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PROPER HOLOMORPHIC DISCS IN $\mathbb{C}^2$

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&1. Results.

Let U be the open unit disc in $\mathbb{C}$ and $T = bU$ the unit circle. The second author proved in [Glo] that for any point p in a Stein manifold X of dimension at least two there exists a proper holomorphic map $f\colon U \to X$ such that $f(0) = p$. We shall call such maps proper holomorphic discs in X. For smoothly bounded pseudoconvex domains in $\mathbb{C}^n$ this was proved earlier in [FG]; the essential addition in [Glo] was crossing the critical points of a strongly plurisubharmonic exhaustion function on X. The methods developed in [FG] and [Glo] actually show the following:

1.1 Theorem. *Let X be a Stein manifold with $\dim X \geq 2$ and $\rho\colon X \to \mathbb{R}$ a smooth exhaustion function which is strongly plurisubharmonic on $\{\rho > M\}$ for some $M \in \mathbb{R}$. Let $h\colon \overline{U} \to X$ be a continuous map which is holomorphic on U such that $\rho(h(e^{i\theta})) > M$ for each $e^{i\theta} \in T$. Let d be a metric on X. For any pair of numbers $0 < r < 1$, $\epsilon > 0$, and for any finite set $A \subset U$ there exists a proper holomorphic map $f\colon U \to X$ satisfying*

(i) $\lim_{|\zeta|\to 1} \rho(f(\zeta)) = +\infty$,
(ii) $\rho(f(\zeta)) > \rho(h(\zeta)) - \epsilon$ *for* $\zeta \in U$,
(iii) $d\big(f(\zeta), h(\zeta)\big) < \epsilon$ *for* $|\zeta| \leq r$, *and*
(iv) f *matches* h *at each point of* A.

We are interested to what extent theorem 1.1 holds if $\rho\colon X \to \mathbb{R}$ is a (strongly) plurisubharmonic function whose sub-level sets are not necessarily relatively compact. Besides its intrinsic interest, we are motivated by the question whether it is possible to avoid any closed complex hypersurface L in a Stein manifold X by proper holomorphic discs in X. Any such L is the zero set of a smooth plurisubharmonic function $\rho\colon X \to \mathbb{R}_+$ which is strongly plurisubharmonic on $\{\rho > 0\} = X \backslash L$, hence a positive answer to the first question gives such discs in $X \backslash L$. In this paper we obtain positive results in certain model situations in $\mathbb{C}^2$, beginning with the following.

1.2 Theorem. *For each $c < 1$ and $M \in \mathbb{R}$ the conclusion of theorem 1.1 holds with $X = \mathbb{C}^2$ and the function $\rho_c\colon \mathbb{C}^2 \to \mathbb{R}$,*

$$\rho_c(z_1, z_2) = \rho_c(x_1 + iy_1, x_2 + iy_2) = x_1^2 + x_2^2 - c(y_1^2 + y_2^2). \tag{1.1}$$

If $c \geq 1$ then for any proper holomorphic map $f\colon U \to \mathbb{C}^2$ the function $\rho_c \circ f$ is unbounded from below on U.

Note that ρ_c is strongly plurisubharmonic if $c < 1$, strongly plurisuperharmonic if $c > 1$, and $\rho_1(z_1, z_2) = \Re(z_1^2 + z_2^2)$ is pluriharmonic.

The second statement in theorem 1.2 (when $c \geq 1$) can be seen by applying part theorem 1.5 (d) below to the function $g = f_1^2 + f_2^2$: its range at each boundary point $e^{i\theta} \in T$ omits at most a polar set in $\mathbb{C}$ and hence its real part $\Re g = \rho_1(f_1, f_2)$ cannot be bounded from below. Since $\rho_c \leq \rho_1$ for $c \geq 1$, the same is true for $\rho_c \circ f$. The first part of theorem 1.2 (for $c < 1$) is proved in section 3 below.

