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# 1-HOMOGENEOUS GRAPHS WITH COCKTAIL PARTY $\mu$-GRAPHS

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# 1-HOMOGENEOUS GRAPHS WITH COCKTAIL PARTY $\mu$-GRAPHS

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**Abstract**

Let $\Gamma$ be a 1-homogeneous graph in the sense of Nomura with diameter $d \geq 2$, i.e., for every edge $xy$ of $\Gamma$ the distance partition

$$\{\{z \in V(\Gamma) \mid \mathrm{dist}(z,y) = i,\ \mathrm{dist}(x,z) = j\} \mid\ 0 \leq i,j \leq \mathrm{diam}(\Gamma)\}$$

is equitable and its parameters do not depend on the edge $xy$. Then $\Gamma$ is distance-regular and also locally strongly regular. The local graphs have the same parameters. Let us denote them by $(v',k',\lambda',\mu')$. Obviously we have $v' = k$, $k' = a_1$, $(v'-k'-1)\mu' = k'(k'-1-\lambda')$ and $c_2 \geq \mu' + 1$, since a $\mu$-graph is a regular graph with valency $\mu'$. There are several interesting examples of 1-homogeneous graphs whose $\mu$-graphs are isomorphic to a few copies of complete multipartite graphs. For example, if equality holds in the above bound and $c_2 \neq 1$, then $\Gamma$ is a Terwilliger graph, i.e., all the $\mu$-graphs are complete, and we have already classified such graphs. In this article we consider the case $c_2 = \mu' + 2 \geq 3$, i.e., the case when the $\mu$-graphs of $\Gamma$ are the Cocktail Party graphs, and obtain that either $\lambda' = 0$, $\mu' = 2$ or $\Gamma$ is one of the following graphs: (i) a Johnson graph $J(2m,m)$ with $m \geq 2$, (ii) a folded Johnson graph $\overline{J}(4m,2m)$ with $m \geq 3$, (iii) a halved $m$-cube with $m \geq 4$, (vi) a folded halved $(2m)$-cube with $m \geq 5$, (v) a Cocktail Party graph $K_{m\times 2}$ with $m \geq 3$, (vi) the Schläfli graph, (vii) the Gosset graph.

CONTENTS

# 1 Introduction

We study 1-homogeneous graphs in the sense of Nomura [16]. Some examples of such graphs are a distance-regular graph with at most one $i$, s.t. $a_i \neq 0$ (e.g. a bipartite graph, a generalized Odd graph, in particular a triangle free strongly regular graphs, and a complete multipartite graph), a regular near $(2n)$-gon (i.e., a distance-regular graph with $a_i = c_i a_1$ for all $i$ and no induced $K_{1,2,1}$), a Taylor graph (i.e., a 2-cover of a complete graph), the Johnson graph $J(2d,d)$, the folded Johnson graph $\overline{J}(4d,2d)$, the halved $n$-cube $H(n,2)$, the folded halved $(2n)$-cube and a 3-valent distance-regular graph. The 1-homogeneous graphs with diameter $d \geq 2$, $a_1 \neq 0$ and $a_d = 0$ are precisely the tight graphs in the sense of [10], in particular such examples are the Patterson graph and 10 tight antipodal distance-regular graphs with diameter four.

Let $\Gamma$ be a 1-homogeneous graph. Then $\Gamma$ is distance-regular, locally strongly regular and, by [11, Prop.2.1], the local graphs have the same parameters. Let us denote them by $(v', k', \lambda', \mu')$. Obviously we have $v' = k$, $k' = a_1$, $(v'-k'-1)\mu' = k'(k'-1-\lambda')$ and $c_2 \geq \mu'+1$, since a $\mu$-graph is a regular graph with valency $\mu'$ by [9, Thm.3.1]. The case $c_2 = \mu'+1 \geq 2$, i.e., the case when $\Gamma$ is a Terwilliger graph, was classified in [11]. The $\mu$-graphs of Terwilliger graphs are complete graphs. Since many of the above mentioned examples of 1-homogeneous graphs have the property that their $\mu$-graphs are complete multipartite graphs, it is natural to study 1-homogeneous graphs with this property. Alternative motivation comes from the study of extended generalized quadrangles, see for example [5] and [22]. We establish some general properties of such graphs that are related to their structure, parameters and eigenvalues. There are many families of 1-homogeneous graphs for which we can show that their $\mu$-graphs are complete multipartite. One such example is the case $c_2 = \mu' + 2 \geq 3$, i.e., the case when the $\mu$-graphs are the Cocktail Party graphs. When this is the case we show that either $\lambda' = 0$ and $\mu' = 2$ or the smallest eigenvalue of each local graph is $-2$ and so by Seidel's classification [17], [3, Thm.3.12.4], either $\lambda' = 0$ and $\mu' = 2$ or each local graph of $\Gamma$ is one of the well known strongly regular graphs. In the latter case we show that $\Gamma$ must be one of the well known distance-regular graphs. Before we state the complete statement of our main result we establish some notation and review basic definitions, for more details see Brouwer, Cohen and Neumaier [3], and Godsil [8]. At the end of this section we describe how is this paper organized.

Let us first recall that an **equitable partition** of a graph is a partition $\pi = \{C_1, \ldots, C_s\}$ of its vertices into cells, such that for all $i$ and $j$ the number $c_{ij}$ of neighbours, which a vertex in the cell $C_i$ has in the cell $C_j$, is independent of the choice of the vertex in $C_i$. Let $\Gamma$ be a connected graph with diameter $d$. For a vertex $x$ of $\Gamma$ we define $\Gamma_i(x)$ to be the set of vertices at distance $i$ from $u$. For $y \in \Gamma_i(x)$ and integers $j$ and $h$ we define $D^h_j(x,y) = \Gamma_j(x) \cap \Gamma_h(y)$ and $p^i_{jh}(x,y) = |D^h_j(x,y)|$. Then $\Gamma$ is **$i$-homogeneous** in the sense of Nomura [16] when the distance partition corresponding to any pair $x$, $y$ of vertices at distance $i$, i.e., the collection of nonempty sets $D^j_h(x,y)$, is equitable, and the parameters corresponding to equitable partitions are independent of vertices $x$ and $y$ at distance $i$. Note that the graph $\Gamma$ is 0-homogeneous if and only if it is **distance-regular**, and that if $\Gamma$ is 1-homogeneous than it is distance-regular.

In a graph $\Gamma$ we say that the intersection number $p^i_{jh}$ does **exist** if $p^i_{jh}(x,y) = p^i_{jh}$ for all pairs of vertices $x$ and $y$ at distance $i$. Of course, if $\Gamma$ is distance-regular, then for all $i$, $j$ and $h$ the numbers $p^i_{jh}$ do exist. Let $a_i(x,y) := p^i_{1i}(x,y)$, $b_i(x,y) := p^i_{1,i+1}(x,y)$ and $c_i(x,y) := p^i_{1,i-1}(x,y)$. For a vertex $x$ of a graph $\Gamma$ we define the **local graph** $\Delta(x)$ as the subgraph of $\Gamma$, induced by the neighbours of $x$. If $\Gamma$ is distance-regular, then $\Delta(x)$ has $k = b_0$ vertices and valency $a_1$. The graph $\Gamma$ is said to be **locally** $\mathcal{C}$, where $\mathcal{C}$ is a graph or a class of graphs, when each local graph of $\Gamma$ induces a graph isomorphic to (respectively is a member of) $\mathcal{C}$.

In order to state the main result of this paper we need to define one more quantity. Let $\Gamma$ be a 1-homogeneous graph with $a_2 \neq 0$. Then there exists an integer $\alpha$ such that for all vertices $x$, $y$ and $z$ of $\Gamma$, $\partial(x,y) = \partial(x,z) = 2$ and $\partial(y,z) = 1$, we have $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)| = \alpha$, see Figure 2.1(b),(c). A strongly regular graph with $a_2 \neq 0$, that is locally strongly regular is 1-homogeneous if and only if the (triple) intersection number $\alpha$ exists, see Figure 1.1(a).

