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Preprint series, Vol. 38 (2000), 721

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ISSN 1318-4865

September 15, 2000

Ljubljana, September 15, 2000

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[1]Supported in part by "Ministrstvo za znanost in tehnologijo Slovenije", proj. no. J1-0496-0101-99.

[2]Supported in part by the grant from Zhengzhou University. Wang is grateful to the Institute of Mathematics, Physics and Mechanics at the University of Ljubljana for hospitality and financial support during his visit that led to the completion of this work.

SEMISYMMETRIC GRAPHS OF ORDER $2p^3$

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Abstract

A regular graph is called *semisymmetric* if it is edge-transitive but not vertex-transitive. It is proved that the Gray graph is the only cubic semisymmetric graph of order $2p^3$, where $p \geq 3$ is a prime.

1 Introduction

Throughout this paper graphs are assumed to be finite, and, unless specified otherwise, simple, undirected and connected. For the group-theoretic concepts and notation not defined here we refer the reader to [3, 9, 21].

Given a graph X we let $V(X)$, $E(X)$ and $\mathrm{Aut}\, X$ be the vertex set, the edge set and the automorphism group of X, respectively. For two adjacent vertices u and v we write $u \sim v$, and use the symbol uv to denote either the edge between u and v, or the arc from u to v. No ambiguity should arise for we always clearly state whether we refer to an edge or to an arc. If a subgroup G of $\mathrm{Aut}\, X$ acts transitively on $V(X)$ and $E(X)$, we say that X is *G-vertex-transitive* and *G-edge-transitive*, respectively. In the special case when $G = \mathrm{Aut}\, X$ we say that X is *vertex-transitive* and *edge-transitive* respectively. It can be shown that a G-edge- but not G-vertex-transitive graph X is necessarily bipartite, where the two parts of the bipartition are orbits of $G \leq \mathrm{Aut}\, X$. Moreover, if X is regular these two parts have equal cardinality. A regular G-edge- but not G-vertex-transitive graph will be referred to as a *G-semisymmetric* graph. In particular, if $G = \mathrm{Aut}\, X$ the graph is said to be *semisymmetric*.

The study of semisymmetric graphs was initiated by Folkman [6] who posed a number of problems which spurred the interest in this topic (see [1, 2, 4, 5, 11, 12, 13, 16]). Among other things he proved that there are no semisymmetric graphs of order $2p$ or $2p^2$, for p a prime.

This paper deals with (non)existence of cubic semisymmetric graphs of order $2p^3$, where p is a prime. The first example of such a graph, the so called Gray graph, has order 54 and is descibed in [1]. Its discovery, according to [1], is due to Marion C. Gray in 1932, thus explaining its name. As shown in [15], it is the smallest cubic semisymmetric graph. Following [17], the Gray graph is a regular 9-fold cover of $K_{3,3}$, with $\mathbb{Z}_3^2$ as the group of covering transformations (see Figure 1 below). For the purpose of this paper we take this as the definition of the Gray graph. Alternative definitions can be found in [17].

Our object is to prove the following result.

Theorem 1.1 *Let X be a cubic semisymmetric graph of order $2p^3$, $p \geq 3$ a prime. Then $p = 3$ and X is isomorphic to the Gray graph.*

The proof of Theorem 1.1 uses a combination of purely group-theoretic and combinatorial techniques, and is given in Section 3.

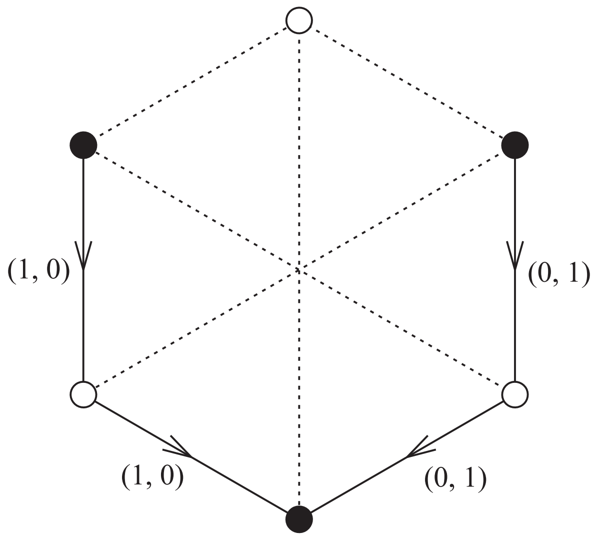

Figure 1: The Gray graph as a cover of $K_{3,3}$.

2 Preliminaries

An epimorphism $\wp : X \to Y$ of connected graphs is a *regular covering projection* if it arises essentially as a factorization $X \to X/G \cong Y$, where the action of $G \leq \operatorname{Aut} X$ is semiregular on both vertices and edges of X. Note that the graph Y may not be simple even if X is. The graph X is called the *covering graph* and Y is the *base graph*. The preimage $\wp^{-1}(v)$, $v \in V(Y)$, corresponds to an orbit of G on $V(X)$ and is called the (*vertex*)*-fibre* over v. Similarly, *edge-fibres* correspond to orbits of G on $E(X)$.

It is well-known that a regular covering projection $X \to Y \cong X/G$ can be reconstructed in terms of *voltage assignments* valued in G as follows (see [10]). First label arbitrarily a vertex in each fibre by $1 \in G$, and then label all other vertices by the right regular action of $G \leq \operatorname{Aut} X$ on each fibre. Consequently, given an arc uv in Y, the origins and termini of arcs in $\wp^{-1}(uv)$ are labelled, respectively, by g and ag $(g \in G)$ for some $a \in G$. This fact is recorded by assigning the *voltage* $\operatorname{vol}(uv) = a \in G$ to the corresponding arc uv. Clearly, inverse arcs carry inverse voltages. The edges of X can thus be retrieved from Y by considering the left regular action of G induced by the above labelling. A given voltage assignment can be modified in such a way that the arcs of an arbitrarily prescribed spanning tree receive trivial voltages, and that the modified assignment is associated with the same covering projection [10]. Moreover, the following proposition holds.