When $c > 0$, ρ_c is not an exhaustion function on $\mathbb{C}^2$. For $0 < c < 1$ theorem 1.2 gives examples of proper holomorphic maps $f: U \to \mathbb{C}^2$ with images $f(U)$ contained in the real cone $\Gamma_c = \{\rho_c > 0\}$ with axis $\mathbb{R}^2 = \{y = 0\}$. Moreover, when $c > 1$ we can apply theorem 1.2 with $-\rho_c(z)/c = y_1^2 + y_2^2 - \frac{1}{c}(x_1^2 + x_2^2)$ to obtain a proper holomorphic map $f: U \to \mathbb{C}^2$ whose image avoids Γ_c. This gives proper holomorphic discs in $\mathbb{C}^2$ avoiding relatively large real cones. On the other hand, no proper holomorphic disc (in fact, no transcendental complex curve) in $\mathbb{C}^2$ can avoid a nonempty open complex cone (see theorem 2 in [SW] and theorem 1.5 below).

Our next result is that there exist proper holomorphic discs in $\mathbb{C}^2$ avoiding a pair of intersecting complex lines.

1.3 Theorem. *There exists a proper holomorphic map $f = (f_1, f_2): U \to \mathbb{C}^2$ whose image $f(U)$ is contained in $(\mathbf{C}^*)^2 = \mathbb{C}^2 \backslash \{zw = 0\}$.*

Writing $f: U \to (\mathbf{C}^*)^2$ as $f = \left(e^{g_1}, e^{g_2}\right) = \left(e^{u_1 + iv_1}, e^{u_2 + iv_2}\right)$, we have $|f|^2 = |f_1|^2 + |f_2|^2 = e^{2u_1} + e^{2u_2}$, and f is proper as a map into $\mathbb{C}^2$ if and only if $\max\{u_1, u_2\}$ tends to $+\infty$ at the boundary of U. Thus theorem 1.3 is equivalent to

1.4 Theorem. *There exists a pair of harmonic functions u_1, u_2 on the disc U such that*

$$\lim_{|\zeta| \to 1} \max\{u_1(\zeta), u_2(\zeta)\} = +\infty.$$

Theorem 1.3 is a special case of theorem 4.1 in section 4 below. A different proof of theorem 1.4 was shown to us by J.-P. Rosay (private communication).

It would be interesting to know whether proper discs in $\mathbb{C}^2$ can avoid any given finite collection of complex lines. Part (d) in theorem 1.5 shows that such a disc cannot avoid a non-polar set of complex lines through the origin (or parallel complex lines) in $\mathbb{C}^2$; the same holds if we replace the disc by any transcendental complex curve (Sibony and Wong [SW], Theorem 2). H. Alexander [Ale] proved in 1975 that for parallel lines in $\mathbb{C}^2$ this is the only obstruction: *If $E \subset \mathbb{C}$ is a closed polar set containing at least two points, there exists a proper holomorphic map $f = (f_1, f_2): U \to \mathbb{C}^2$ such that $f_1: U \to \mathbb{C} \backslash E$ is a universal covering map of the disc onto $\mathbb{C} \backslash E$.* We don't know whether an analogue of Alexander's result holds for complex lines through the origin.

In the remainder of this section we discuss the boundary behavior of proper holomorphic maps $f = (f_1, f_2): U \to \mathbb{C}^2$ at the circle $T = \{|\zeta| = 1\}$. We must recall some basic notions from the theory of cluster sets of meromorphic functions on the disc; we refer to Chapter 8 in the monograph [CL] (see also section 5 below).

Let g be a meromorphic function on U. A point $e^{i\theta} \in T$ at which the (unrestricted) cluster set of g equals $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called a **Weierstrass point** of g. If the (restricted) cluster set of g at $e^{i\theta}$ within each conical region in U with vertex $e^{i\theta}$ equals $\overline{\mathbb{C}}$ then $e^{i\theta}$ is called a **Plessner point** of g. A point $e^{i\theta}$ at which g has a non-tangential limit (a limit as $\zeta \to e^{i\theta}$ within any cone in U with vertex $e^{i\theta}$) is called a **Fatou point** of g, and the set of all Fatou point is the **Fatou set** of g. The **range** of g at $e^{i\theta}$, denoted $R(g, e^{i\theta})$, consists of all $\alpha \in \overline{\mathbb{C}}$ such that $g(\zeta_j) = \alpha$ for points in a sequence $\zeta_j \in U$ with $\lim_{j\to\infty} \zeta_j = e^{i\theta}$.

