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# THE EXACT ISOPERIMETRIC INEQUALITY FOR TERNARY AND QUATERNARY CUBES

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# The Exact Isoperimetric Inequality for Ternary and Quaternary Cubes

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**Abstract**

We extend the well-known edge-isoperimetric inequality of Harper, Bernstein and Hart to ternary and quaternary cubes. More generally, let $Q$ be the graph with vertex set $V = \prod_{i=1}^{n}[k_i]$ in which $x \in V$ is joined to $y \in V$ if for some $i$ we have $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. If $k_1 \geq \cdots \geq k_n$ and $k_2 \leq 4$, we prove that for any $0 \leq m \leq |V|$, no $m$-set of vertices of $Q$ is joined to the rest of $Q$ by fewer edges than the set of the first $m$ vertices of $Q$ in the lexicographic ordering on $V$.

# 1 Introduction

Given a graph $G$ and an integer $m$, at least how many edges must join a set of $m$ vertices of $G$ to the rest of the graph? For a set $A$ of vertices of a graph $G$, write $\partial A$ for the *edge-boundary* of $A$:

$$\partial A = \{uv \in E(G) \colon u \in A, v \notin A\}.$$

An inequality of the form

$$|\partial A| \geq f(m) \quad \text{for all } A \subseteq V(G) \text{ with } |A| = m \tag{1}$$

is called an *(edge-)isoperimetric inequality* for the graph $G$. Harper [5], Lindsey [8], Bernstein [1] and Hart [6] determined the best possible isoperimetric inequality for the discrete (binary) cube. Namely, let $Q^n$ be the graph on the vertex set $\{0,1\}^n$ in which $x \in \{0,1\}^n$ and $y \in \{0,1\}^n$ are joined if they differ in precisely one coordinate. The *lexicographic* (or *binary*) ordering on $Q^n$ is the one in which $(x_i)_{i=1}^n$ precedes $(y_i)_{i=1}^n$ if for some $i$ we have $x_i < y_i$ and $x_j = y_j$ for all $j < i$. Harper, Lindsey, Bernstein and Hart proved that no $m$-set of vertices of $Q^n$ has smaller edge-boundary than the set of the first $m$ vertices of $Q^n$ in the lexicographic ordering on $Q^n$ (see Bollobás and Leader [2] for a very simple proof of this theorem).

The cube $Q^n$ is an example of a discrete *grid*, namely the graph with vertex set $\prod_{i=1}^n [k_i]$ in which two vertices $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ are joined if for some $i$ we have $|x_i - y_i| = 1$ and $x_j = y_j$ for $j \neq i$. Bollobás and Leader [3] considered the edge-isoperimetric problem for the discrete grid $[k]^n$. They obtained a very sharp isoperimetric inequality which for quite a few values of $k$, $n$ and $m$ is exact. The exact isoperimetric inequality for the grid $\prod_{i=1}^n [k_i]$, or even $[k]^n$, however, remains unknown for all grids other than $[2]^n$.

In this paper we obtain the exact isoperimetric inequality for the grids $[3]^n$ and $[4]^n$ (and more generally, the grids $[4]^a \times [3]^b \times [2]^c$ and $[\ell] \times [4]^a \times [3]^b \times [2]^c$ with $\ell \geq 5$). The lexicographic ordering $\preceq$ on the grid $\prod_{i=1}^n [k_i]$ is defined as for the cube: $(x_i)_{i=1}^n$ precedes $(y_i)_{i=1}^n$ if for some $i$ we have $x_i < y_i$ and $x_j = y_j$ for $j < i$. We prove the following theorem.

**Theorem 1** *Suppose that $n \geq 2$, $k_1 \geq \cdots \geq k_n \geq 2$ and $k_2 \leq 4$. Let $A \subseteq \prod_{i=1}^n [k_i]$ be an arbitrary set of vertices and let $I$ be the set of the first $|A|$ elements of $\prod_{i=1}^n [k_i]$ in the lexicographic ordering. Then we have $|\partial A| \geq |\partial I|$.*

The conditions on the $k_i$ in the theorem are all essential; the conclusions of the theorem do not hold for any grid $\prod_{i=1}^n [k_i]$ in which at least two $k_i$'s are $\geq 5$. This is shown in Section 3.

For edge-isoperimetric inequalities on some other families of graphs see Lindsey [8], Clements [4], Kleitman, Krieger and Rothschild [7] and Bollobás and Leader [3]. See also Mohar [9].