Proposition 2.1 [18] *Leaving the voltages of a spanning tree trivial and replacing the voltage assignments on the cotree arcs by their images under an automorphism of the voltage group results in an equivalent covering projection.*

Let $\wp : X \to Y \cong X/G$ be a regular covering projection. If $\varphi \in \operatorname{Aut} Y$ and $\tilde{\varphi} \in \operatorname{Aut} X$ satisfy $\tilde{\varphi}\wp = \wp\varphi$ we call $\tilde{\varphi}$ the *lift* of φ, and φ the *projection* of $\tilde{\varphi}$. (For the purpose of this paper, all functions are composed from left to right.) Concepts such as the lift of a group of automorphisms and the projection of a group of automorphisms are self-explanatory. The lifts and the projections of groups are of course subgroups in $\operatorname{Aut} X$ and $\operatorname{Aut} Y$, respectively. In particular, $\operatorname{CT}(p) = G$ is the lift of the identity group and is known as the *group of covering transformations*. Clearly, if G is normal in $\operatorname{Aut} X$ then $\operatorname{Aut} X$ does project (however, the projection need not be onto). The problem whether an automorphism φ lifts can be grasped in terms of voltages as follows. First observe that the voltage assignment on arcs extends to an assignment on all walks in a natural way. Define the mapping $\varphi^{\#} : G \to G$, relative to a chosen base vertex v, by the rule

$$(\operatorname{vol}(C))^{\varphi^{\#}} = \operatorname{vol}(C^{\varphi}),$$

where C ranges over all fundamental closed walks at v. Note that if G is abelian, $\varphi^{\#}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generating the cycle space of X.

Proposition 2.2 [14] *Let $\wp : X \to Y \cong X/G$ be a regular covering projection. Then an automorphism φ of Y lifts along $\wp$ if and only if $\varphi^{\#}$ extends to an automorphism of G.*

Corollary 2.3 *If X is a bipartite graph admitting an abelian subgroup G of automorphisms acting regularly on each of the bipartition sets, then X is vertex-transitive.*

PROOF. Consider the regular covering projection $X \to X/G$. The base graph is a dipole dip_d, that is, a graph with two vertices and d parallel edges, where d is the valency of X. Let τ be the reflection of this dipole mapping each arc to its inverse. Then $g^{\tau^{\#}} = g^{-1}$. Since taking inverses in an abelian group is an automorphism, the reflection τ lifts. Hence the graph X is vertex-transitive. ■

Following [8], the *coset graph* $\Gamma(G; H_0, H_1)$ of a group G with respect to finite subgroups H_0 and H_1 is a bipartite graph with $\{H_0 g \mid g \in G\}$ and $\{H_1 g \mid g \in G\}$ as its bipartition sets of vertices, where $H_0 g$ is adjacent to $H_1 g'$ whenever $H_0 g \cap H_1 g' \neq \emptyset$. The following facts may be extracted from [8]:

(i) $\Gamma(G; H_0, H_1)$ is regular of valency d if and only if $H_0 \cap H_1$ has equal index d in both H_0 and H_1;

(ii) $\Gamma(G; H_0, H_1)$ is connected if and only if $G = \langle H_0, H_1 \rangle$;

(iii) G acts on $\Gamma(G; H_0, H_1)$ by right multiplication. Moreover, this action is faithful if and only if a normal subgroup of G which is contained in $H_0 \cap H_1$ coincides with the identity group;

(iv) In the case when the action of G is faithful, the coset graph $\Gamma(G; H_0, H_1)$ is G-edge-transitive but not G-vertex-transitive.

Proposition 2.4 *Let X be a regular graph and $G \leq \operatorname{Aut} X$. If X is G-semisymmetric, then X is isomorphic to the coset graph $\Gamma(G; G_u, G_v)$, where u and v are adjacent vertices.*

PROOF. Define a mapping $\varphi : u^g \mapsto G_u g$, $v^g \mapsto G_v g$, $g \in G$. Then φ is well-defined. It is easy to see that φ is bijective. We verify that φ is a graph isomorphism of X onto $\Gamma(G; G_u, G_v)$.

Indeed, it is easy to see that X is bipartite. Moreover, let u^g and v^h be two adjacent vertices in X, where $g, h \in G$. Then $u \sim v^{hg^{-1}}$. Since G is edge-transitive, $v^{hg^{-1}} = v^{g'}$ for some $g' \in G_u$. Hence there exists some $h' \in G_v$ such that $hg^{-1}g'^{-1} = h'$. So $h'^{-1}h = g'g \in G_u g \cap G_v h$, that is, $G_u g$ is adjacent to $G_v h$ in $\Gamma(G; G_u, G_v)$. Conversely, if $G_u g$ is adjacent to $G_v h$ in $\Gamma(G; G_u, G_v)$, then there exist some $g' \in G_u$ and $h' \in G_v$ such that $g'g = h'h$. Since u and v are adjacent and $G \leq \operatorname{Aut} X$, it follows that $u^g = u^{g'g}$ is adjacent to $v^{h'h} = v^h$. Therefore φ is the desired graph automorphism. ■

From this we can easily derive the following proposition.

Proposition 2.5 *The vertex stabilisers of a connected G-semisymmetric cubic graph have order $2^r \cdot 3$. Moreover, if u and v are two adjacent vertices, then $G = \langle G_u, G_v \rangle$, and the edge stabiliser $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v.*

The precise structure of the pair of admissible vertex stabilisers (G_u, G_v) was completely determined in [8]. In this article, however, a much weaker result will be needed.