1.5 Theorem. *Let $f = (f_1, f_2): U \to \mathbb{C}^2$ be a proper holomorphic map of the disc to $\mathbb{C}^2$. Let P, Q be nonconstant holomorphic polynomials on $\mathbb{C}^2$ whose leading order homogeneous parts have no common divisor. Denote by g any of the following (meromorphic) functions: (i) f_1 or f_2, (ii) f_1/f_2, (iii) $P(f_1, f_2)$, (iv) $P(f_1, f_2)/Q(f_1, f_2)$. Then*
(a) the Fatou set of g has Lebesgue measure zero in T,
(b) every point of T is a Weierstrass point of g,
(c) almost every point of T is a Plessner point of g, and
(d) for every $e^{i\theta} \in T$ the set $\mathbb{C} \backslash R(g, e^{i\theta})$ is polar.

Theorem 1.5 is proved in section 5. Part (d) can be interpreted as a result on polynomial hulls as follows. We define the polynomial hull $\widehat{K}$ of an arbitrary subset $K \subset \mathbb{C}^n$ as the intersection of all closed set in $\mathbb{C}^n$ of the form $\{\Re P \le 0\}$ containing K, where P is a holomorphic polynomial. For compact sets this coincides with the usual definition of the polynomial hull. Clearly $\widehat{K}$ is contained in the closed convex hull of K. Theorem 1.5 (d) immediately implies

1.6 Corollary. *If $f: U \to \mathbb{C}^2$ is a proper holomorphic map then for each open set $D \subset \mathbb{C}$ intersecting T the polynomial hull of $f(U \cap D)$ equals $\mathbb{C}^2$ (and hence its closed convex hull also equals $\mathbb{C}^2$).*

Theorem 1.5 does not generalize directly to proper maps $f: U \to \mathbb{C}^n$ for $n > 2$. Namely, if $(f_1, f_2): U \to \mathbb{C}^2$ is proper holomorphic and if f_3 is any holomorphic function on U then $(f_1, f_2, f_3): U \to \mathbb{C}^3$ is also proper holomorphic; thus the addition of the third component need not enlarge the cluster set at any boundary point.

In the Appendix we comment on the proof of theorem 1.1 in [Glo]. Let $\rho: X \to \mathbb{R}$ be a strongly plurisubharmonic Morse exhaustion function on a Stein manifold X of dimension ≥ 2. We show that one can push the boundary of an analytic disc in X over a critical level of ρ by using the gradient flow of ρ. This creates a non-holomorphic contribution which can be cancelled off during a later stage of the lifting procedure (this was the crucial observation in [Glo]).

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&2. Lifting holomorphic discs.

In this section we describe a general method for lifting the boundary of an analytic disc in $\mathbb{C}^n$ to a higher level set of a strongly plurisubharmonic function $\rho\colon \mathbb{C}^n \to \mathbb{R}$. This method was developed in [FG], but for our present needs we need more precise estimates for the amount of possible lifting at each step of the process.

2.1 Proposition. *Let $\lambda\colon T \times \overline{U} \to \mathbb{C}^n$ be a continuous map such that for each $\zeta \in T$ the map $\lambda_\zeta = \lambda(\zeta, \cdot)\colon \overline{U} \to \mathbb{C}^n$ is holomorphic in U and $\lambda_\zeta(0) = 0$. Given numbers $\epsilon > 0$ and $0 < r < 1$, there exists a holomorphic polynomial map $h\colon \mathbb{C} \to \mathbb{C}^n$ satisfying*

(i) $\operatorname{dist}\big(h(\zeta), \lambda_\zeta(T)\big) < \epsilon \quad (\zeta \in T)$,

(ii) $\operatorname{dist}\big(h(t\zeta), \lambda_\zeta(\overline{U})\big) < \epsilon \quad (\zeta \in T,\ r \le t \le 1)$, *and*

(iii) $|h(\zeta)| < \epsilon \ \ (|\zeta| \le r)$.