# 2 Proof of Theorem 1

Given a finite linearly ordered set $(S, \leq)$ and $a, b \in S$, we shall use the following notation: $[a, b]_\leq = \{x \in S : a \leq x \leq b\}$, and $[b]_\leq = \{x \in S : x \leq b\}$. An *initial segment* of $S$ is a set of the form $[b]_\leq$ for some $b \in S$. For $0 \leq m \leq |S|$, $[m]_\leq$ shall denote the initial segment of $S$ of order $m$. The *immediate successor* of $a \in S \setminus \max S$ shall be denoted $a^+$. If $a$ and $b$ are integers, we shall write $[a, b] = \{a, a+1, \ldots, b\}$ and $[b] = [1, b]$.

If $A$ is a subset of $\prod_{i=1}^n [k_i]$, $1 \leq i \leq n$ and $1 \leq j \leq k_i$, we define the *$i$-section $A_{i|j}$ of $A$ at $j$* as follows

$$\begin{aligned} A_{i|j} &= \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in [k_1] \times \cdots \times [k_{i-1}] \times [k_{i+1}] \times \cdots \times [k_n]: \\ &\qquad (a_1, \ldots, a_{i-1}, j, a_{i+1}, \ldots, a_n) \in A\}. \end{aligned}$$

For $1 \leq i \leq n$ and $A \subseteq \prod_{i=1}^n [k_i]$, the *$i$-compression* $C_i(A)$ of $A$ is defined by prescribing its $i$-sections: $C_i(A)_{i|j} = [|A_{i|j}|]_{\preceq}$. A set $A \subseteq \prod_{i=1}^n [k_i]$ is *$i$-compressed* if $C_i(A) = A$ and it is *compressed* if for all $1 \leq i \leq n$ we have $C_i(A) = A$.

We begin by characterizing compressed sets in grids of dimension $\geq 3$ and by proving Theorem 1 for compressed sets.

**Lemma 2** *Suppose that $n \geq 3$ and $k_1, \ldots, k_n \geq 2$. Then a set $A \subseteq \prod_{i=1}^n [k_i]$ is compressed if, and only if, $A$ is either an initial segment of $\prod_{i=1}^n [k_i]$ in the lexicographic ordering, or, for some integers $j$, $x$ and $y$ with $2 \leq j \leq k_1$ and $1 \leq x \leq y \leq k_n - 1$ we have*

$$A = [(j-1, k_2, \ldots, k_{n-1}, y)]_{\preceq} \cup [(j, 1, \ldots, 1), (j, 1, \ldots, 1, x)]_{\preceq}. \tag{2}$$

*Proof.* It is routine to check that initial segments and sets of the form (2) are indeed compressed (furthermore, we shall not use this fact in the rest of the proof).

Suppose then that $n \geq 3$, $k_1, \ldots, k_n \geq 2$ and that $A \subseteq \prod_{i=1}^n [k_i]$ is compressed but is not an initial segment of $\prod_{i=1}^n [k_i]$. To complete the proof, we must show that (2) holds for some $j$, $x$ and $y$. Since $A$ is not an initial segment, there exists a vertex $a \in \prod_{i=1}^n [k_i]$ such that $a \notin A$ and $a^+ \in A$. Since $A$ is 1-compressed, $a$ and $a^+$ must differ in the first coordinate and this is only possible if $a = (j-1, k_2, \ldots, k_n)$ and $a^+ = (j, 1, \ldots, 1)$ for some $j$ with $2 \leq j \leq k_1$. Since $a^+ \in A$ and $A$ is $n$-compressed, we must have $(j-1, k_2, \ldots, k_{n-1}, 1) \in A$. Let $y \in [k_n]$ be maximal such that $(j-1, k_2, \ldots, k_{n-1}, y) \in A$; clearly we have $1 \leq y \leq k_n - 1$. Since $(j-1, k_2, \ldots, k_{n-1}, y) \in A$ and $A$ is 2-compressed, we have $[j-2] \times \{(k_2, \ldots, k_n)\} \subseteq A$ and so, since $A$ is 1-compressed, $[j-2] \times \prod_{i=2}^n [k_i] \subseteq A$. Since $(j-1, k_2, \ldots, k_{n-1}, y) \in A$, $(j-1, k_2, \ldots, k_{n-1}, y+1) \notin A$ and $A$ is 1-compressed, we have $A_{1|j-1} = [(k_2, \ldots, k_{n-1}, y)]_{\preceq}$.

