$A^3 \vdash\dashv A^{2n+1} \text{ and dually, } 3A \vdash\dashv (2n+1)A$$

We shall display below some derivations of the first equivalence given. The rest being equally trivial.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{|A^2 \vdash A^2| \qquad |A^2 \vdash A^2|}{|A^2, A^2 \vdash A^4| \qquad |A^2 \vdash A^2|}}{=\!= \atop |(A^2)^{(n)} \vdash A^{2n}|}}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!= \atop (A^2)^{(2n)}| \quad \vdash A^{2n}|}}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!= \atop A^{(4n)}| \quad \vdash A^{2n}|}}{A^{(2)}| \quad \vdash A^{2n}|}}{|A^2 \vdash A^{2n}|}$$

where a double line indicates several successive applications of a specific rule of inference.

$$
\begin{array}{c}
|A \vdash A| \qquad\qquad |A \vdash A| \\ \hline
A^{(2)}, A^{(2)}| \quad \vdash A^2| \qquad\qquad |A \vdash A| \\ \hline
A^{(2)}, A^{(2)}|A \vdash A^3| \qquad\qquad |A \vdash A| \\ \hline
A^{(2)}, A^{(2)}, A^{(2)}, A^{(2)}| \quad \vdash A^4| \qquad\qquad |A \vdash A| \\ \hline
\cdots \qquad\quad \cdots \qquad\quad \cdots \qquad\quad \cdots \\ \hline
\cdots \qquad\quad \cdots \qquad\quad \cdots \qquad\quad \cdots \\ \hline
A^{(4n)}| \quad \vdash A^{2n}| \\ \hline
A^{(2)}| \quad \vdash A^{2n}| \\ \hline
|A^2 \vdash A^{2n}|
\end{array}
$$

where the dot lines indicate several successive interchanging applications of restricted and promoted right $\otimes$-rules. Analogously, a derivation for any other ordering of brackets covered by A^{2n} can easily be made.

For the other turnstile direction of the equivalence under consideration, it suffices to make the following derivation:

$$
\begin{array}{c}
|A^2 \vdash A^2| \\ \hline
(A^2)^{(2)}| \quad \vdash A^2| \\ \hline
A^{(4)}| \quad \vdash A^2| \\ \hline
A^{(2n)}| \quad \vdash A^2| \\ \hline
|A^{2n} \vdash A^2|
\end{array}
$$

In case $n = 2$ there is no application of weakening rule.

Finally, notice also that for any $n \geq 1$ also $|A \otimes A \vdash A^{2n+1}|$, and dually $|(2n+1)A \vdash A \parr A|$ are $\mathbf{CLL_a^2}$-derivable.

To sum up, this indeed means that for any $n \geq 1$ the following are $\mathbf{CLL_a^2}$-derivable:

$$|A \otimes A \vdash A^n|, \text{ and dually, } |nA \vdash A \parr A|.$$

3 Embedding of the System with 2-Bounded Structural Rules into Classical Linear Logic

In what follows we shall show that the axiomatic systems $\mathbf{CLL_a^2}$ can faithfully be embedded into **CLL**. For that purpose we shall first define a translation between the sets of formulas of the underlying languages.

DEFINITION 3.1 *A translation* $(.)^t : For_{CLL^2_a} \to For_{CLL}$ *is given inductively as follows:*

- $P^t = P$, *for any propositional letter;*
- $1^t = 1$, $\bot^t = \bot$, $\top^t = \top$ *and* $0^t = 0$*;*
- $(\neg A)^t = \neg A^t$*;*
- $(A\&B)^t = A^t \& B^t$*;*
- $(A \oplus B)^t = A^t \oplus B^t$
- $(A \otimes B)^t = \begin{cases} !C^t & ; \quad \textit{if } A \otimes B \textit{ is of the form } C^{2k}, \textit{ with } k \geq 1 \\ !C^t \otimes C^t & ; \quad \textit{if } A \otimes B \textit{ is of the form } C^{2k+1}, \textit{ with } k \geq 1 \\ A^t \otimes B^t & ; \quad \textit{otherwise} \end{cases}$
- $(A \parr B)^t = \begin{cases} ?C^t & ; \quad \textit{if } A \parr B \textit{ is of the form } 2kC, \textit{ with } k \geq 1 \\ ?C^t \parr C^t & ; \quad \textit{if } A \parr B \textit{ is of the form } (2k+1)C, \textit{ with } k \geq 1 \\ A^t \parr B^t & ; \quad \textit{otherwise} \end{cases}$

In what follows, we shall use some more abbreviations: given a multiset Γ, let $\Gamma^t = \{A^t; A \in \Gamma\}$, i.e. Γ^t is the corresponding multiset of translated copies of formulas in Γ; moreover, given multisets Π and Σ of an even number of copies of formulas, let $!\Pi^t$ and $?\Sigma^t$ be the **sets** of modalized translations of formulas in Π and Σ by "!" and "?" respectively.