Proof. It suffices to show that λ can be approximated uniformly on $T \times \overline{U}$ by maps of the form

$$\tilde\lambda(\zeta, w) = \frac{w}{\zeta^M} \sum_{j=1}^{N} A_j(\zeta) w^{j-1}, \tag{2.1}$$

where the A_j's are holomorphic polynomials and M, N are positive integers. The polynomial map

$$h(\zeta) = \tilde\lambda(\zeta, \zeta^K) = \zeta^{K-M} \sum_{j=1}^{N} A_j(\zeta) \zeta^{(j-1)K}$$

then satisfies proposition 2.1 provided that the approximation of λ by $\tilde\lambda$ is sufficiently close and the integer $K \ge M$ is chosen sufficiently large.

We begin by replacing λ by $(\zeta, w) \mapsto \lambda(\zeta, sw)$ for a suitable $s < 1$ sufficiently close to 1. Denoting the new map again by λ we may thus assume that λ_ζ is holomorphic in a larger disc $|w| < 1/s$ for each $\zeta \in T$. We expand λ in Taylor series with respect to w and approximate it uniformly on $bU \times T$ by a Taylor polynomial $\lambda_N(\zeta, w) = \sum_{j=1}^{N} a_j(\zeta) w^j$ with continuous coefficients $a_j\colon T \to \mathbb{C}^n$. (The coefficient a_0 is zero since $\lambda(\zeta, 0) = 0$.) Finally we approximate each a_j uniformly on T by a map $A_j(\zeta)/\zeta^M$ for some holomorphic polynomial A_j and some integer N which can be chosen to be independent of j. This gives the desired approximation of λ by a map of the form (2.1). ♠

2.2 Corollary. *Let $g_0: \overline{U} \to \mathbb{C}^n$ be a continuous map that is holomorphic in U and let λ be as in proposition 2.1. Suppose that $\rho: \mathbb{C}^n \to \mathbb{R}$ is a real continuous function such that for some constants $C_0 < C_1$ and $0 < r < 1$ we have*

(a) $\rho\big(g_0(\zeta) + \lambda(\zeta, w)\big) = C_1 \quad (\zeta \in T,\ w \in T)$,

(b) $\rho\big(g_0(\zeta) + \lambda(\zeta, w)\big) > C_0 \quad (\zeta \in T,\ w \in \overline{U})$, *and*

(c) $\rho\big(g_0(\zeta)\big) > C_0 \quad (r \le |\zeta| \le 1)$.

Then for each $\epsilon > 0$ there exists a holomorphic polynomial map $g: \mathbb{C} \to \mathbb{C}^n$ satisfying

(i) $\big|\rho(g(\zeta)) - C_1\big| < \epsilon \quad (\zeta \in T)$,

(ii) $\rho(g(\zeta)) > C_0 \quad (r \le |\zeta| \le 1)$, *and*

(iii $\big|g(\zeta) - g_0(\zeta)\big| < \epsilon \quad (|\zeta| \le r)$.