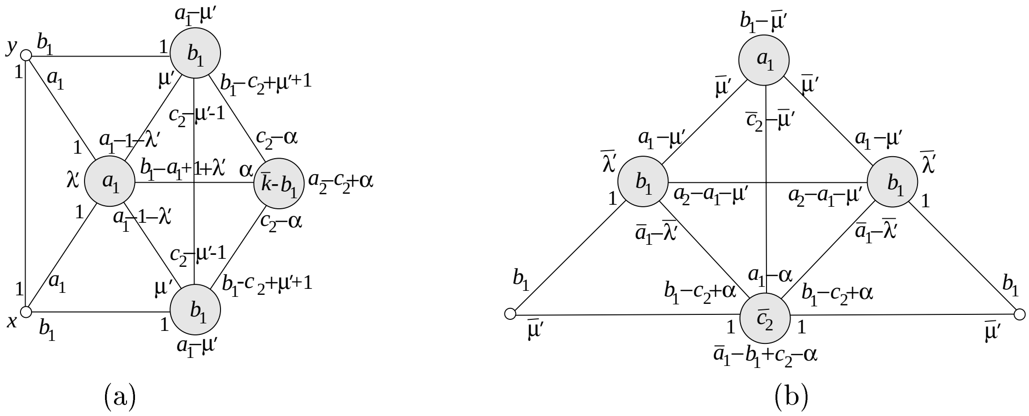


Figure 1.1: Let $\Gamma$ be a strongly regular graph $(v, k, a_1, c_2)$ with $a_2 = k - c_2 \neq 0$, that is locally strongly regular graph $(v', k', \lambda', \mu')$, and for which $\alpha$ exists. (a) 1-homogeneous partition of $\Gamma$; (b) 2-homogeneous partition of the complement of $\Gamma$, where $\overline{a_1} = v - 2k + c_2 - 2$, $\overline{c_2} = v - 2k + a_1$, $\overline{\lambda'} = k - 2a_1 + \mu' - 2$, $\overline{\mu'} = k - 2a_1 + \lambda'$, $\overline{k} = kb_1/c_2$ and $\overline{k} - b_1 = a_2 b_1/c_2$.

**Theorem 1.1** *Let $\Gamma$ be a 1-homogeneous graph with diameter $d \geq 2$. Let for all vertices $x$ of $\Gamma$ the local graph $\Delta(x)$ be a strongly regular graph with parameters $(v', k', \lambda', \mu')$. If $c_2 = \mu' + 2 \geq 3$, then either $\lambda' = 0$, $\mu' = 2$, $d \geq 3$ and $\alpha = 1$, or $\Gamma$ is one of the following graphs:*

*(i) a Johnson graph $J(2m, m)$ with $m \geq 2$,*

*(ii) a folded Johnson graph $\overline{J}(4m, 2m)$ with $m \geq 3$,*

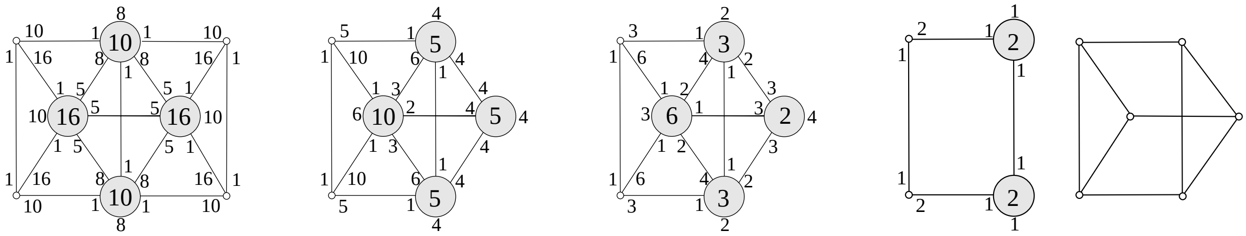
*(iii) a halved $m$-cube with $m \geq 4$,*

*(iv) a folded halved $(2m)$-cube with $m \geq 5$,*

*(v) a Cocktail Party graph $K_{m\times 2}$ with $m \geq 3$,*

*(vi) the Schläfli graph with intersection array $\{16, 5; 1, 8\}$,*

*(vii) the Gosset graph with intersection array $\{27, 10, 1; 1, 10, 27\}$.*



(a) the Gosset graph (b) the Schläfli graph (c) the Clebsch graph (d) the 3-prizm

Figure 1.2: 1-homogeneous partitions of graphs that form a tower: (a) the Gosset graph is a unique distance-regular graph with intersection array $\{27, 10, 1; 1, 10, 27\}$, an antipodal 2-cover of the complete graph $K_{28}$, and it is locally Schläfli graph, see [3, pp. 103, 313]; (b) the Schläfli graph is a unique strongly regular graph $(27, 16, 10, 8)$ and it is locally halved 5-cube, see [3, p.103]; (c) the halved 5-cube, also known as the Clebsch graph, is a unique strongly regular graph $(16, 10, 6, 6)$ and it is locally $J(5, 2)$, i.e., the complement of the Petersen graph, see [3, p.264] (so the local graph is not 1-homogeneous), the Johnson graph $J(5, 2)$ is a unique strongly regular graph $(10, 6, 3, 4)$ and is locally the 3-prizm; (d) the 3-prizm has two distance partitions corresponding to an edge.

It worths to mention that a strongly regular graph $\Gamma$ is 1-homogeneous if and only if its complement is 2-homogeneous (the second subconstituent of the complement of $\Gamma$ is isomorphic to the complement of a local graph of $\Gamma$ and for vertices $x$ and $y$ of $\Gamma$ at distance 2 a vertex in $D^1_1(x, y)$ has $a_2 - \alpha$ neighbours in the set $D^2_2(x, y)$), see Figure 1.1(b) and Figure 1.3 for some examples, and note that the 1-homogeneous partition of the Gosset graph is at the same time also its 2-homogeneous partition (actually, in general, a 1-homogeneous 2-cover with diameter $D$ is also $(D - 1)$-homogeneous).

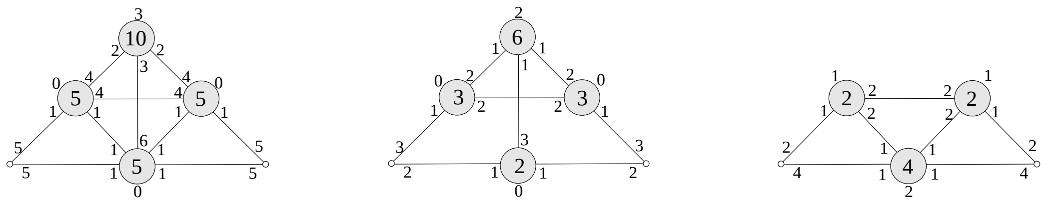


Figure 1.3: 2-homogeneous partitions of the complement of the Schläfli, the complement of the Clebsch and the Johnson graph $J(5, 2)$.

Let $\Gamma$ be a distance-regular graph with diameter $d \geq 2$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then we say that $\Gamma$ is **tight** in the sense of [10] whenever it is not bipartite and $k(a_1 + b^+ b^-) = (a_1 - b^+)(a_1 - b^-)$, where $b^+ = -1 - b_1/(1 + \theta_d)$ and $b^- = -1 - b_1/(1 + \theta_1)$.

**Corollary 1.2** *Let $\Gamma$ be a tight graph of diameter $d \geq 3$. Let for all vertices $x$ of $\Gamma$ the local graph $\Delta(x)$ be a strongly regular graph with parameters $(v', k', \lambda', \mu')$. If $c_2 = \mu' + 2$, then either $\lambda' = 0$, $\mu' = 2$, and $\alpha = 1$ or $\Gamma$ is one of the following graphs:*

*(i) a Johnson graph $J(2m, m)$, with $m \geq 2$,*

*(ii) a halved $(2m)$-cube, with $m \geq 2$,*

*(iii) a Cocktail Party graph $K_{m \times 2}$, with $m \geq 3$,*

*(iv) the Gosset graph with intersection array $\{27, 10, 1; 1, 10, 27\}$.*

*Proof.* Since $\Gamma$ is tight graph, it is locally connected and $a_d = 0$ by [10, Thm.12.6 and Thm.11.7]. Now $\mu' = 0$, implies that the local graph is a clique, and hence $\Gamma$ is a complete graph as well, so we have $\mu' \geq 1$ and $c_2 = \mu' + 2 \geq 3$. Hence, $\Gamma$ is one of the graphs in the list of Theorem 1.1 with $a_d = 0$ and we are done. ∎

Our study is part of a larger project to classify 1-homogeneous graphs with complete multipartite $\mu$-graphs. There are also very interesting examples of 1-homogeneous graphs with $\mu$-graphs that consist of several copies of complete multipartite graphs or even the 2-extension of the 5-cube.

The paper is organized in the following way. In Section 2 we introduce four local conditions that are satisfied by a 1-homogeneous graph having all the $\mu$-graphs equal to the complete multipartite graph $K_{t\times n}$, Then we establish some basic properties of graphs that satisfy these four conditions. The most important such property is that the intersection parameter $\alpha$ can only be $t$ or $t-1$. Let $\Gamma$ be a graph that satisfies those four conditions. In Section 3 we study the smallest eigenvalue of the local graphs of $\Gamma$. If $\alpha = t$, then $-n$ is the smallest eigenvalue of $\Gamma$. If $\alpha = t-1$, then either $n \neq 2$ or $\lambda' = 0$ and $\mu' = 2$. This sets the stage for our classification of 1-homogeneous graphs with $c_2 = \mu' + 2$. In Section 4 we determine all such graphs that are additionally locally grid graphs or locally triangular graphs. In Section 5 we prove the main theorem.