We are now ready to state the

THEOREM 3.2 (Embedding Theorem) $\mathbf{CLL^2_a} \hookrightarrow \mathbf{CLL}$ *in the following sense:*

$$\Pi|\Gamma \vdash_{CLL^2_a} \Delta|\Sigma \quad iff \quad !\Pi^t, \Gamma^t \vdash_{CLL} \Delta^t, ?\Sigma^t.$$

Proof. (soundness): by induction on the lenght of $\mathbf{CLL^2_a}$-derivations. We shall here display some cases of the induction step considering promoted rigt $\otimes$ rules, as well as structural and switching rules.

For the promoted right $\otimes$-rule as the last applied rule within a $\mathbf{CLL^2_a}$-derivation we are to distinguish two cases, one of them displayed below:

$$\frac{\Pi_1|\Gamma_1 \vdash A^{2k+1}, \Delta_1|\Sigma_1 \qquad \Pi_2|\Gamma_2 \vdash A^{2m+1}, \Delta_2|\Sigma_2}{\Pi_1, \Pi_2, \Gamma_1^{(2)}, \Gamma_2^{(2)}| \quad \vdash A^{2(k+m+1)}|\Delta_1^{(2)}, \Delta_2^{(2)}, \Sigma_1, \Sigma_2}$$

for $k, m \geq 1$.

$$\dfrac{\dfrac{\dfrac{\dfrac{!\Pi_1^t, \Gamma_1^t \vdash !A^t \otimes A^t, \Delta_1^t, ?\Sigma_1^t}{!\Pi_1^t, !\Gamma_1^t \vdash !A^t \otimes A^t, ?\Delta_1^t, ?\Sigma_1^t}}{!\Pi_1^t, !\Gamma_1^t \vdash !(!A^t \otimes A^t), ?\Delta_1^t, ?\Sigma_1^t} \qquad !(!A^t \otimes A^t) \vdash !A^t}{!\Pi_1^t, !\Gamma_1^t \vdash !A^t, ?\Delta_1^t, ?\Sigma_1^t}}{!\Pi_1^t, !\Pi_2^t, !\Gamma_1^t, !\Gamma_2^t | \quad \vdash !A^t, ?\Delta_1^t, ?\Delta_2^t, ?\Sigma_1^t, ?\Sigma_2^t}$$

Next, assuming that the last applied rule in a $\mathbf{CLL_a^2}$-derivation is one of the structural rules, say contraction or detensoring, while a reader may try his/her skill with weakening:

Left contraction rule:

$$\frac{\Pi, A^{(2n)}|\Gamma \vdash \Delta|\Sigma}{\Pi, A^{(2k)}|\Gamma \vdash \Delta|\Sigma}$$

for any $n > k \geq 1$.

$$\frac{!\Pi^t, !A^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t}{\Pi, !A^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t}$$

Left detensoring rule:

$$\frac{\Pi, (A^n)^{(2)}|\Gamma \vdash \Delta|\Sigma}{\Pi, A^{(2n)}|\Gamma \vdash \Delta|\Sigma}$$

for any $n > 1$.

$$\frac{!\Pi^t, !(A^n)^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t}{!\Pi^t, !A^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t}$$

To see that the rule just obtained is indeed **CLL**-derivable under the induction hypothesis that its premiss is **CLL**-derivable a reader is to spell out the following two cases:
if $n = 2k$, then $(A^{2k})^t = !A^t$, and if $n = 2k+1$, then $(A^{2k+1})^t = !A^t \times A^t$.

Switching from a linear part to a structural part:

$$\frac{\Pi|\Gamma, A \vdash \Delta|\Sigma}{\Pi, A^{(2)}|\Gamma \vdash \Delta|\Sigma}$$

$$\frac{!\Pi^t, \Gamma^t, A^t \vdash \Delta^t, ?\Sigma^t}{!\Pi^t, !A^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t}$$

Switching from a structural part to a linear part:

$$\frac{\Pi, A^{(2n)}|\Gamma \vdash \Delta|\Sigma}{\Pi|A^{2n}, \Gamma \vdash \Delta|\Sigma}$$

$$\frac{!\Pi^t, !A^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t}{!\Pi^t, !A^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t}$$

(faithfullness):
Assume now that $!\Pi^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t$ is **CLL**-derivable. Then we want to show that the sequent $\Pi|\Gamma \vdash \Delta|\Sigma$ is $\mathbf{CLL^2_a}$-derivable or equivalently, as easily seen, that the sequent $|\Pi^2, \Gamma \vdash \Delta, 2\Sigma|$ is $\mathbf{CLL^2_a}$-derivable. The proof goes by induction on the lenght of a **CLL**-derivation replacing just all modalized occurrences of formulas $!A$ and $?A$ in it by $A \otimes A$ and $A \parr A$ respectively.