Proof. Take $g(\zeta) = \tilde{g}_0(\zeta) + h(\zeta)$, where $\tilde{g}_0$ is a polynomial approximation of g_0 and h is a suitably chosen map provided by proposition 2.1. ♠

Assume now that $\rho: \mathbb{C}^n \to \mathbb{R}$ is a function of class $\mathcal{C}^2$. For each fixed z we write

$$\rho_z(w) = \rho(z + w) - \rho(z) = \Re Q_z(w) + \mathcal{L}_z(w) + o(|w|^2), \tag{2.2}$$

where

$$Q_z(w) = 2\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(z) w_j w_k$$

$$\mathcal{L}_z(w) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k$$

(Q_z is the *Levi polynomial* and $\mathcal{L}_z$ is the *Levi form* of ρ at z). The set

$$\Lambda_z = \{w \in \mathbb{C}^n : Q_z(w) = 0\} \tag{2.3}$$

is a quadratic complex hypersurface in $\mathbb{C}^n$ and we have $\rho_z(w) = \mathcal{L}_\rho(z; w) + o(|w|^2)$ for $w \in \Lambda_z$. For $c > 0$ we denote by $B(z; c)$ the connected component of the sublevel set $\{w \in \Lambda_z : \rho_z(w) < c\}$ which contains the point $0 \in \Lambda_z$. If ρ is strongly plurisubharmonic near z (i.e., its Levi form $\mathcal{L}_z$ at z is positive definite) and if $\partial\rho(z) \ne 0$ (so that the hypersurface Λ_z is smooth near 0), then for all sufficiently small $c > 0$ the set $B(z; c)$ is diffeomorphic to the real $(2n-2)$-dimensional ball. Moreover, if $C > 0$ is such that the function $\rho_z|\Lambda_z$ has no critical points on $B(z; C)$ other than the point 0, Morse theory shows that for $0 < c \le C$ the sets $B(z; c)$ are complex manifolds diffeomorphic to the $(2n-2)$-ball. (We include the singularities of Λ_z among the critical points of $\rho_z|\Lambda_z$.) In particular, when $n = 2$, these sets are complex one-dimensional and hence conformally equivalent to the disc. We state the next proposition only for $n = 2$ since we shall only need this case.

2.3 Proposition. *Let $g_0\colon \overline{U} \to \mathbb{C}^2$ be a continuous map that is holomorphic in U and let $\rho\colon \mathbb{C}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ function which is strongly plurisubharmonic in a neighborhood of $g_0(T)$ and has no critical points on $g_0(T)$. Suppose that $C\colon T \to (0,\infty)$ is a continuous function such that the function $\rho_{g_0(\zeta)}|\Lambda_{g_0(\zeta)}$ (2.2) has no critical points on $B(g_0(\zeta); C(\zeta))\backslash\{0\}$ for each $\zeta \in T$. Then for each $\epsilon > 0$ and $0 < r < 1$ there is polynomial map $g\colon \mathbb{C} \to \mathbb{C}^2$ satisfying*

(i) $|\rho(g(\zeta)) - \rho(g_0(\zeta)) - C(\zeta)| < \epsilon \quad (\zeta \in T)$,

(ii) $\rho(g(\zeta)) > \rho(g_0(\zeta)) - \epsilon \quad (\zeta \in \overline{U})$, *and*

(iii) $|g(\zeta) - g_0(\zeta)| < \epsilon \quad (|\zeta| \le r)$.

Proof. We have seen above that for each $\zeta \in T$ the set $B(g_0(\zeta); C(\zeta)) \subset \Lambda_{g_0(\zeta)}$ is conformally equivalent to the disc U. Decreasing $C(\zeta)$ slightly (so that $B(g_0(\zeta); r)$ is still biholomorphic to U for some $r > C(\zeta)$) we can obtain a parametrization $\lambda_\zeta\colon \overline{U} \to \overline{B}(g_0(\zeta); C(\zeta))$ $(\zeta \in T)$, depending continuously on $(\zeta, w) \in T \times \overline{U}$, such that λ_ζ is holomorphic in U and $\lambda_\zeta(0) = g_0(\zeta)$ for each $\zeta \in T$. The result now follows from proposition 2.1 applied to the family of discs λ_ζ. ♠