Summarizing, we see that

$$A \cap \left( [j-1] \times \prod_{i=2}^{n} [k_i] \right) = [(j-1, k_2, \ldots, k_{n-1}, y)]_{\preceq}. \tag{3}$$

Since $(j-1, k_2, \ldots, k_{n-1}, y+1) \notin A$ and $A$ is $n$-compressed, we have $(j, 1, \ldots, 1, y+1) \notin A$. Let $x \in [k_n]$ be maximal such that $(j, 1, \ldots, 1, x) \in A$; clearly we have $1 \leq x \leq y$. Since $(j, 1, \ldots, 1, x+1) \notin A$ and $A$ is 2-compressed, we have $([j+1, k_1] \times \{(1, \ldots, 1)\}) \cap A = \emptyset$ and so, since $A$ is 1-compressed, $A \cap ([j+1, k_1] \times \prod_{i=2}^{n} [k_i]) = \emptyset$. Since $(j, 1, \ldots, 1, x) \in A$, $(j, 1, \ldots, 1, x+1) \notin A$ and $A$ is 1-compressed, we have $A_{1|j} = [(1, \ldots, 1, x)]_{\preceq}$. Summarizing, we see that

$$A \cap \left( [j, k_1] \times \prod_{i=2}^{n} [k_i] \right) = [(j, 1, \ldots, 1), (j, 1, \ldots, 1, x)]_{\preceq}. \tag{4}$$

Combining (3) and (4) we obtain (2). Since $2 \leq j \leq k_1$ and $1 \leq x \leq y \leq k_n - 1$, the lemma is thus proved. □

**Remark.** Curiously enough, the proof of Lemma 2 reveals that a set $A \subseteq \prod_{i=1}^{n} [k_i]$ is compressed if, and only if, it is 1-, 2- and $n$-compressed.

**Lemma 3** *Let $n \geq 3$, $k_1, \ldots, k_n \geq 2$ be integers and let $A \subseteq \prod_{i=1}^{n} [k_i]$ be a compressed set. Let $I$ be the set of the first $|A|$ elements of $\prod_{i=1}^{n} [k_i]$ in the lexicographic ordering. Then we have $|\partial A| \geq |\partial I|$.*

*Proof.* By Lemma 2 we may assume that for some $2 \leq j \leq k_1$ and $1 \leq x \leq y \leq k_n - 1$ we have

$$A = [(j-1, k_2, \ldots, k_{n-1}, y)]_{\preceq} \cup [(j, 1, \ldots, 1), (j, 1, \ldots, 1, x)]_{\preceq}.$$

On a little reflection we see that we have

$$|\partial A| = \begin{cases} \prod_{i=2}^{n} k_i - (k_n - y) + (k_n - y + x)(n-2) + 2 & \text{if } j = 2, \\ \prod_{i=2}^{n} k_i + (k_n - y + x)(n-2) + 2 & \text{if } 3 \leq j \leq k_1 - 1, \\ \prod_{i=2}^{n} k_i - x + (k_n - y + x)(n-2) + 2 & \text{if } j = k_1. \end{cases}$$

*Case (i)* $x + y < k_n$. Then we have $I = [(j-1, k_2, \ldots, k_{n-1}, x+y)]_{\preceq}$ and so

$$|\partial I| = \begin{cases} \prod_{i=2}^{n} k_i - (k_n - (x+y)) + (k_n - (x+y))(n-2) + 1 & \text{if } j = 2, \\ \prod_{i=2}^{n} k_i + (k_n - (x+y))(n-2) + 1 & \text{if } 3 \leq j \leq k_1. \end{cases}$$

In either of the cases $j = 2$, $3 \leq j \leq k_1 - 1$ and $j = k_1$ we see that $|\partial A| > |\partial I|$.

*Case (ii)* $x + y = k_n$. In this case we have $I = [j-1] \times \prod_{i=2}^{n}[k_i]$ and so $|\partial I| = \prod_{i=2}^{n} k_i < |\partial A|$.

*Case (iii)* $x + y > k_n$. In this case we have $I = [(j, 1, \ldots, 1, x+y-k_n)]_{\preceq}$ and so

$$|\partial I| = \begin{cases} \prod_{i=2}^{n} k_i + (x+y-k_n)(n-2) + 1 & \text{if } 2 \leq j \leq k_1 - 1, \\ \prod_{i=2}^{n} k_i - (x+y-k_n) + (x+y-k_n)(n-2) + 1 & \text{if } j = k_1. \end{cases}$$

In either of the cases $2 \leq j \leq k_1 - 1$ and $j = k_1$ we see that $|\partial A| > |\partial I|$. □

**Remark.** It is clear from the proof of Lemma 3 that equality holds in the lemma if, and only if, $A$ is itself an initial segment of $\prod_{i=1}^{n}[k_i]$ in the lexicographic order.