## 2 Preliminaries

Let $\Gamma$ be a distance-regular graph diameter $d$. For vertices $x$ and $y$ of $\Gamma$ at distance $i$, $1 \leq i \leq d$, we define the sets

$$C_i(x,y) = \Gamma_{i-1}(x) \cap \Gamma(y), \qquad A_i(x,y) = \Gamma_i(x) \cap \Gamma(y) \quad \text{and} \quad B_i(x,y) = \Gamma_{i+1}(x) \cap \Gamma(y).$$

and say that $\Gamma$ has the **CAB$_j$ property**, when the partition $\mathrm{CAB}_i(x,y) = \{C_i(x,y), A_i(x,y), B_i(x,y)\}$ of the local graph of $y$ is equitable for each pair of vertices $x$ and $y$ of $\Gamma$ at distance $i \leq j$. Since the graph $\Gamma$ with $a_1 \neq 0$ is 1-homogeneous graph if and only if it has $\mathrm{CAB}_d$ property, see [11, Thm.3.1], we can now take a local approach to 1-homogeneous graphs. We establish some basic properties of graphs that satisfy a few local regularity conditions such as the $\mathrm{CAB}_1$ property.

Let $\Gamma$ be a graph. As usually, we denote the distance between vertices $x$ and $y$ of $\Gamma$ by $\partial(x,y)$. If $\Gamma$ is a regular graph with $v$ vertices and valency $k$ in which any two vertices at distance two have precisely $\mu = \mu(\Gamma)$ common neighbours, then it is called **edge-regular** with parameters $(v,k,\mu)$, see [3, p.3]. The multipartite graph $K_{t\times n}$ is the complement of $t$ cliques of size $n$, i.e., the multipartite graph $K_{n_1,n_2,\dots,n_t}$ with $n_1 = n_2 = \cdots = n_t = n$. Let $\Gamma$ be a graph with diameter at least two that satisfies the following conditions:

**(A)** *the intersection numbers $k, c_2, a_1$ and $a_2$ exist, i.e., for every vertex $x$ of $\Gamma$ the partition $\{x\} \cup \Gamma(x) \cup \Gamma_2(x)$ is equitable and its parameters are independent of $x$, see Figure 2.1(a),*

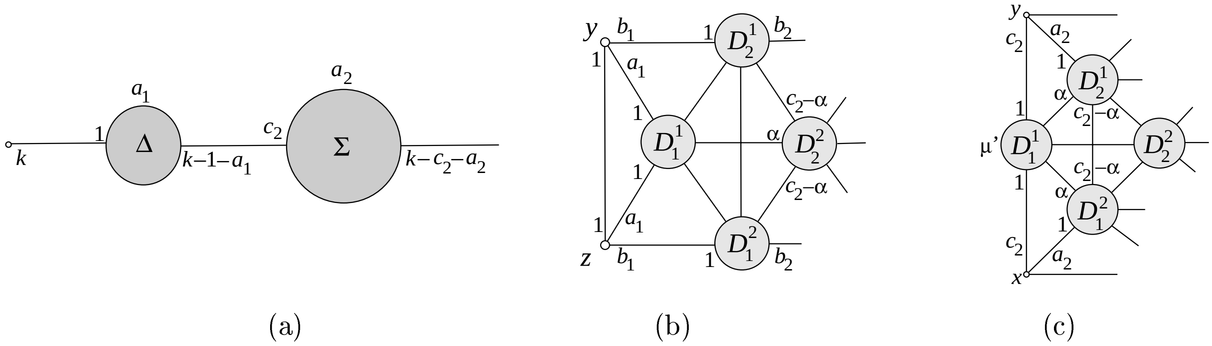



*Figure 2.1: (a) The distance distribution corresponding to a vertex and the intersection numbers $k$, $a_1$, $c_2$ and $a_2$; (b) The distance distribution corresponding to $y$ and $z$, $\partial(y,z) = 1$. Then we have $|D_1^1(y,z)| = a_1$ and $|D_1^2(y,z)| = |D_1^2(y,z)| = b_1$. (c) The distance distribution corresponding to $x$ and $y$, $\partial(x,y) = 2$. Then we have $|D_1^1(x,y)| = c_2$ and $|D_1^2(x,y)| = |D_1^2(x,y)| = a_2$.*

**(B)** *$a_2 \neq 0$ and there exists an integer $\alpha \geq 1$ such that for all vertices $x$, $y$ and $z$ of $\Gamma$, $\partial(x,y) = \partial(x,z) = 2$ and $\partial(y,z) = 1$, we have $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)| = \alpha$, see Figure 2.1(b),(c) (since $\alpha \leq a_1$ we have also $a_1 \neq 0$),*

**(C)** *there are positive integers $n$ and $t$ such that each $\mu$-graph of $\Gamma$ is the complete multipartite graph $K_{t\times n}$.*

The condition **(A)** is obviously satisfied in the case $\Gamma$ is distance-regular, and the condition **(B)** is satisfied when $\Gamma$ with $a_2 \neq 0$ has additionally the $\text{CAB}_2$ property ($\alpha$ is equal to the number of neighbours that a vertex of $A_2$ has in $C_2$), therefore also for 1-homogeneous graphs. Certain examples of tight graphs satisfy the condition **(C)**, cf. [10] and [12]. If $\Gamma$ is 1-homogeneous with a diameter at least two, then $c_2 = \mu' + 1$ if and only if $\Gamma$ is a Terwilliger graph if and only if $\Gamma$ satisfies the condition **(C)** with $n = 1$. Such graphs with $c_2 > 1$ have been classified in [11, Thm.4.10].

A $\mu$-graph of $\Gamma$ has $c_2 = nt$ vertices. Let $x$ be a vertex of $\Gamma$. Since we assumed that $\Gamma$ has diameter at least two, the local graph $\Delta(x)$ is not a complete graph. Any two vertices of $\Delta(x)$ at distance two have $\mu' = (t-1)n$ common neighbours in $\Delta(x)$ (this is also the valency of the $\mu$-graph), hence $\Gamma$ is co-edge-regular. It was quite natural to assume $a_2 \neq 0$ in **(C)**, since otherwise $a_1 = \mu'$ and each local graph is complete multipartite, hence $\Gamma$ itself is complete multipartite by [3, Prop.1.1.5]. By counting the edges between $D_1^1(x,y)$ and $D_2^1(x,y)$, we find $\alpha a_2 = c_2(a_1 - \mu')$. We sum up the above observations in the following result.

**Lemma 2.1** *Let $\Gamma$ be a non-complete graph satisfying the conditions* **(A)**–**(C)**. *Then $c_2 = nt$, each local graph of $\Gamma$ is co-edge-regular with parameters $(v', k', \mu')$, where*

$$v' = k, \quad k' = a_1, \quad \text{and} \quad \mu' = n(t-1)$$

*and $\alpha a_2 = c_2(a_1 - \mu')$.* ∎

Let us now establish a few more basic facts about $\Gamma$.

**Lemma 2.2** *Let $\Gamma$ be a non-complete graph satisfying the conditions* **(A)**–**(C)**, *and let $x$ and $y$ be vertices of $\Gamma$ at distance two. For all $z \in D_2^1(x,y)$, the subgraph induced by $\Gamma(z) \cap D_1^1(x,y)$ is complete.*

*Proof.* Suppose the opposite. Then $|D_1^1(x,y) \cap \Gamma(z)| \geq 2$ and there exist two nonadjacent vertices $u, v \in D_1^1(x,y) \cap \Gamma(z)$. Then $D_1^1(u,v) \supseteq \{x,y,z\}$ and the subgraph induced by $D_1^1(u,v)$ is not complete multipartite, since $y$ and $z$ are in the same coclique as $x$ and are adjacent. ∎

Similarly, as in the case of Terwilliger 1-homogeneous graphs [11, Lemma 4.1], there are also only two possibilities for $\alpha$ in the present situation.