If $K_0 \subset\subset K_1 \subset\subset \mathbb{C}^2$ is a pair of compact sets such that ρ is strongly plurisubharmonic and has no critical points on K_1, there is a constant $C > 0$ such that $\rho_z|\Lambda_z$ has no critical points on $B(z; C)\backslash\{0\}$ for each $z \in K_0$. Hence proposition 2.3 provides a uniform lifting of the boundary of an analytic disc (with respect to ρ) as long as the boundary remains in K_0. If the set $A(c_0, c_1) = \{x \in X\colon c_0 \le \rho(x) \le c_1\}$ is compact for some $c_0 < c_1$ and if ρ is strongly plurisubharmonic and without critical points on this set, proposition 2.3 allows us to lift the boundary of an analytic disc in X from the level $\rho = c_0$ to the level $\rho = c_1$. Unfortunately this breaks down in general if the level sets of ρ are not compact. In this case we need a more precise analysis which we shall do for the function (1.1).

2.4 Proposition. *Let ρ_c be the function (1.1). If $c < 1$ there exists a number $a = a(c) > 0$ with the following property: For each continuous map $h\colon \overline{U} \to \mathbb{C}^2$, holomorphic in U, such that $m(h) = \inf\{\rho_c(h(\zeta))\colon |\zeta| = 1\} > 0$, and for each pair of numbers $\epsilon > 0$ and $0 < r < 1$ there exists a holomorphic polynomial map $g\colon \mathbb{C} \to \mathbb{C}^2$ satisfying*

(i) $m(g) \ge (1 + a)m(h)$,

(ii) $\rho_c(g(\zeta)) > \rho_c(h(\zeta)) - \epsilon \quad (|\zeta| \le 1)$, *and*

(iii) $|g(\zeta) - h(\zeta)| < \epsilon \quad (|\zeta| \le r)$.

Proof. Note ρ_c is strongly plurisubharmonic when $c < 1$. Fix such a c and write $\rho = \rho_c$. The only critical point of ρ is $z_1 = z_2 = 0$. Proposition 2.4 follows immediatelly from proposition 2.3 and the following

2.5 Lemma. *Let $\rho = \rho_c$ for some $c < 1$ be given by (1.1). There is a constant $a = a(c) > 0$ such that for each $z \in \mathbb{C}^2$ with $\rho(z) > 0$ the function $\rho_z|\Lambda_z$ has no critical points on $B(z; a\rho(z))\backslash\{0\}$.*

Proof. A calculation shows that $\rho_z(w) = \Re Q_z(w) + \mathcal{L}_z(w)$, where

$$Q_z(w) = 2(x_1 + icy_1)w_1 + 2(x_2 + icy_2)w_2 + \frac{1}{2}(1+c)\big(w_1^2 + w_2^2\big)$$
$$\mathcal{L}_z(w) = \frac{1}{2}(1-c)(|w_1|^2 + |w_2|^2) = \frac{1}{2}(1-c)|w|^2.$$

It suffices to consider the case $0 < c < 1$. If $w \in \Lambda_z$ then

$$\rho_z(w) = \rho(z+w) - \rho(z) = \frac{1}{2}(1-c)|w|^2 \tag{2.4}$$

The critical points of $\rho_z|\Lambda_z$ are precisely those points $w \in \Lambda_z$ at which the complex gradients ∂Q_z and $\partial\rho_z$ (with respect to the variable $w = (w_1, w_2) \in \mathbb{C}^2$) are $\mathbb{C}$-linearly dependent. This set will include any singular points of Λ_z. By (2.4) we may replace $\partial\rho_z$ by $\partial|w|^2$. Set $h(x+iy) = x + icy$, so $|h(x+iy)|^2 = x^2 + c^2y^2$. We have

$$\partial Q_z(w) = \big(2h(z_1) + (1+c)w_1, 2h(z_2) + (1+c)w_2\big), \qquad \partial|w|^2 = (\overline{w}_1, \overline{w}_2).$$

This gives the following system of two equations for w, in which the first is the colinearity equation between ∂Q_z and $\partial|w|^2$ (after conjugation) and the second is $Q_z(w) = 0$:

$$\begin{aligned} 2\overline{h(z_2)}w_1 - 2\overline{h(z_1)}w_2 &= -(1+c)(w_1\overline{w}_2 - \overline{w}_1 w_2) \\ 4h(z_1)w_1 + 4h(z_2)w_2 &= -(1+c)(w_1^2 + w_2^2). \end{aligned} \tag{2.5}$$