Proposition 2.6 *Let u and v be two adjacent vertices of a connected G-semisymmetric cubic graph. If G_u and G_v are both abelian, then $G_u \cong G_v \cong \mathbb{Z}_3$.*

PROOF. If both G_u and G_v are abelian, then $G_u \cap G_v$, the common Sylow 2-subgroup of G_u and G_v, is contained in $Z(G)$ as $G = \langle G_u, G_v \rangle$. As $G_u \cap G_v$ is the stabiliser of the edge uv in G and since G is edge-transitive on X, any edge stabiliser of G is some G-conjugate of $G_u \cap G_v$. It follows that $G_u \cap G_v = 1$, which implies $G_u \cong G_v \cong \mathbb{Z}_3$. ∎

In general, the action of the full automorphism group of a semisymmetric graph on its bipartition sets need not be faithful [4]. However, for cubic graphs this action is always faithful.

Proposition 2.7 *Let X be a cubic semisymmetric graph. Then* $\mathrm{Aut}\, X$ *acts faithfully on each of the bipartition sets of X.*

PROOF. Let V_0 and V_1 be the two bipartition sets of X, and let $u \in V_0$ and $v \in V_1$ be adjacent.

Assume that $A = \mathrm{Aut}\, X$ is not faithful, say, on V_0. Then the kernel $C_A(V_0)$ of the action of A on V_0 is nontrivial. By Proposition 2.5, each vertex stabiliser of A is a $\{2,3\}$-group. Since $C_A(V_0)$ is the intersection of vertex stabilisers A_w for $w \in V_0$, it follows that $C_A(V_0)$ is also a $\{2,3\}$-group. If 3 divides $|C_A(V_0)|$, then there exists some element of order 3 in $C_A(V_0)$. Since an element of order 3 in a vertex stabiliser of a semisymmetric cubic graph is transitive on its neighbours, it follows that $C_A(V_0)$ is transitive on the neighbourhood $X(w)$ of any vertex $w \in V_0$. By the connectivity of X we have $X \cong K_{3,3}$, a contradiction since $K_{3,3}$ is vertex-transitive. Hence $C_A(V_0)$ is a 2-group. It follows that $C_A(V_0) \leq A_u \cap A_v = A_{uv}$. Since $C_A(V_0)$ is normal in A (and A is faithful on $E(X)$), it follows that $C_A(V_0) = 1$, contrary to our assumption. ∎

3 Proof of Theorem 1.1

The following lemma which gives a detailed information on the local p-subgroups of the full automorphism group of cubic semisymmetric graphs of order $2p^3$ is esential to the proof of our main theorem. (Note that the proof of this lemma depends on the classification of finite simple groups, applied to groups of order $2^r \cdot 3 \cdot p^3$.)

Lemma 3.1 *Let X be a cubic semisymmetric graph of order $2p^3$, $p \geq 3$ a prime. Then $A = \mathrm{Aut}\, X$ contains a subgroup G isomorphic to $\mathbb{Z}_p^2$, acting semiregularly on vertices and edges of X. Moreover,*

(i) if $p > 3$ then G is normal in $\mathrm{Aut}\, X$; and

(ii) if $p = 3$ then G is normal in a Sylow 3-subgroup of $\mathrm{Aut}\, X$, and the quotient graph X/G is isomorphic to $K_{3,3}$.

PROOF. Denote the bipartition sets of X by V_0 and V_1 and let $u \in V_0$, and $v \in V_1$ be adjacent.

We first consider the case $p > 3$. By Propositions 2.5 and 2.7, $|A| = 2^r \cdot 3 \cdot p^3$ for some integer r. Let P be a Sylow p-subgroup of A. Then P acts regularly on each V_i, and hence X is a regular P-cover of the dipole dip_3 with three parallel edges. The covering projection $X \to \mathrm{dip}_3$ can be reconstructed in terms of the voltage group P, where the voltages on the three arcs from V_0 to V_1 are 1, a and b. Since X is connected, we have $P = \langle a, b \rangle$. By Corollary 2.3, P is not abelian. It is well known that there are exactly two nonisomorphic nonabelian groups of order p^3, given by respective presentations

$$M(p) = \langle a, b \mid a^p = b^p = c^p = 1, c = [a, b], ac = ca, bc = cb \rangle$$

and

$$M_1(p) = \langle x, y \mid x^{p^2} = y^p = 1, [x, y] = x^p \rangle.$$

We show that $P = M_1(p)$. Suppose, on the contrary, that $P = M(p)$. One can show that in this case $\mathrm{Aut}P$ acts transitively on the set of ordered pairs of generators of P.

Therefore, without changing the covering projection we may assume that the generators of the voltage group P are the elements a and b as in the above presentation of $M(p)$. Let $\tau \in \mathrm{Aut}\,\mathrm{dip}_3$ be the reflection which maps each of the arcs to its inverse. Then $a^{\tau^{\#}} = a^{-1}$ and $b^{\tau^{\#}} = b^{-1}$. It is easy to see that $\tau^{\#}$ extends to a group automorphism of P. Hence τ lifts, implying that X is vertex-transitive, a contradiction. Therefore $P = M_1(p)$.

For a set π of prime divisors of the order of A, let $O_\pi(A)$ denote the largest normal subgroup of A whose order is divisible only by primes in the set π. Furthermore, as usual, let π' denote the set of all prime divisors of the order of A not in π. Now, $Q = O_p(A)$ is the maximal normal p-subgroup of A. Note that since $O_{p'}(A)$ is contained in every vertex stabiliser, we have $O_{p'}(A) = 1$.