**Lemma 2.3** *Let $\Gamma$ be a non-complete graph satisfying the conditions* **(A)**–**(C)**. *Then the following holds*

*(i)* $\alpha \in \{t-1, t\}$, *i.e.,* $t-1 \leq \alpha \leq t$,

*(ii)* $\Gamma$ *is a Terwilliger graph, i.e.,* $c_2 = \mu' + 1$ *if and only if* $n = 1$,

*(iii)* $\Gamma$ *is locally connected if and only if* $t \neq 1$, *in which case every local graph has diameter two.*

*Proof.* Lemma 2.2 implies $\alpha \leq t$. Let $x$ and $y$ be vertices of $\Gamma$ at distance two. Let $z \in D_1^2(x,y)$ and $A = \Gamma(z) \cap D_1^1(x,y)$. Suppose $\alpha \leq t-2$. Then there are two adjacent vertices $u, v \in D_1^1(x,y)$ such that the subgraph induced by $\{u\} \cup \{v\} \cup A$ is complete and $\partial(u,z) = 2 = \partial(v,z)$. Then the set $\Gamma(u) \cap D_1^1(v,z)$ contains $A \cup \{x\}$, which means that $|A| = \alpha \geq |A \cup \{x\}|$. Contradiction! Hence $\alpha \geq t-1$. Let $t \neq 1$ and $w_1$, $w_2$ be nonadjacent vertices of the local graph $\Delta(x)$. Then $x \in D_1^1(w_1,w_2)$, so $\partial_\Gamma(w_1,w_2) = 2$ and there is $(t-1)n$ neighbours of $x$ in the $\mu$-graph of $w_1$ and $w_2$ and hence also in the local graph $\Delta(x)$. Hence $\Delta(x)$ has diameter two. The statements (ii) and (iii) now follow directly from Lemma 2.1. ∎

Since the 1-homogeneous graphs with $t = 1$ have been classified in [16], we assume from now on $t \geq 2$. Lemmas 2.1 and 2.3 imply that we can calculate $a_2$ in terms of $a_1$, $n$ and $t$:

$$a_2 = \begin{cases} na_1 - (t-1)n^2 & \text{if } \alpha = t, \\ t\dfrac{a_1 n}{t-1} - n^2 t & \text{if } \alpha = t-1. \end{cases} \tag{1}$$

Let $\Gamma$ be a graph with diameter $d \geq 2$ that satisfies conditions **(A)**, **(B)**, **(C)** and the following condition:

**(D)** *the local graphs of $\Gamma$ are strongly regular with parameters $(v', k', \lambda', \mu')$.*

We have already mentioned that $\alpha \leq a_1$, so condition **(B)** implies $a_1 \neq 0$. In Lemma 2.1 we expressed $v'$, $k'$ and $\mu'$ in terms of $a_1$, $n$ and $t$, therefore we can do the same for $\lambda'$:

$$\lambda' = a_1 - 1 + \mu' - \frac{\mu'(k-1)}{a_1} = a_1 - 1 + n(t-1) - \frac{n(t-1)(k-1)}{a_1}. \tag{2}$$

If $d = 2$, then $\Gamma$ is strongly regular by **(A)** and 1-homogeneous by **(B)** and **(D)**.

## 3 Eigenvalues of local graphs

Let $\Gamma$ satisfy conditions **(A)**–**(D)**. We study the smallest eigenvalue of a local graph of $\Gamma$.

Let $x_1, \ldots, x_n$ be vertices of a graph $\Gamma$. Then we denote the intersection $\Gamma(x_1) \cap \cdots \cap \Gamma(x_n)$ by $\Gamma(x_1, \ldots, x_n)$ and the corresponding induced subgraph by $\Delta(x_1, \ldots, x_n)$.

**Lemma 3.1** *Let $\Gamma$ be a non-complete graph satisfying conditions* **(A)**–**(D)***, with $t \geq 2$. For an edge $xy$ of $\Gamma$, the subgraph $\Delta(x, y)$ is co-edge-regular with parameters $(v'', k'', \mu'')$, where*

$$v'' = k', \qquad k'' = \lambda', \qquad \text{and} \qquad \mu'' = n(t-2),$$

*$t \geq 3$ implies the subgraph $\Delta(x, y)$ has diameter two, and it contains an equitable partition with quotient matrix*

$$\begin{pmatrix} n(t-2) & \lambda' - n(t-2) \\ \alpha - 1 & \lambda' - \alpha + 1 \end{pmatrix}.$$

*In particular,*

$$(\alpha - 1)\Big(a_1 - n(t-1)\Big) = \Big(\lambda' - (t-2)n\Big)n(t-1). \tag{3}$$

*Proof.* The verification of co-edge-regularity is similar as in the case of Lemma 2.1. Let $z \in D_2^1(x, y)$. Since the valency of $\Delta(x, y)$ is $\lambda'$, the partition

$$\{\Gamma(x) \cap \Gamma(y) \cap \Gamma(z), \Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)\}$$

is an equitable partition of $\Delta(x, y)$ with the required quotient matrix. The first set in the above partition has $a_2' = k' - \mu' = a_1 - n(t-1)$ vertices, while the other one has $\mu' = n(t-1)$ vertices. We obtain (3) by two way counting of edges that are connecting vertices from different parts of the above partition. ∎

The relation (3) gives us:

$$\lambda' = 1 - n - \alpha + n(t-1) + \frac{a_1(\alpha - 1)}{n(t-1)}, \tag{4}$$

hence $n(t-1) \mid a_1(\alpha - 1)$. The above relation and (2) imply that one can express $k$ in terms of $n$, $\alpha$, $t$ and $a_1$.

**Theorem 3.2** *Let $\Gamma$ be a non-complete graph satisfying conditions* **(A)**–**(D)** *with $\alpha = t \geq 2$. Then, for all vertices $x$ of $\Gamma$, the smallest eigenvalue of $\Delta(x)$ equals $-n$.*

*Proof.* It follows directly from Equation (3) that $a_1 - n(t-1) = (\lambda' - (t-2)n)n$. Now using that $n(t-1) = \mu'$ and $a_1 = k'$ it follows that $(k' - \mu') = (\lambda' - \mu' + n)n$ and hence $-n$ is the negative eigenvalue of the local graph $\Delta(x)$ with parameters $(v', k', \lambda', \mu')$. ∎

**Theorem 3.3** *Let $\Gamma$ be a non-complete graph satisfying conditions* **(A)**–**(D)** *with $\alpha = t-1 \geq 2$. Then there exists a positive integer $a$ such that $-a-n$ is the smallest eigenvalue of every local graph and*

$$n(\lambda' - n(t-2)) = a(t-2)(\lambda' - n(t-3) + a). \tag{5}$$

*In particular, $a(t-2) < n$.*

*Proof.* Fix a vertex $x$ of $\Gamma$. Let $s$ be the smallest eigenvalue of the local graph $\Delta(x)$. Then

$$\mu' - k' = (\lambda' - \mu' - s)s.$$

On the other hand, $k' = a_1$ and $\mu' = n(t-1)$ by Lemma 2.1, so by (3) and $\alpha = t-1$, we have

$$(\lambda' - \mu' + n)n(t-1) = (t-2)(\lambda' - \mu' - s)(-s).$$

This means that $-s > n$, in particular $s$ is negative. Set $a := -n-s$ in the above identity and we obtain (5). To show that $a$ is integral, suppose the opposite. Then $\Delta(x)$ is a conference graph with parameters $(4\mu'+1, 2\mu', \mu'-1, \mu')$, so $k = 4\mu'+1$, $a_1 = 2\mu'$ and $\lambda' = \mu'-1$. Applying (4) we obtain $n = t-1$, so $k = 4(t-1)^2+1$, $a_1 = b_1 = 2(t-1)^2$, $c_2 = nt = (t-1)t$, $\lambda' = t(t-2)$ and $k_2 = kb_1/c_2 = 2(4t^2-8t+5)(t-1)/t$, which implies $t|10$, thus $t=5$. Let $y \in \Gamma(x)$. Then $|D_2^1(x,y)| = b_1$, $|D_2^2(x,y)| = a_1(b_1 - b_1')/\alpha = 2(t-1)^3$ and

$$0 \leq k_2 - |D_2^2(x,y)| - |D_2^1(x,y)| = -2\frac{(t-1)(-5t^2+8t-5+t^3)}{t},$$

which is impossible. Therefore $\Delta(x)$ is not a conference graph, and so $a$ is integral.

Suppose $a(t-2) \geq n$. Then by (5) we obtain

$$\lambda' - n(t-2) > \lambda' - n(t-3) + a, \qquad \text{i.e.,} \qquad -n \geq a,$$

which is not possible since $a$ is a positive integer. Therefore, we have $a(t-2) < n$. ∎

Note that the assumptions $\alpha = t-1$ and condition **(B)** imply $t = \alpha + 1 \geq 2$.