It suffices to obtain a good lower estimate for the norm $|w|$ of any nonzero solution of (2.5) in terms of $|z|$. We apply Cramer's formula to express w_1 and w_2 from the linear part in terms of the right hand side terms in (2.5). The determinant of the matrix of coefficients is $W(z) = 8\big(|h(z_1)|^2 + |h(z_2)|^2\big) \geq c'|z|^2$ where $c' > 0$ depends only on c. If we replace one of the columns of the coefficient matrix by the right hand side then each term in the corresponding determinant is of the form constant times $h(z_j)w_k w_l$ for some $j, k, l \in \{1, 2\}$. Hence we can estimate the determinant from above by the Cauchy-Schwarz inequality and thus obtain the following estimate for the solutions of (2.5):

$$|w_j| \leq \frac{c_2\big(|h(z_1)|^2 + |h(z_2)|^2\big)^{1/2}|w|^2}{W(z)} \leq \frac{c_3|w|^2}{|z|} \quad (j = 1, 2).$$

This gives $|w| \leq c_4|w|^2/|z|$ and therefore $|w| \geq c_5|z|$ for any nonzero solution w of (2.5), where $c_5 > 0$ depends only on c. Since $w \in \Lambda_z$, (2.4) gives

$$\rho(z+w) \geq \rho(z) + c_6|z|^2 \geq \rho(z) + c_7\rho(z)$$

for some $c_7 > 0$. Thus any constant $a < c_7$ satisfies lemma 2.5. ♠

&3. Proper discs in cones in $\mathbb{C}^2$ with real axis.

In this section we prove theorem 1.2. If the constant M in the theorem is negative, we first apply the procedure described in [Glo] to cross the critical point of ρ_c at $(0,0)$ and thus push the boundary of the given initial analytic disc h to the set $\rho_c > 0$ while changing h as little as desired on $\{|\zeta| \le r\}$. Hence it suffices to prove theorem 1.2 for $M \ge 0$. In this case the result follows immediatelly from the following.

3.1 Theorem. *Let $c < 1$, $M \ge 0$, and let $\rho = \rho_c$ be the function (1.1). Given a continuous map $h\colon \overline{U} \to \mathbb{C}^2$, holomorphic in U, such that $\rho(h(\zeta)) > M$ for $|\zeta| = 1$, there exists for each $\epsilon > 0$ and $0 < r_1 < 1$ a proper holomorphic map $f\colon U \to \mathbb{C}^2$ satisfying*
(i) $\lim_{|\zeta|\to 1} \rho_c(f(\zeta)) = +\infty$,
(ii) $\rho(f(\zeta)) > \rho(h(\zeta)) - \epsilon \quad (|\zeta| < 1)$, *and*
(iii) $|f(\zeta) - h(\zeta)| < \epsilon \quad (|\zeta| \le r_1)$.

Proof. It suffices to consider the case $0 < c < 1$. Fix numbers $M > 0$, $0 < r < 1$, $\epsilon > 0$ and a map h as in the statement of theorem 3.1 and write $M_1 = M$, $\epsilon_1 = \epsilon$, $f_1 = h$. Let $a > 0$ be the number given by proposition 2.4 for the pair c and M_1. Set

$$M_k = (1+a)^{k-1} M_1, \quad \epsilon_k = \epsilon/2^{k-1}, \quad k = 2, 3, 4, \ldots$$

We inductively construct a sequence of polynomial maps $f_k\colon \overline{U} \to \mathbb{C}^2$ and a sequence of numbers $0 < r_1 < r_2 < r_3 < \ldots < 1$ with $\lim_{k\to\infty} r_k = 1$ such that the following hold for each $k \ge 2$:
(a_k) $\rho(f_k(\zeta)) > M_k \quad (r_k \le |\zeta| \le 1)$,
(b_k) $\rho(f_k(\zeta)) > \rho(f_{k-1}(\zeta)) - \epsilon_{k-1} \quad |\zeta| \le 1)$, and
(c_k) $|f_k(\zeta) - f_{k-1}(\zeta)| < \epsilon_{k-1} \quad (|\zeta| \le r_{k-1})$.