Suppose first that $Q = 1$. Let N be a minimal normal subgroup of A. In general, N is a direct product of isomorphic simple groups, which must be nonabelian, by our assumption. But 3^2 does not divide $|A|$ and so N is a nonabelian simple $\{2, 3, p\}$-group. By the classification of finite simple $\{2, 3, p\}$-groups, we have $N \cong \mathrm{PSL}_2(p)$, where $p = 5$ or 7. In particular, $N \neq A$ and $\mathrm{Out}(N) = \mathbb{Z}_2$. On the other hand, by [19, Theorem 6.11], we have that $A/C_A(N)$ is isomorphic to a subgroup of $\mathrm{Aut}\, N$. Since $C_A(N) \cap N = 1$, we have that $C_A(N)N = C_A(N) \times N$, and so $C_A(N) \times N$ is a subgroup of index at most 2 in A, implying that $C_A(N)$ is nontrivial. Moreover, since $O^3(A)$ (the smallest normal subgroup of A such that the order of the quotient $A/O^3(A)$ is a 3-group) is a subgroup of N, we have that $C_A(N)$ is a $\{2, p\}$-group. Thus, $C_A(N)$ is solvable, and so either $O_p(C_A(N)) \neq 1$ or $O_{p'}(C_A(N)) \neq 1$. Since these two groups are characteristic in $C_A(N)$, they are both normal subgroups of A. Since $O_{p'}(A) = 1$, this contradicts our assumption that $O_p(A) = 1$.

Suppose now that $Q \cong \mathbb{Z}_p$. Then $Q = Z(P)$ for every Sylow p-subgroup P of A. Set $C = C_A(Q)$. Then $P \leq C$, and $O_p(C/Q) = 1$. Let $O_{p'}(C/Q) = M/Q$, where $M \leq C$. As M/Q is characteristic in C/Q and $C/Q \lhd A/Q$, we have $C/Q \lhd A/Q$. Hence M is normal in A. Since Q is a normal Sylow p-subgroup of M, it follows that Q has a p'-complement M_1 in M. Consequently, $M = QM_1 = Q \times M_1$ and so $M_1 = O_{p'}(M)$ is normal in A. Thus $M_1 = 1$ and $O_{p'}(C/Q) = 1$. Hence C/Q is nonsolvable. Now we let N/Q be a minimal normal subgroup of C/Q. By the same argument as in the preceding paragraph (replacing A and N by C/Q and N/Q, respectively) a similar contradiction is obtained.

Suppose now that $Q \cong \mathbb{Z}_{p^2}$. Set $C = C_A(Q)$. Note that Q is a normal Sylow p-subgroup of C. Let C_1 be a p'-complement of Q in C. Then $C = QC_1 = Q \times C_1$, and so $C_1 = O_{p'}(C)$. Since $O_{p'}(C)$ is normal in A and $O_{p'}(A) = 1$, it follows that $C_1 = 1$. Thus $C = Q$ and so, by [19, Theorem

6.11], we have that A/Q is isomorphic to a subgroup of $\mathrm{Aut}\, Q \cong \mathbb{Z}_{p(p-1)}$. This implies that the vertex stabilisers A_u and A_v are both abelian. In view of Proposition 2.6 we have $A_u \cong A_v \cong \mathbb{Z}_3$. Now $|A| = 3p^3$, and by Sylow's theorem it is easily seen that A has a normal Sylow p-subgroup, contradicting $Q \cong \mathbb{Z}_{p^2}$.

To summarize, we have proved thus far that either Q is elementary abelian of order p^2 or $Q = P$ is normal in A. In the first case we take the group G to be Q. In the second case we have that $\Omega_1(P)$, the group generated by all elements of order p in P, is a characteristic subgroup of P, and hence normal in A. Since $P = M_1(p)$, we have $\Omega_1(P) \cong \mathbb{Z}_p^2$, and we take $G = \Omega_1(P)$ in this case.

Moreover, since $p > 3$, we have that $G \cong \mathbb{Z}_p^2$ is always semiregular on both vertices and edges because the vertex stabilisers and edge stabilisers of X are all p'-groups.

Let us now consider the case $p = 3$. We have $|P| = 3^4$, $|V_0| = |V_1| = 3^3$, and $|A| = 2^r 3^4$. Let $i \in \{0, 1\}$. Since A is transitive on V_i, and $|V_i| = 3^3$, it follows by [21, Theorem 3.4] that also the Sylow p-subgroup P of A is transitive on V_i. Further, P is nonabelian for its action on V_i is not regular. Note that each Sylow 2-subgroup of A is an edge stabiliser. Consequently, P acts edge-transitively and hence semisymmetrically on X.

Let G be a normal subgroup of order 9 in P. We first show that G must be semiregular on each V_i. For if G is not semiregular, say on V_0, then G has nontrivial intersection with $P_u \cong \mathbb{Z}_3$, for some $u \in V_0$, and hence $P_u \leq G$. Choose an arbitrary $w \in V_0$. By the normality of G in P and the transitivity of P on V_0 it follows that $P_w \leq G$. Since P_w is transitive on the neighbourhood $N(w)$ of w, it follows that $N(w)$ is contained in only one orbit of G on V_1. Moreover, for any other vertex w' in the same orbit of G on V_0, the neighbourhoods $N(w)$ and $N(w')$ are contained in the same orbit of G on V_1. By the connectivity of X, the group G is transitive on V_1, which is impossible as $|G| = 9$ and $|V_1| = 27$. Therefore G is semiregular on each V_i, and $X \to X/G$ is a regular covering projection, where X/G is a cubic graph of order 6. Since G is normal in P, the group P projects onto an edge-transitive subgroup of X/G. Consequently, the quotient graph X/G is isomorphic to $K_{3,3}$.

Note that the argument of the preceding paragraph implies that no normal subgroup of order 9 of P contains a vertex stabiliser P_w, $w \in V(X)$.