**Corollary 3.4** *Let $\Gamma$ be a non-complete graph satisfying conditions* **(A)**–**(D)** *with $\alpha = t-1$ and $n = 2$. Then $\alpha = 1$, $\lambda' = 0$ and $\mu' = 2$.*

*Proof.* Suppose $\alpha > 1$, i.e., $t \geq 3$. By Theorem 3.3, we obtain $a(t-2) < n = 2$, and therefore $a = 1$ and $t = 3$. This implies that every local graph in $\Gamma$ is strongly regular with parameters $(57, 16, 5, 4)$. However, this is not possible, since for these parameters we do not have integral eigenvalue multiplicities. Therefore, condition **(B)** implies $\alpha = 1$, and so $t = 2$. Hence $\mu' = 2$ and $\lambda' = 0$ by Lemma 2.1 and relation (4). ∎

**Remark 3.5** *There are only three known strongly regular graphs with $\lambda = 0$ and $\mu = 2$. These are the quadrangle, the folded 5-cube with intersection array $\{5,4;1,2\}$ and the Gewirtz graph with intersection array $\{10,9;1,2\}$. According to Böinck [1] the antipodal distance-regular graph with intersection array $\{16,10,1;1,5,16\}$, constructed in [3, Prop.12.5.3], is the only distance-regular graph that is locally the folded 5-cube. This graph satisfies the conditions* **(A)** *and* **(D)**, *but not the condition* **(C)**, *since $\mu$-graphs are pentagons. It is also not 1-homogeneous by [10, Thm.11.7] and [12, Thm.2.2]. The Soicher1 graph (antipodal distance-regular graph with diameter 4) and its antipodal quotient are locally Gewirtz and they are both 1-homogeneous, see [10], [12]. Both have disconnected $\mu$-graphs. In the first graph they are the disjoint union of two copies of $K_{2,2}$ and in the last one they are the disjoint union of six copies of $K_{2,2}$.*

## 4 Locally grid and locally triangular graphs

Before we start to study locally grid and locally triangular graphs we need to introduce some basic notions about codes. Let $\Gamma$ be a graph with diameter $d$ and the vertex set $X$. A **code** $C$ in $\Gamma$ is a nonempty subset of $X$. Then the **distance of a vertex** $x \in X$ **to** $C$ and the **covering radius** of $C$ respectively are

$$\partial(x,C) := \min\{\partial(x,y) \mid y \in C\} \quad \text{and} \quad t(C) := \max\{\partial(x,C) \mid x \in X\}.$$

Let $C_i$ be the set of vertices at distance $i$ from $C$ and $t = t(C)$. The code $C$ is **completely regular** when the partition $\{C_i \,|\, i = 0,\dots,t\}$ is equitable. This definition is due to Neumaier [14], who showed that it is equivalent to the original Delsarte's definition, that the code $C$ is completely regular when for each vertex $x$ of $\Gamma$ and for each $i \in \{0,1,\dots,t\}$, the intersection number $|C \cap \Gamma_i(x)|$ depends only on $\partial(x,C)$, see [6] or [3, p.351]. A partition $\pi$ of a graph $\Gamma$ gives rise to the **quotient graph** $G/\pi$ with cells as vertices and two distinct cells $C_i$ to $C_j$ adjacent if there is an edge of $\Gamma$ joining some vertex of $C_i$ to some vertex in $C_j$. An equitable partition $\pi$ is **uniformly regular** if there are constants $e_{01}$ and $e_{11}$ such that

$$c_{ij} = \begin{cases} e_{01} & \text{if } i = j, \\ e_{11} & \text{if } C_i \sim C_j \text{ in } \Gamma/\pi. \end{cases}$$

The line graph of $K_{m,n}$, i.e., the graph $K_m \times K_n$, will be called the $(m \times n)$-**grid**.

**Proposition 4.1** *Let $\Gamma$ be a distance-regular graph with diameter at least 2. If $\Gamma$ is locally the $(m \times n)$-grid and $c_2 = 4$, then $\Gamma$ is the Johnson graph $J(n+m,n)$, or $m = n$ and $\Gamma$ is the folded Johnson graph $\overline{J}(2m,m)$.*

*Proof.* By [3, Thm.9.1.3], the graph $\Gamma$ is the Johnson graph $J(n+m,n)$, or $m = n$ and $\Gamma$ is a quotient of the Johnson graph $J(2m,m)$. More precisely, in the latter case we can partition the vertex set of $J(2m,m)$ into a uniform partition $\pi := \{C_i \mid i = 1,\dots,\binom{2m}{m}/2\}$, where $|C_i| = 2$. By

[3, Thm.11.1.6], we obtain that $\pi$ is completely regular, i.e. the sets $C_i = \{x_i, y_i\}$ are completely regular with the same intersection numbers. Suppose $\partial(x_i, y_i) = h < d = m$. Then, by $b_h \neq 0$, there exists a neighbour $v$ of $x_i$ that is at distance $h+1$ from $y_i$. Therefore, each neighbour of a vertex in $C_i$ is at distance $h+1$ from the other vertex of $C_i$. Hence $h = 1$ (since otherwise $c_h = 0$) and $a_1 = 0$. Since $a_1 = 2m - 2$, this is not possible, thus $\partial(x_i, y_i) = d$ for every $i$. It follows that $\Gamma$ is the folded Johnson graph $\overline{J}(2m, m)$. ∎

The last part of the above proof was motivated by the proof of [13, Thm.2.3.3].

The line graph of the complete graph $K_n$ is the **triangular graph** $T(n)$, i.e., the Johnson graph $J(n, 2)$. Note that $T(1)$ is an empty graph, $T(2)$ is $K_1$, $T(3)$ is $K_3$ and $T(4)$ is the complete multipartite graph $K_{2,2,2}$, i.e., the octahedron, and $T(5)$ is the complement of the Petersen graph.

**Proposition 4.2** *Let $\Gamma$ be a distance-regular graph with diameter $d \geq 3$ and let*

*(i) $\Gamma$ be locally a triangular graph,*

*(ii) $\Gamma$ have the $CAB_i$ property for some $i \in \{2, \dots, d-1\}$.*

*Then there exists an integer $n$, such that the graph $\Gamma$ is locally the triangular graph $T(n)$ and for $1 \leq j \leq i$ and for all vertices $x$ and $y$ at distance $j$ the induced subgraph on $C_j(x, y)$ is the triangular graph $T(2j)$. Furthermore, if the distance between vertices $x$ and $y$ is $i+1$, then the subgraph induced on $C_{i+1}(x, y)$ is a disjoint union of triangular graphs $T(2i+2)$.*

*Proof.* By [3, Prop.4.3.9 and Lemma 4.3.10], cf. [15], [20] and [21], condition (i) implies

- there exists an integer $n$ such that the graph $\Gamma$ is locally $T(n)$,
- $\Gamma$ is the halved graph of a bipartite graph $\Gamma'$ with intersection numbers $c_i(\Gamma') = i$ for $i \leq 3$, and
- the $\mu$-graphs in $\Gamma$ are isomorphic to the disjoint union of at most $\lfloor n/4 \rfloor$ copies of $K_{2,2,2}$.

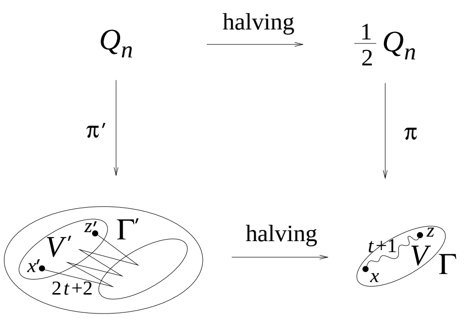



Figure 4.1: The halved graph of $Q_n$ is denoted by $\frac{1}{2}Q_n$ and called the **halved $n$-cube**.

So $k = n(n-1)/2$ and $a_1 = 2(n-2)$. Since $c_3(\Gamma') = 3$ and $a_2(\Gamma') = 0$, any 3-claw in $\Gamma'$ determines a unique 3-cube by [3, Lemma 4.3.5(ii)]. Therefore, by [3, Prop.4.3.6 and Cor.4.3.7], the $n$-cube $Q_n$ covers $\Gamma'$. More precisely, there exists a map $\pi' : V(Q_n) \longrightarrow V(\Gamma')$ that preserves distances $\leq 3$. It induces a map $\pi : V(\frac{1}{2}Q_n) \longrightarrow V(\Gamma)$, that preserves adjacency, see Figure 4.1. Let us denote by $V'$ the set of vertices of $\Gamma'$ corresponding to the vertices of $\Gamma$.

Let us define $c'_m(u',v') := c_m(\Gamma')(u',v')$ for vertices $u'$ and $v'$ at distance $m$ in $\Gamma'$ and $m = 1,2,\ldots,\mathrm{diam}(\Gamma')$. Suppose we have shown for an integer $t$, where $1 \leq t < i$, and for all $j \in \{1,\ldots,t\}$ that

(a) the subgraph of $\Gamma$ induced by $C_j(x,y)$ is the triangular graph $T(2j)$ for all $x,y \in V(\Gamma)$ with $\partial(x,y) = j$,

(b) $c'_m(x',y') = m$ for all $m \in \{1,\ldots,2t{+}1\}$, $x' \in V'$ and $y' \in \Gamma'_m(x')$.