We shall here consider just some cases of the induction step involving the modality !. The rest is trivial.

Assume the last applied rule of a **CLL**-derivation is !-weakening:

$$\frac{!\Pi^t, \Gamma^t \vdash \Delta^t, ?\Sigma^t}{!\Pi^t, \Gamma^t, !A^t \vdash \Delta^t, ?\Sigma^t}$$

covered by applications of $\mathbf{CLL^2_a}$-switching rule from structural to linear part and weakening rule:

$$\cfrac{\cfrac{|\Pi^2, \Gamma \vdash \Delta, 2\Sigma|}{A^{(2)}|\Pi^2, \Gamma \vdash \Delta, 2\Sigma|}}{|\Pi^2, A \otimes A, \Gamma \vdash \Delta, 2\Sigma|}$$

Assuming the last applied rule of a **CLL**-derivation is !-contraction:

$$\frac{!\Pi^t, \Gamma^t, !A^t, !A^t \vdash \Delta^t, ?\Sigma^t}{!\Pi^t, \Gamma^t, !A^t \vdash \Delta^t, ?\Sigma^t}$$

covered by applications of $\mathbf{CLL^2_a}$-switching rule, left contraction rule, detensoring rules and successive applications of the other switching rule, as shown below:

$$\cfrac{\cfrac{\cfrac{\cfrac{|\Pi^2, \Gamma, A \otimes A, A \otimes A, \Delta, 2\Sigma|}{\overline{(A^2)^{(4)}|\Pi^2, \Gamma \vdash \Delta, 2\Sigma|}}}{\overline{A^{(8)}|\Pi^2, \Gamma \vdash \Delta, 2\Sigma|}}}{A^{(2)}|\Pi^2, \Gamma \vdash \Delta, 2\Sigma|}}{|\Pi^2, \Gamma, A \otimes A \vdash \Delta, 2\Sigma|}$$

Assume the last applied rule of a **CLL**-derivation of is left deriliction:

$$\frac{!\Pi^t, \Gamma^t, A^t \vdash \Delta^t, ?\Sigma^t}{!\Pi^t, \Gamma^t, !A^t \vdash \Delta^t, ?\Sigma^t}$$

covered exactly by applications of both $\mathbf{CLL^2_a}$-switching rules, as witnessed below:

$$\cfrac{\cfrac{|\Pi^2, \Gamma, A \vdash \Delta, \Sigma|}{A^{(2)}|\Pi^2, \Gamma \vdash \Delta, 2\Sigma|}}{|\Pi^2, \Gamma, A \otimes A \vdash \Delta, 2\Sigma|}$$

Assume the last applied rule of a **CLL**-derivation is right promotion:

$$\frac{!\Gamma^t \vdash A^t, ?\Delta^t}{!\Gamma^t, \vdash !A^t, ?\Delta^t}$$

covered by applications of $\mathbf{CLL^2_a}$-switching rules, left and right contraction rules, applications of detensoring rule, and an application of promoted right $\otimes$ rule with the same premisses, as displayed below:

$$\cfrac{\cfrac{\cfrac{\cfrac{|\Gamma^2 \vdash A, 2\Delta| \qquad |\Gamma^2 \vdash A, 2\Delta|}{(\Gamma^2)^{(4)}| \quad \vdash A \otimes A|(2\Delta^{(4)})}}{\Gamma^{(8)}| \quad \vdash A \otimes A|\Delta^{(8)}}}{\Gamma^{(2)}| \quad \vdash A \otimes A|\Delta^{(2)}}}{|\Gamma^2 \vdash A \otimes A, \Delta^2|}$$

$\diamondsuit$

As an important consequence of the proof of faithfulness of the embedding $\mathbf{CLL^2_a} \hookrightarrow \mathbf{CLL}$ and the fact that **CLL** enjoys the cut-elimination property we end up with the following

COROLLARY 3.3 *The cut-free system of* $\mathbf{CLL^2_a}$ *can faithfully be embedded into* **CLL**.

COROLLARY 3.4 *The system* $\mathbf{CLL^2_a}$ *enjoys the cut-elimination property.*

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