We now show that $G \cong \mathbb{Z}_3^2$. Suppose not. Then $G \cong \mathbb{Z}_9$. By [19, Theorem 6.11] the quotient $P/C_P(G)$ is isomorphic to a subgroup of $\mathrm{Aut}\, G \cong$

$\mathbb{Z}_6$. It follows that either $C_P(G) = P$ or $C_P(G)$ is abelian of order 27.

If $C_P(G) = P$, that is, $G \leq Z(P)$, then $G = Z(P)$ as P is not abelian. Recall that G and P_u have trivial intersection. It follows that $\langle G, P_u \rangle \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Now $\Omega_1(\langle G, P_u \rangle)$ is characteristic in $\langle G, P_u \rangle$ and hence normal in P. Note that $\Omega_1(\langle G, P_u \rangle)$ contains P_u and is of order 9. This contradicts the fact that P has no such subgroups. Therefore, $C_P(A)$ is abelian of order 27. In other words, $C_P(G)$ is isomorphic either to $\mathbb{Z}_9 \times \mathbb{Z}_3$ or to $\mathbb{Z}_{27}$.

Suppose that $C_P(G) \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. By Corollary 2.3, $C_P(G)$ cannot be regular on both the V_0 and V_1. Consequently, it contains a vertex stabiliser P_w for some $w \in V(X)$. The same argument as above now leads to a contradiction.

It remains to exclude the case $C_P(G)$ cyclic. In fact, we claim that P contains no cyclic subgroups of order 27. Assuming that P contains a subgroup R isomorphic to $\mathbb{Z}_{27}$, we have by Corollary 2.3 that R cannot be transitive on both V_0 and V_1. Suppose that R is intransitive on V_0. Then $P_u \leq R$ for some $u \in V_0$. Since R has index 3 in P, it is normal in P. By the transitivity of P on V_0, the group R contains P_w for all $w \in V_0$. As R is cyclic and $P_u \cong \mathbb{Z}_3$, the groups P_w, $w \in V_0$, coincide and are therefore contained in the kernel of A on V_0. This contradicts Proposition 2.7, completing the proof of Lemma 3.1. ∎

Proof of Theorem 1.1. We shall distinguish two cases.

CASE 1: $p > 3$.

By Lemma 3.1, the automorphism group $A = \mathrm{Aut}\, X$ has a normal subgroup G isomorphic to $\mathbb{Z}_p^2$, acting semiregularly on vertices and edges of X. Consider the regular covering projection $\wp : X \to X/G$. Since G is normal in A, the group A projects to a subgroup of automorphisms of the graph $Y = X/G$ isomorphic to A/G, acting semisymmetrically on Y. But Y has order $2p$ and by the classical result of Folkman [6], we have that Y is a bipartite, vertex-transitive and therefore also an arc-transitive graph. The structure of such graphs is well known. We may identify the vertex set of Y with two copies of $\mathbb{Z}_p$, where the elements of the second copy are conveniently dinstinguished from the elements of the first copy by the symbol $'$. Furthermore there exists an element $s \in \mathbb{Z}_p^* \setminus \{1\}$ such that $s^2 + s + 1 = 0$. The edges of Y are then of the form $y(y+1)'$, $y(y+s)'$, $y(y+s^2)'$, for $y \in \mathbb{Z}_p$.

We reconstruct our covering by assigning the trivial voltage to the arcs of the natural hamiltonian path from 0 to $1'$. Furthermore, the voltage of the arc $01'$ is denoted by a, and the voltages of the chordal arcs from $i(s-1)$ to $(i(s-1)+s^2)'$, $i \in \mathbb{Z}_p$, are denoted by a_i (see Figure 2).

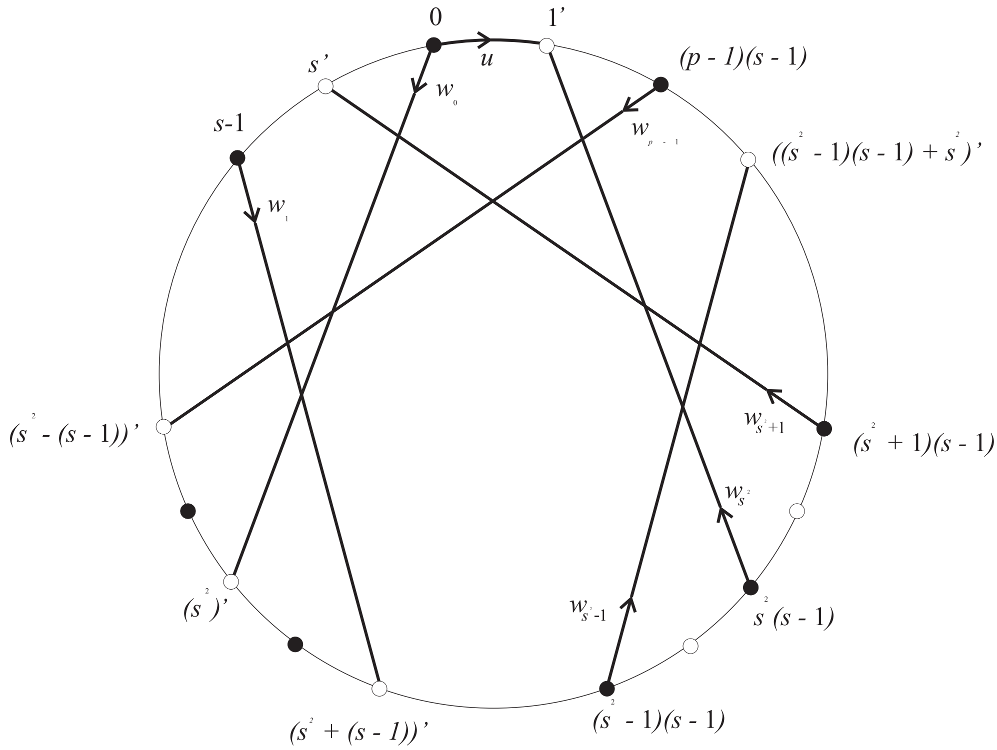

Figure 2: The voltage assignment in the case $p > 3$.