Conditions (a) and (b) are certainly true for $t = 1$ by the observation made at the beginning of this proof and since $C_1(x,y)$ contains only one vertex, which means that it induces $T(2)$.

Before continuing with the induction, we need to introduce some new notations. Let $m$ be a positive integer, $x' \in V'$ and $y' \in \Gamma'_m(x')$. We define $c'_m := c'_m(x',y')$, if $c'_m(x',y') = c'_m(u',v')$ for all $u' \in V'$ and $v' \in \Gamma'_m(u')$, i.e., if $c'_m$ exists. For vertices $u$ and $v$ at distance $s$ in a graph $X$ we denote by $I_X(u,v)$ the **interval graph**, that is the subgraph of $X$ induced by the set $\{w \in V(X) \mid \partial_X(u,w) + \partial_X(v,w) = s\}$, i.e., the set of vertices that lie on a shortest path between $u$ and $v$.

Let $x$ be a vertex of $\Gamma$ and $x'$ the corresponding vertex of $\Gamma'$. Without loss of generality we may choose $\pi'$ to map the vector $\mathbf{0}$ to $x'$. Since $c_m(Q_n) = m = c'_m$ for all $m \in \{1,2,\ldots,2t{+}1\}$, by the induction assumption and since both graphs $\Gamma'$ and $Q_n$ are bipartite, the map $\pi'$ preserves distances $\leq 2t{+}1$ when at least one of the vertices is from $V'$, so also the map $\pi$ preserves distances $\leq t$, and the words of weight $m$ in $Q_n$ are in 1-1 correspondence with the vertices at distance $m$ from $x'$ in $\Gamma'$.

Let $z \in \Gamma_{t+1}(x)$ and let $z'$ be the corresponding vertex of $\Gamma'$. Then $z' \in \Gamma_{2t+2}(x')$ and since $c'_{2t+1} = 2t{+}1$ and $\Gamma'$ is bipartite, the words of weight $2t{+}2$ in the preimage $\pi'^{-1}(z')$ are mutually disjoint. Moreover, as $2t{+}1 \geq 3$, we have

$$c_{t+1} = c_{t+1}(x,z) = \frac{c'_{2t+2}(x',z')\, c'_{2t+1}}{c'_2} = \frac{c'_{2t+2}(x',z')\,(2t+1)}{2},$$

and therefore $c'_{2t+2}$ exists and it is equal to $2c_{t+1}/(2t{+}1)$. The interval graph $I_{\Gamma'}(x',z')$ consist of $p := c'_{2t+2}/(2t+2)$ copies of the $(2t{+}2)$-cubes sharing only the vertices $x'$ and $z'$ with each other.

Let $\Sigma$ be the subgraph of $\Gamma$ induced by the set $C_{t+1}(x,z)$. Then $\Sigma$ consists of $p$ disjoint graphs and each of them is the halved graph of the second neighbourhood of the $(2t{+}2)$-cube. The halved graph of the second neighbourhood of the $s$-cube is the Johnson graph $J(s,2)$, i.e., the triangular graph $T(s)$. It follows that the graph $\Sigma$ is a disjoint union of $p$ copies of the triangular graph $T(2t{+}2)$. Since $\Gamma$ has the $\mathrm{CAB}_{t+1}$ property the set $C_{t+1}(x,z)$ is a completely regular code with covering radious 1 or 2 in the triangular graph $T(n)$. The later graph can be considered as the line graph of $K_n$, and the $p$ copies of $T(2t{+}2)$ correspond to $p$ distinguished disjoint $(2t{+}2)$-cliques of $K_n$. Since $t < i \leq d-1$, we have $t+1 \neq d$ and the covering radious equals 2, so the distinguished cliques do not cover all its vertices. Every edge of $K_n$ corresponding to a vertex of $A_{t+1}(x,z)$

connects a vertex from one of the distingushed $(2t+2)$-cliques with one of the remaining vertices of $K_n$, or, if $p > 1$, it connects vertices from two distinct such $(2t+2)$-cliques. However, the later is not possible by the $\text{CAB}_{t+1}$ property, thus we conclude that $p = 1$ and so $c'_{2t+2} = 2t+2$.

Now we will show that $c'_{2t+3} = 2t+3$. By the fact that $C_{t+1}(x,z)$ is a completely regular code in $T(n)$, it follows that $b_{t+1} = \binom{n-2t-2}{2}$. We have chosen $x$ and $z$ to be vertices of $\Gamma$ at distance $t+1$ and $x'$, $z'$ as their corresponding vertices of $\Gamma'$ at distance $2t+2$, hence

$$\frac{(n-2t-2)(n-2t-3)}{2} = b_{t+1} = b_{t+1}(x,z) = \frac{1}{c'_2} \sum_{y' \in B_{2t+2}(x',z')} b'_{2t+3}(x',y'),$$

As $\Gamma'$ is a bipartite graph with $c'_2 = c_2(\Gamma') = 2$, by [3, Prop.1.9.1], it follows that $c'_i(u',v') \geq i$ for all vertices $u'$ and $v'$ of $\Gamma'$ at distance $i$. So, by $|B_{2t+2}(x',z')| = b'_{2t+2} = n - c'_{2t+2} = n-(2t+2)$ and $b'_{2t+3}(x',y') = n - c'_{2t+3}(x',y') \leq n-(2t+3)$, we conclude $b'_{2t+3}(x',y') = n-2t-3$, which implies $c'_{2t+3} = 2t+3$. Now the proposition follows by induction. ∎

For the convience of the reader we give a proof of the following result.

**Theorem 4.3 (A. E. Brouwer)** *Let $\Gamma$ be a bipartite distance-regular graph with diameter $d \geq 4$ and $c_i = i$ for $i \leq d-1$. Then $\Gamma$ is a $d$-cube, a folded $(2d)$-cube or if $d = 4$, the coset graph of the extended binary Golay code.*

*Proof.* For $d = 4$ it follows from [4, Thm.5.10 and Thm.5.12] that $k \in \{4, 8, 24\}$. For $k = 4$ we get the 4-cube, for $k = 8$ we get the folded 8-cube, and for $k = 24$ it follows, by [2], that $\Gamma$ is the coset graph of the binary Golay code. By [3, Thm.11.1.6 and Cor.4.3.7], there exists a completely regular code $C$ with the following distance partition


Figure 4.2

For $d \geq 5$ this means that $C$ is uniformly packed with minimum distance at least 10. They were classified by van Tilborg, who showed $|C| \leq 2$. The result follows now. ∎

**Theorem 4.4** *Let $\Gamma$ be a distance-regular graph with diameter $d \geq 2$. Then*

*(i) $\Gamma$ is locally a triangular graph,*

*(ii) $\Gamma$ has the $CAB_d$ property and*

*if and only if $\Gamma$ is the halved $n$-cube, $n \geq 4$, or $\Gamma$ is the folded halved $n$-cube with $n = 2m$, $m \in \mathbb{N}$ and $m \geq 4$, or $\Gamma$ is the halved coset graph of the extended binary Golay code.*

*Proof.* Let $\Gamma$ be locally a triangular graph and let $\Gamma$ has the $\text{CAB}_d$ property. Then, similarly as in the proof of Proposition 4.2, we start with the following:

(a) there exists an integer $n$ such that the graph $\Gamma$ is locally $T(n)$,

(b) $\Gamma$ is the halved graph of a bipartite graph $\Gamma'$ with intersection numbers $c_i(\Gamma') = i$ for $i \le 3$, and

(c) the $\mu$-graphs in $\Gamma$ are isomorphic to the disjoint union of at most $\lfloor n/4 \rfloor$ copies of $K_{2,2,2}$.