*Proof of Theorem 1.* We proceed by induction on $n$. The case $n = 2$ can easily be checked by hand (observe that it is sufficient to prove the theorem for compressed sets $A$; see also [10] for another proof). So suppose that $n \geq 3$.

We claim that for any $A \subseteq \prod_{i=1}^{n}[k_i]$ and any $1 \leq i \leq n$ we have

$$|\partial A| \geq |\partial C_i(A)|. \tag{5}$$

To prove the claim, observe that

$$|\partial A| = \sum_{j=1}^{k_i} |\partial A_{i|j}| + \sum_{j=2}^{k_i} |A_{i|j} \triangle A_{i|j-1}|,$$

and, similarly for $B = C_i(A)$,

$$|\partial B| = \sum_{j=1}^{k_i} |\partial B_{i|j}| + \sum_{j=2}^{k_i} |B_{i|j} \triangle B_{i|j-1}|.$$

By induction hypothesis we have, for all $1 \leq j \leq k_i$, $|\partial A_{i|j}| \geq |\partial B_{i|j}|$. Also, since the $(B_{i|j})_{j=1}^{k_i}$ are nested, we have $|A_{i|j} \triangle A_{i|j-1}| \geq |B_{i|j} \triangle B_{i|j-1}|$ for all $2 \leq j \leq k_i$. Thus, we have $|\partial A| \geq |\partial B|$ and (5) is proved.

The grid $\prod_{i=1}^{n}[k_i]$ is linearly ordered (lexicographically). The *binary* order on $\mathcal{P}(\prod_{i=1}^{n}[k_i])$ is the one in which $P$ precedes $Q$ if $\max(P \triangle Q) \in Q$. Observe that for any $S \subseteq \prod_{i=1}^{n}[k_i]$ and any $1 \leq i \leq n$, $C_i(S)$ precedes $S$ in the binary order on $\mathcal{P}(\prod_{i=1}^{n}[k_i])$. Therefore there exists a finite sequence of sets $A_1 = A$, $A_2 = C_{i_1}(A_1)$, $\ldots$, $A_\ell = C_{i_{\ell-1}}(A_{\ell-1})$ such that $A_\ell$ is compressed. By (5) we have

$$|\partial A| \geq |\partial A_2| \geq \cdots \geq |\partial A_\ell|.$$

Since compressions preserve cardinality, we have $|I| = |A| = |A_2| = \cdots = |A_\ell|$ and so by Lemma 3 we have $|\partial A_\ell| \geq |\partial I|$. Therefore, $|\partial A| \geq |\partial I|$ and so the theorem is proved. $\square$

## 3 Other Grids

We stated in the introduction that the conditions on the $k_i$ in Theorem 1 are all essential, namely that the theorem does not hold for any other choice of $k_i$ (we are excluding the trivial case $n = 1$ from our consideration; of course Theorem 1 holds for 1-dimensional grids; we are also excluding the (uninteresting) possibility of some $k_i$ being 1). We now prove this claim.

First suppose that $k_1 \geq \cdots \geq k_n \geq 2$ and $k_2 \geq 5$. Let $A \subseteq \prod_{i=1}^{n}[k_i]$ be the set $[2] \times [2] \times \prod_{i=3}^{n}[k_i]$ and let $I$ be the set of the first $|A|$ elements of $\prod_{i=1}^{n}[k_i]$ in the lexicographic order. It is easily seen that $I = [1] \times [4] \times \prod_{i=3}^{n}[k_i]$ and so

$$|\partial A| = 4\prod_{i=3}^{n} k_i < 5\prod_{i=3}^{n} k_i = |\partial I|.$$

The assertion of Theorem 1 therefore does not hold if $k_1 \geq \cdots \geq k_n \geq 2$ and $k_2 \geq 5$.

Perhaps a version of Theorem 1 holds for a wider class of $k_i$ if we order the $k_i$ in some other way? It is easy to see that this is not the case. Suppose that $k_1, \ldots, k_n \geq 2$ and for some $j \in [n-1]$ we have $k_j < k_{j+1}$. Let $A \subseteq \prod_{i=1}^n [k_i]$ be the set $[1]^{j-1} \times [k_j] \times [1] \times \prod_{i=j+2}^n [k_i]$ and let $I$ be the set of the first $|A|$ elements of $\prod_{i=1}^n [k_i]$ in the lexicographic order. It is easily seen that $I = [1]^j \times [k_j] \times \prod_{i=j+2}^n [k_i]$ and so

$$|\partial A| = j|A| < j|A| + \frac{|A|}{k_j} = |\partial I|.$$

The assertion of Theorem 1 therefore also fails to hold if the $k_i$ are not ordered in a decreasing order.

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