Since A/G acts semisymmetrically on Y, at least two automorphisms of Y lift: the cyclic rotation α, mapping according to the rule $y \mapsto y+s-1$, $y' \mapsto (y+s-1)'$, for $y \in \mathbb{Z}_p$, and the element σ of order 3 fixing 0 and $0'$, mapping according to the rule $y \mapsto sy$, $y' \mapsto (sy)'$. The requirement that α and σ lift imposes strong restrictions to the cotree voltages. Recall that by Proposition 2.2 an automorphism of Y lifts along $\wp$ if and only if it induces an automorphism of the voltage group G. Regarding G as a vector space over $\mathbb{Z}_p$ we may view $\mathrm{Aut}\, G$ as a group of linear transformations of G.

Considering the action of α and σ on the fundamental cycles we find the images of the voltages a and a_i under $\alpha^{\#}$ and $\sigma^{\#}$, thus restricting the possibilities for the voltage assignments a and a_i.

In our analysis we shall distinguish two different cases.

SUBCASE 1.1: $a \neq 0$.

Set

$$b_i = \begin{cases} a_i, & i \leq s^2 \\ a_i + a, & i > s^2 \end{cases}.$$

One can check that $\alpha^{\#}$ fixes a and maps $b_i \mapsto b_{i+1}$. We claim that $b_i = (i\lambda, 1)$, which forces

$$a_i = \begin{cases} (i\lambda, 1), & i \leq s^2 \\ (i\lambda - 1, 1), & i > s^2 \end{cases}. \tag{1}$$

Indeed, by the connectivity of the covering graph X, it follows that $G = \langle a, a_0, a_1, \ldots, a_{p-1} \rangle = \langle a, b_0, b_1, \ldots, b_{p-1} \rangle$. So b_i is nonzero for each $i \in \mathbb{Z}_p$. Moreover, if b_0 and a are linearly dependent, then $\langle b_0 \rangle = \langle a \rangle$. As $\alpha^{\#}$ maps b_i to b_{i+1}, we have by induction that $\langle b_i \rangle = \langle a \rangle$ for all $i \in \mathbb{Z}_p$, a contradiction since G is not cyclic. We can therefore choose $\{a, b_0\}$ as a basis of G. There is an automorphism of G mapping a to $(1,0)$ and b_0 to $(0,1)$. By Proposition 2.1 we may therefore assume $a = (1,0)$ and $b_0 = (0,1)$.

Let $b_1 = (\lambda, \mu)$, where $\lambda, \mu \in \mathbb{Z}_p$. By writing the cordinates as row vectors, the matrix of $\alpha^{\#}$ with respect to the above basis is

$$M_\alpha = \begin{bmatrix} 1 & 0 \\ \lambda & \mu \end{bmatrix}.$$

Clearly, $(\alpha^{\#})^p$ fixes a and each b_i. So $(\alpha^{\#})^p = \mathrm{id}$, forcing $\mu^p = 1$. But since $\mu \neq 0$ and p is a prime, we have $\mu = \mu^p$ and so $\mu = 1$. This implies $b_i = (i\lambda, 1)$, and (1) follows as claimed.

We now investigate how $\sigma^{\#}$ imposes further restrictions to our voltage assignment. As we shall see, $\lambda = 0$, implying that our covering graph X is uniquely determined. To this end we first describe the action of $\sigma^{\#}$ on the voltages of the "small" base cycles, determined by the chords. Explicitly, the cycle $\mathcal{C}_i$, $i \in \mathbb{Z}_p$, of length $2s + 2$ with initial vertex $i(s-1)$ is of the form

$$\mathcal{C}_i = C_{i,1}\, C'_{i,2}\, C_{i,3}\, C'_{i,4}\, \ldots\, C_{i,2s+1}\, C'_{i,2s+2}\, C_{i,1},$$

where

$$C_{i,1} = i(s-1), \quad C_{i,2} = i(s-1) + s^2,$$

and for $k = 1, 2, \ldots, s$

$$C_{i,2k+1} = (i + s - k + 1)(s - 1), \quad C_{i,2k+2} = (i + s - k + 1)(s - 1) + 1.$$

It can be seen that σ maps $\mathcal{C}_i$ to the cycle

$$\mathcal{C}_i^\sigma = D_{i,1}\, D'_{i,2}\, D_{i,3}\, D'_{i,4}\, \ldots\, D_{i,2s+1}\, D'_{i,2s+2}\, D_{i,1},$$

where

$$D_{i,1} = i(s - 1)s, \quad D_{i,2} = i(s - 1)s + 1,$$

and for $k = 1, 2, \ldots, s$

$$D_{i,2k+1} = (i + s - k + 1)(s - 1)s, \quad D_{i,2k+2} = (i + s - k + 1)(s - 1)s + s.$$

Note that $\text{vol}(\mathcal{C}_i) = b_i$ for each $i \in \mathbb{Z}_p$. To calculate the voltage of the mapped cycle $\mathcal{C}_i^\sigma$, observe that the chords of Y are distinguished from the edges on the natural hamiltonian cycle by the difference of their endvertices. The chordal arcs are of the form $y(y + s^2)'$ for $y \in \mathbb{Z}_p$ (see Figure 2). The cycle $\mathcal{C}_i^\sigma$ has exactly $s + 1$ chordal arcs, namely, $D'_{i,2k}D_{i,2k+1}$ for $k = 1, 2, \ldots, s$ and $D'_{i,2s+2}D_{i,1}$. Since the only arc in $\mathcal{C}_i^\sigma$ of the form $y(y + 1)'$ is $D_{i,1}D'_{i,2}$, it follows that among all base cycles $\mathcal{C}_i$, $i \in \mathbb{Z}_p$, the cycle $\mathcal{C}_0$ is the only one whose image under σ contains $01'$. By computation,

$$\text{vol}(\mathcal{C}_0^\sigma) = a - a_{s^2} - a_{s(s-1)} - a_{s(s-2)} - \ldots - a_s - a_0, \tag{2}$$

and, for $i \neq 0$,

$$\text{vol}(\mathcal{C}_i^\sigma) = -a_{s(s+i)} - a_{s(s+i-1)} - a_{s(s+i-2)} - \ldots - a_{s(i+1)} - a_{si}. \tag{3}$$