If $d \ge 3$, then, by Proposition 4.2, for vertices $x$ and $y$ at distance $i \in \{1, \dots, d-1\}$ the subgraph induced by $C_i(x,y)$ is the triangular graph $T(2i)$, and for vertices $x$ and $y$ at distance $d$ the subgraph induced by $C_d(x,y)$ is a disjoint union of exactly $n/(2d)$ copies of the triangular graphs $T(2d)$. Let us show that the same statement is true also when $d = 2$. We only need to check it for $i = 2$. Let $x$ and $y$ be vertices of $\Gamma$ at distance 2. Then we want to show that the subgraph induced by $C_2(x,y)$, i.e., the $\mu$-graph of $x$ and $y$, is a disjoint union of the triangular graphs $T(4)$. Since $K_{2,2,2}$ is isomorphic to $T(4)$, this statement coincides with the above property (c). It follows, $\Gamma$ is a distance-regular graph with intersection numbers

$$c_i = \binom{2i}{2}, \quad b_i = \binom{n-2i}{2} \quad \text{for} \quad i \le d-1, \quad \text{and} \quad c_d \in \left\{ \binom{2d}{2}, \frac{n}{2d}\binom{2d}{2} \right\}.$$

The first case can happen only when $2d \in \{n, n-1\}$. But then $|V(\Gamma)| = 2^{n-1}$ and by [3, Cor.4.3.8(ii)] the graph $\Gamma$ is the halved $n$-cube. So we may assume that we are in the second case. This can only happen when $d \ge 3$ and it is easy to see that $2d = d'$, where $d'$ is the diameter of $\Gamma'$. We are going to show $\Gamma'$ is a distance-regular graph with intersection numbers $c'_i = i$ for $i \le d-1$ and $c_{d'} = n$. As in the proof of Proposition 4.2, let $V'$ be the set of the vertices of $\Gamma'$ of the corresponding to vertices of $\Gamma$ and we have shown that $c'_i(x', y') = i$ for $i \le d-1$ and $x' \in V'$, $y' \in V(\Gamma')$ at distance $i$. Furthermore, by assumptions we have $c_{d'} = n$.
Let $x' \in V'$ and $y' \in V(\Gamma')$. Then

$$|\Gamma_{2i}(x')| = \binom{n}{2i}, \quad |\Gamma_{2i}(y')| \le \binom{n}{2i}, \quad |\Gamma_{2d}(x')| = \binom{n-1}{2d-1} \quad \text{and} \quad |\Gamma_{2i}(y')| \le \binom{n-1}{2d-1},$$

as $c'_i(u', v') \ge i$ for all $u', v' \in V(\Gamma')$ at distance $i$. But

$$\sum_{i=0}^{d} |\Gamma'_{2i}(x')| = \sum_{i=0}^{d} |\Gamma'_{2i}(y')|$$

and therefore $c'_i = i$ for $i \le 2d-1$ and $c'_{2d} = n$. Hence $\Gamma'$ is a distance-regular graph with intersection numbers $c'_i = i$ for $i \le d-1$ and $c_{d'} = n$, so the result follows now by Theorem 4.3. The result follows now. ∎

## 5 Proof of the main result

By Gardiner [7], in an antipodal distance-regular graph $\Gamma$ with diameter $D$ a vertex $x$, which is at distance $i \le \lfloor D/2 \rfloor$ from one vertex in an antipodal class, is at distance $D - i$ from all other vertices in this antipodal class, hence

$$\Gamma_{D-i}(x) = \bigcup \{\Gamma_D(y) \mid y \in \Gamma_i(x)\} \quad \text{for } i = 0, 1, \dots, \lfloor D/2 \rfloor. \tag{6}$$

If $\Gamma$ is 1-homogeneous and $x$, $y$ are its adjacent vertices, then it is not hard to conclude by (6) that, by taking antipodal quotient of $\Gamma$, the cells $D^{d-j}_{d-i}(x,y)$ and $D^j_i(x,y)$ fold together for $0 \le i,j \le \lfloor d/2 \rfloor$. However, it is even more effective to follow antipodal folding through CAB$_i$ partitions of $\Gamma$ and its antipodal quotient.

**Theorem 5.1** *Let $\Gamma$ be an antipodal graph with diameter $D \ge 4$ that is locally connected and $\Sigma$ its antipodal quotient graph with diameter $d$. Then for $i \le d-1$ the graph $\Sigma$ has the CAB$_i$ property if and only if $\Gamma$ has the CAB$_i$ property, and the following are equivalent.*

*(i)* *The graph $\Gamma$ is 1-homogeneous and $D = 2d$.*

*(ii)* *The graph $\Gamma$ has the CAB$_d$ property and $D = 2d$.*

*Moreover, if graph $\Gamma$ has the CAB$_d$ property, then the folded graph $\Sigma$ is 1-homogeneous if and only if $D = 2d$.*

*Proof.* The first part of the statement and (i) $\Longleftrightarrow$ (ii) follow from the fact that a CAB$_i$ partition of $\Sigma$, the corresponding CAB$_{D-i}$ partitions of $\Sigma$ and the corresponding CAB$_i$ partition of $\Gamma$ are isomorphic by the covering projections for $i = 1, ..., d-1$. It remains to consider the case $i = d$, see Figure 5.1.


Figure 5.1: The CAB$_d$ partition in $\Gamma$ (left) and the CAB$_d$ partition in the folded graph $\Sigma$ (right).

Let $y_1, \dots, y_r$ be the vertices of an antipodal class of $\Gamma$, let $x$ be at distance $d$ from $y_1$, and let $\hat{x}$ and $\hat{y}$ be the projections of $x$ and $y_1$ respectively. Consider the following partition

$$\Omega = D^{d-1}_1(x,y_1) \cup \dots \cup D^{d-1}_1(x,y_r) \cup \Big(\Gamma(x) \setminus \bigcup_{i=1}^{r} D^{d-1}_1(x,y_i)\Big)$$

of the local graph of $x$, see Figure 5.2. If $D = 2d$, then the first $r$ sets have size $c_d(\Gamma)$, the last one has size $a_d(\Gamma)$, and there are no edges between $D^{d-1}_1(x,y_i)$ and $D^{d-1}_1(x,y_j)$ when $i \ne j$, which means that by [11, Thm.2.4] the partition $\Omega$ is equitable if and only if the partitions CAB$_d(x,y_i)$ for $i = 1, \dots, r$ are equitable, in which case $\Omega$ has the following quotient matrix

$$\begin{pmatrix} g & 0 & 0 & \dots & 0 & a_1-g \\ 0 & g & 0 & \dots & 0 & a_1-g \\ 0 & 0 & g & \dots & 0 & a_1-g \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & g & a_1-g \\ & & & & & \\ h & h & h & \dots & h & a_1-rh \end{pmatrix} \tag{7}$$

for some integers $g$ and $h$. If (i)–(ii) holds, then $g = \gamma_d = a_1 - \delta_d$, $h = \alpha_d = \beta_d/(r-1)$ and the CAB$_d$ partition of $\Sigma$ is equitable with the quotient matrix $\left(\begin{smallmatrix}\gamma_d, & \delta_d \\ r\alpha_d, & a_1-r\alpha\end{smallmatrix}\right)$.

Finally, let us suppose the folded graph $\Sigma$ is 1-homogeneous, i.e., $\Sigma$ has the $\mathrm{CAB}_d$ property, and let $\left(\begin{smallmatrix} g, & a_1-g \\ a, & a_1-a \end{smallmatrix}\right)$ be the quotient matrix of the $\mathrm{CAB}_d(\hat{x}, \hat{y})$ partition in $\Sigma$. Suppose $D$ is odd. Then the local graph is disconnected, see Figure 5.2 (right), and we have $g = a_1$ and $a = 0$. By [11, Prop.2.2], the set $C_d(\hat{y}, \hat{x})(\Sigma)$ is independent in $\Sigma$, which is not possible because $a_1 \geq 2$ by **(B)**. Therefore, we have $D = 2d$ by [3, Prop.4.2.2]. ∎

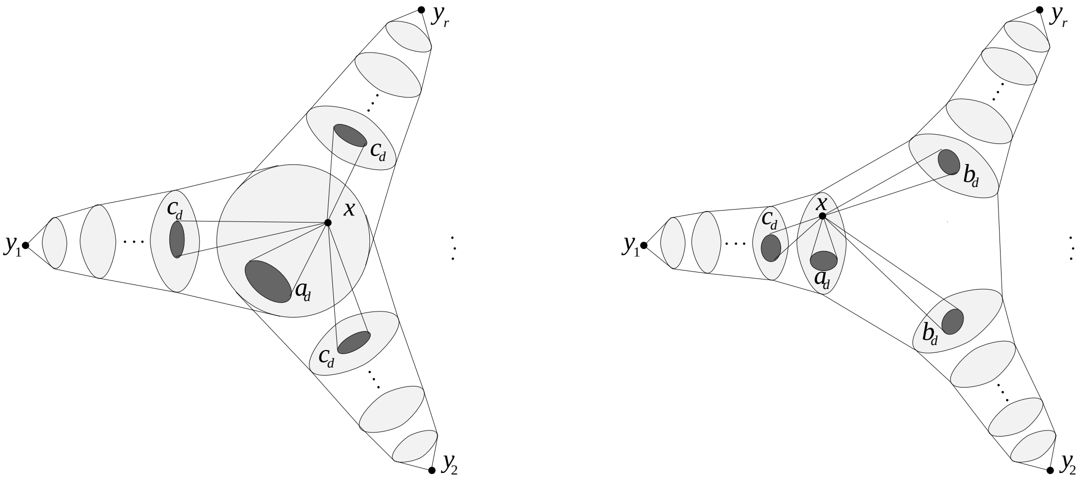


Figure 5.2: The partition corresponding to the distance distribution of the antipodal class $\{y_1, \ldots, y_r\}$ in the case when $D$ is even (left) and the case when $D$ is odd (right). We have chosen $r$ to be three. Inside this partition there is a partition of the neighbourhood of the vertex $x$.