Similarly, the natural hamiltonian cycle $\mathcal{C}$, which has voltage a, is mapped to the cycle $\mathcal{C}^\sigma$ with

$$\text{vol}(\mathcal{C}^\sigma) = -a_{s(p-1)} - a_{s(p-2)} - \ldots - a_s - a_0 = -\sum_{i=0}^{p-1} a_i = s.$$

Combining this with (2), we have that the matrix of $\sigma^\#$ with respect to the basis $\{a, b_0\}$ is

$$M_\sigma = \begin{bmatrix} s & 0 \\ s\lambda/2 + 1 & s^2 \end{bmatrix}.$$

Combining this with a direct computation using (3), it follows that $\sigma^{\#}$ maps $b_1 = a_1$ to $(\lambda, 1)M_\sigma = (3s\lambda/2 + 1, s^2) = ((1 + s/2)\lambda + 1, s^2)$. Since $s \neq 1$ we get $\lambda = 0$. Therefore for $i \in \mathbb{Z}_p$

$$a_i = \begin{cases} (0,1), & i \leq s^2 \\ (-1,1), & i > s^2 \end{cases} . \tag{4}$$

We now show that the unique covering graph obtained by this voltage assignment is not semisymmetric. To this end we show that the reflection τ interchanging y with $(1-y)'$ lifts.

Obviously, the natural hamiltonian cycle $\mathcal{C}$ is reversed by this reflection, and so $\mathrm{vol}(\mathcal{C}^\tau) = -a$. As for the cycles $\mathcal{C}_i$, $i \in \mathbb{Z}_p$, observe that the chordal arc $C_{i,1}C'_{i,2}$ is mapped to the chordal arc $C'_{s^2-i,2}C_{s^2-i,1}$. Also, the arc $01'$ appears only on cycles $\mathcal{C}_i$ with $s^2 < i < p$. Moreover, the reflection τ preserves the set of cycles $\mathcal{C}_i$ containing $01'$ and the set of cycles $\mathcal{C}_i$ not containing $01'$. This implies that $\mathrm{vol}(\mathcal{C}_i^\tau) = -b_{s^2-i}$, for $i \in \mathbb{Z}_p$.

Finally, using Proposition 2.2, one can check that τ lifts, implying that the covering graph is not semisymmetric.

Subcase 1.2: $a = 0$.

We inherit the notation from Subcase 1.1. Since $\alpha^{\#}$ extends to an automorphism of G, either all a_i are zero or all a_i are nonzero. Because of the connectivity of the covering graph X the second possibility must occur. Furthermore, the voltages a_i, $i \in \mathbb{Z}_p$, are pairwise linearly independent. Suppose, on the contrary, that a_i and a_j are linearly dependent, for some $i < j$. Since α^{j-i} induces a cyclic permutation of order p on the indices of a_k, $k \in \mathbb{Z}_p$, all the voltages a_k are linearly dependent. This contradicts the fact that $G = \langle a_0, a_1, \ldots, a_{p-1}\rangle$ is not cyclic. It follows that all the subgroups $\langle a_i \rangle$, $i \in \mathbb{Z}_p$, are pairwise distinct. But there are exactly $p+1$ subspaces of dimension 1 in G, and so $\alpha^{\#}$ must fix the only remaining 1-subspace generated by some $b \in G$.

Now choose $\{b, a_0\}$ as a basis of G, and let the corresponding matrix of $\alpha^{\#}$ be

$$M_\alpha = \begin{bmatrix} \mu & 0 \\ \lambda & \nu \end{bmatrix},$$

for some $\lambda \in \mathbb{Z}_p$ and $\mu, \nu \in \mathbb{Z}_p^*$.

Since $(\alpha^{\#})^p = \mathrm{id}$, we have $M_\alpha^p = I$. It follows that $\mu^p = \nu^p = 1$ and so $\mu = \nu = 1$. It is easily seen that $a_i = (i\lambda, 1)$.

As in the previous subcase, a short computation involving the action of $\sigma^{\#}$ gives $\lambda = 0$. Therefore $a_i = (0,1)$ for $i \in \mathbb{Z}_p$, forcing $G \cong \mathbb{Z}_p$, a contradiction.

We conclude that there is no cubic semisymmetric graph of order $2p^3$ for $p > 3$.

CASE 2: $p = 3$.

By Lemma 3.1, a Sylow 3-subgroup P of A has a normal subgroup G isomorphic to $\mathbb{Z}_3^2$ and acting semiregularly on vertices and edges of X. A quotienting by the action of G results in a regular covering projection of X onto $Y = X/G \cong K_{3,3}$. Let the bipartition sets of Y be $\{0,2,4\}$ and $\{1,3,5\}$. Since G is normal in P and X is cubic, it follows that the group P projects to a subgroup of automorphisms of Y acting semisymmetrically on Y. Hence the two automorphisms $\varphi = (024)$ and $\psi = (135)$ do have a lift. Also, $\tau_1 = (01)(23)(45)$ and $\tau_2 = (01)(25)(34)$ do not lift, for otherwise the covering graph would be vertex-transitive.

In order to reconstruct this covering by voltages valued in G, let us choose a tree with trivial voltages as shown in Figure 3, and let a, b, c and d be the voltages of the remaining cotree arcs 32, 34, 25 and 45, respectively.

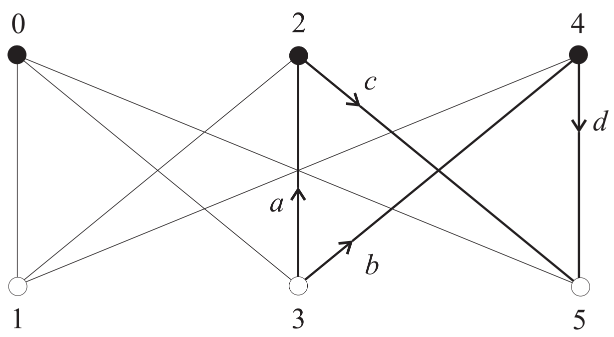

Figure 3: The voltage assignment in the case $p = 3$.

The requirement that φ and ψ lift imposes strong restrictions to the cotree voltages. In fact, we now show that such a covering projection must give rise to the Gray graph. The reader may verify, by checking the fundamental cycles 03210, 03410, 01250 and 01450 of the graph Y, that $\varphi^{\#}$, $\psi^{\#}$, $\tau_1^{\#}$ and $\tau_2^{\#}$ map the voltages a, b, c and d as follows (we use the additive notation for the operation in the group G regarded as a vector space over $\mathbb{Z}_3$.

	a	b	c	d
$\varphi^\#$	$-a+b$	$-a$	$-c+d$	$-c$
$\psi^\#$	$-a-c$	$-b-d$	a	b
$\tau_1^\#$	$-a$	c	b	$-d$
$\tau_2^\#$	d	$-b$	$-c$	a

Table 1: The mappings $\varphi^\#$, $\psi^\#$, $\tau_1^\#$ and $\tau_2^\#$.

By the connectivity of X we have $G = \langle a, b, c, d\rangle$. Observe from Table 1 that none of a, b, c, d is trivial. We shall distinguish two cases.

Suppose first that a and b are linearly dependent. It follows from the row of $\varphi^\#$ in Table 1 that $b = -a$. Then from the row of $\psi^\#$ we have $d = -c$. Now $G = \langle a, c\rangle$. There exists an automorphism of the voltage group G taking a and c to $(-1, 0)$ and $(0, -1)$, respectively. By Proposition 2.1 we may therefore assume $a = (-1, 0)$ and $c = (0, -1)$. Therefore the covering graph X, obtained from $Y = K_{3,3}$ for the case when a and b are linearly dependent, is unique. We now show that it is indeed semisymmetric, and in fact isomorphic to the Gray graph. By recalculating the cotree voltages relative to the spanning tree with edges 01, 12, 03, 25, 14, we get an equivalent voltage assignment where the arcs 23 and 34 receive voltage $(0, 1)$, whereas the arcs 05 and 54 receive voltage $(1, 0)$. It is known that such an assignment gives rise to the Gray graph (see [17]).

We now consider the case when a and b are linearly independent. Choose $\{a, b\}$ as a basis of G. From Table 1 we get that the matrix for $\varphi^\#$, with respect to the basis $\{a, b\}$, is

$$M_\varphi = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let us express c and d in terms of $\{a, b\}$. Regarding a, b, c and d just as symbols we have $(c, d) = (a, b)M$, where M is a 2×2 matrix over $\mathbb{Z}_3$. Then $(c, d)M_\varphi = (a, b)MM_\varphi$. Since $\varphi^\#$ is a linear transformation, $(c^{\varphi^\#}, d^{\varphi^\#}) = (a^{\varphi^\#}, b^{\varphi^\#})M = (a, b)M_\varphi M$. From the Table 1 we see that $(c^{\varphi^\#}, d^{\varphi^\#}) = (c, d)M_\varphi$, and hence $M_\varphi M = MM_\varphi$. It is easy to check that $M \in \{\pm I, \pm M_\varphi, \pm M_\varphi^2\}$.

We now consider how the automorphism $\psi^{\#}$ restricts the voltage assignments. Again from Table 1 we get that the matrix M_ψ for $\psi^{\#}$ with respect to the basis $\{a, b\}$ equals $-M - I$. We now have $(c^{\psi^{\#}}, d^{\psi^{\#}}) = (a^{\psi^{\#}}, b^{\psi^{\#}})M = (a, b)M_\psi M$. On the other hand, Table 1 also implies $(c^{\psi^{\#}}, d^{\psi^{\#}}) = (a, b)I$.

Consequently, $M_\psi M = I$, that is, $M^2 + M + I = 0$. This implies $M \in \{I, M_\varphi, M_\varphi^2\}$. One can check that if $M = M_\varphi$, then τ_2 lifts, and that if $M = M_\varphi^2$, then τ_1 lifts. Thus, in these two cases, the covering graph is not semisymmetric. Therefore, $M = I$. Then $a = c$ and $b = d$.

There exists an automorphism of the voltage group G taking a and b to $(1, 0)$ and $(0, -1)$, respectively. By Proposition 2.1 we may assume $a = (1, 0)$ and $b = (0, -1)$. As above, recalculating the cotree voltages relative to the spanning tree with edges 01, 03, 05, 14 and 52, we get that the covering graph is again isomorphic to the Gray graph.

We conclude that X is unique and isomorphic to the Gray graph, completing the proof of Theorem 1.1. ■

We remark that it is not hard to see that the above covering graph of Y can be described in such a way that it makes its isomorphism with the Gray graph, as defined in [1], self-evident. Just substitute the vertices 0 and 2 by three copies of $K_{3,3}$, regard the vertices 1, 3 and 5 as inserted edge-vertices, and take the vertex 4 to be the nine vertices joining the edge-vertices.

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