**Remark 5.2** *There is an antipodal distance-regular graph with diameter 4 and intersection array* $\{6, 4, 2, 1; 1, 1, 4, 6\}$, *called the halved Foster graph, which is locally disconnected, therefore it is not a tight graph in the sense of [10], and hence not* 1*-homogeneous, see [10, Thm.11.7]). However, its antipodal quotient is the complement of the triangular graph* $T(6)$, *i.e., the incidence graph of the generalized quadrangle* $GQ(2, 2)$, *cf. [3, p.223], and it is 1-homogeneous. Therefore, the assumption on local connectivity in the above result is necessary.*

**Theorem 5.3** *A graph* $\Gamma$ *is an* 1*-homogeneous graph with diameter at least* 2*, which satisfies condition* **(C)** *with* $n = 2$ *and condition* **(B)** *with* $\alpha = t \geq 2$ *if and only if* $\Gamma$ *is one of the following:*

*(i) a Johnson graph* $J(2m, m)$ *with* $m \geq 3$,

*(ii) a folded Johnson graph* $\overline{J}(4m, 2m)$ *with* $m \geq 3$,

*(iii) a halved* $m$*-cube with* $m \geq 5$,

*(iv) a folded halved* $(2m)$*-cube with* $m \geq 5$,

*(v) the Schläfli graph with intersection array* $\{16, 5; 1, 8\}$,

*(vi) the Gosset graph with intersection array* $\{27, 10, 1; 1, 10, 27\}$.

*Proof.* Let $\Gamma$ be an 1-homogeneous graph with diameter at least two, which satisfies the condition **(C)** with $n = 2$ and the condition **(B)** with $\alpha = t \geq 2$. Let $x$ be a vertex of $\Gamma$. Then the subgraph

$\Delta(x)$ is a connected strongly regular graph by [11, Thm. 3.1 and Prop. 2.1] and the smallest eigenvalue of $\Delta(x)$ is $-2$ by Theorem 3.2. By Seidel's classification [17], [3, Thm.3.12.4], the local graph $\Delta(x)$ is either

- a triangular graph $T(m)$ with $m \geq 5$,
- a $(m \times m)$-grid with $m \geq 3$,
- a Cocktail Party graph $K_{m\times 2}$ with $m \geq 2$,
- the Petersen graph,
- the Clebsch graph,
- the Schläfli graph,
- the Shrikhande graph, or
- one of the three Chang graphs.

The $\mu$-graphs of the Shrikhande graph, the Petersen graph and all the three Chang graphs are not all isomorphic to $K_{(t-1)\times 2}$. If $\Gamma$ is locally $K_{m\times 2}$ with $m \geq 2$, then $\Gamma$ is the Cocktail Party graph $K_{(m+1)\times 2}$ by [3, Prop.1.1.5], with $a_2 = 0$, so the condition **(B)** is not satisfied.

If $\Gamma$ is locally Clebsch graph, see [3, p.104], then $\mu' = 6$, $\Gamma$ is the Schläfli graph, see [3, p.312], which has $a_1 = 10$, $a_2 = 8$, $c_2 = 8 = \mu' + 2$ (so it satisfies the condition **(C)** with $n = 2$ and $t = 4$), and $\alpha = 4$ (so it is really 1-homogeneous graph, cf. [11, Thm.3.9]).

If $\Gamma$ is locally Schläfli graph, then $\Gamma$ is the Gosset graph, see [3, p.313], which is 1-homogeneous graph, see [10, Thm. 11.7 and 12.6], and has $a_1 = 16$, $c_2 = 10 = \mu' + 2$ (so it satisfies the condition **(C)** with $n = 2$ and $t = 5$), and $\alpha = 5$.

Suppose $\Gamma$ is locally $(m \times m)$-grid with $m \geq 3$. Then the eigenvalues of the local graphs are $a_1 = 2m - 2$, $m - 2$ and $-2$. Furthermore, $\mu' = 2 = n(t-1)$ and $c_2 = nt = 4$ by Lemma 2.1 and $\Gamma$ is either a Johnson graph $J(2m, m)$ or a folded Johnson graph $\overline{J}(2m, m)$ by Proposition 4.1. The first graph is 1-homogeneous by [10, Thm. 11.7 and 12.6], with $\alpha = 2$ and has $c_2 = \mu' + 2$ (so it satisfies the condition **(C)** with $n = 2$ and $t = 2$). Suppose $\Gamma$ is the second graph and $\theta_1, \theta_d$ respectively its second largest and its smallest eigenvalue. Then the 1-homogeneous property of $\Gamma$ implies that $-1 - b_1/(1+\theta)$ for some $\theta \in \{\theta_1, \theta_d\}$ is an eigenvalue of all local graphs of $\Gamma$ by [11, Thm.3.9]. The later can only happen when $m$ is even by [3, Prop.9.1.5]. The folded Johnson graph $\overline{J}(8, 4)$ is 1-homogeneous with $\alpha = 4$ by Theorem 5.1 and [12, Cor.4.8], so the condition **(B)** is not satisfied for $\alpha = t = 2$. Thus $m = 2s$ with $s \geq 3$. By Theorem 5.1, the folded Johnson graph $\overline{J}(4s, 2s)$ (obtained by folding 1-homogeneous antipodal graph of even diameter) is 1-homogeneous with $\alpha = 2$ and it has $c_2 = \mu' + 2$ (so it satisfies the condition **(C)** with $n = 2$ and $t = 2$).

Finally, we suppose $\Gamma$ is locally triangular graph $T(m)$ with $m \geq 5$. Then $\mu' = 4$, $t = 3$ and $\Gamma$ is either

- the halved $m$-cube, or
- the folded halved $m$-cube with $m$ even and $m \geq 8$, or

- the halved coset graph of the extended binary Golay code,

by [11, Thm.3.1] and Theorem 4.4. The first graph in the case $2|m$ and the second graph in the case $4|m$, $m \neq 8$, are, by [10, Thm. 12.6 and 11.7] and by Theorem 5.1 respectively, 1-homogeneous with $\alpha = 3$ and have $c_2 = 6 = \mu' + 2$ (so they satisfy the condition **(C)** with $n = 2$ and $t = 3$). The folded halved 8-cube is 1-homogeneous with $\alpha = 6$ by Theorem 5.1 and [12, Cor.4.8], so $\alpha \neq t$. By a direct counting argument, we verify that also the halved $(2s+1)$-cubes with $s \geq 2$ and the folded halved $(4s+2)$-cubes with $s \geq 2$ are 1-homogeneous. The third graph is the halved graph of the distance-regular graph with intersection array $\{24, 23, 22, 21; 1, 2, 3, 24\}$ by [3, Thm.11.3.2] and is therefore a strongly regular graph with parameters $(2048, 276, 44, 36)$ by [3, Prop.4.2.2(i)], so $c_2 = 36 \neq 6 = \mu' + 2$.

The converse is straightforward, since we have already verified that the graphs in the above list have the required properties. ∎

*Proof of the main theorem.* Let $\Gamma$ be a 1-homogeneous graph with diameter $d \geq 2$. Let for all vertices $x$ of $\Gamma$ the local graph $\Delta(x)$ be a strongly regular graph with parameters $(v', k', \lambda', \mu')$ and $c_2 = \mu' + 2 \geq 3$. Since $\mu'$ is the valency of the $\mu$-graphs, each $\mu$-graph is the Cocktail Party graph $K_{(c_2/2)\times 2}$, $c_2$ is even and $\mu' \geq 2$. Therefore, $\Gamma$ satisfies the conditions **(A)** and **(C)** with $n = 2$ and $t = c_2/2 \geq 2$. If $a_2 = 0$, then $b_2 = 0$ by $\mu' \neq 0$ and connectivity of $\Gamma$, thus $d = 2$ and $\Gamma$ is the Cocktail party graph $K_{m\times 2}$ with $m = t+1 \geq 3$. Now we assume $a_2 \neq 0$. Since $\Gamma$ is 1-homogeneous and $\mu' \neq 0$, the condition **(B)** is satisfied, $\mu' = n(t-1)$ by Lemma 2.1, $\alpha \in \{t-1, t\}$ by Lemma 2.3. If $\alpha = t$, then by Theorem 5.3 we obtain that $\Gamma$ has to be one of the listed examples except (v). If $\alpha = t - 1$, then By Corollary 3.4 we have $\alpha = 1$, $t = 2$ $\mu' = 2$ and $\lambda' = 0$, in which case $d = 2$ implies, see Figure 1.1(a) $a_1 = 2$ or 5. In the first case we obtain the octahedron that implies $a_2 = 0$ and the second one has already been considered in Remark 3.5 and is not possible.

The converse is straightforward. ∎

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