The construction proceeds as follows. By assumptions the condition (a_1) holds for $|\zeta| = 1$. By continuity we can increase r_1 such that (a_1) holds for $r_1 \le |\zeta| \le 1$. Proposition 2.4 gives a map f_2 such that $\rho(f_2(\zeta)) > M_2$ for $|\zeta| = 1$ and such that (b_2) and (c_2) hold. By continuity we can choose a number $r_2 < 1$ sufficiently close to 1 such that (a_2) holds for $r_2 \le |\zeta| \le 1$.

This process can be continued inductively. If we already have f_{k-1}, proposition 2.4 gives the next map f_k which satisfies (a_k) initially only for $|\zeta| = 1$, and it satisfies (b_k) and (c_k). By continuity we can choose $r_k < 1$ sufficiently close to 1 so that (a_k) holds. We can thus insure that $\lim_{k\to\infty} r_k = 1$.

Condition (c) insures that $f = \lim_{k\to\infty} f_k\colon U \to \mathbb{C}^2$ exists uniformly on compacts in U. For $|\zeta| \le r_1$ we have

$$|f(\zeta) - f_1(\zeta)| \le \sum_{k=1}^{\infty} |f_{k+1}(\zeta) - f_k(\zeta)| < \sum_{k=1}^{\infty} \epsilon_{k+1} = \epsilon.$$

This proves (iii) since $h = f_1$. For a fixed $\zeta \in U$ and $k \geq 1$ we have

$$\begin{aligned}\rho(f(\zeta)) &= \lim_{j\to\infty} \rho(f_j(\zeta)) = \rho(f_k(\zeta)) + \sum_{j=k}^{\infty}\big(\rho(f_{j+1}(\zeta)) - \rho(f_j(\zeta))\big) \\ &> \rho(f_k(\zeta)) - \sum_{j=k}^{\infty} \epsilon_{j+1} = \rho(f_k(\zeta)) - \epsilon_k.\end{aligned}$$

For $k = 1$ we get (ii) in the theorem. For points ζ in the annulus $r_k \leq |\zeta| < 1$ we get $\rho(f(\zeta)) > \rho(f_k(\zeta)) - \epsilon_k > M_k - \epsilon$. Since $\lim_{k\to\infty} M_k = \infty$, this implies (i) and completes the proof of theorem 3.1. ♠

&4. Proper discs in $\mathbb{C}^2$ which omit a pair of lines.

Theorem 1.3 follows from the following more precise result.

4.1 Theorem. *Let $n \geq 2$. Given a continuous map $h = (h_1, h_2, \ldots, h_n): \overline{U} \to \mathbb{C}^n$ which is holomorphic in U and given a number $0 < r < 1$ such that the components h_j have no zeros in $\{\zeta: r \leq |\zeta| \leq 1\}$, there exists for each $\epsilon > 0$ a proper holomorphic map $f = (f_1, f_2, \ldots, f_n): U \to \mathbb{C}^n$ such that the f_j's have no zeros in $\{\zeta: r \leq |\zeta| < 1\}$ and $|f(\zeta) - h(\zeta)| < \epsilon$ for $|\zeta| \leq r$.*

We shall give details only for $n = 2$. By factoring out the (finitely many) zeros of the h_j's we can reduce to the case when the h_j's have no zeros on $\overline{U}$. We seek a solution in the form $f = \big(e^{g_1}, e^{g_2}\big) = \big(e^{u_1+iv_1}, e^{u_2+iv_2}\big)$ for some holomorphic map $g = (g_1, g_2): U \to \mathbb{C}^2$. Set

$$\rho(x_1 + iy_1, x_2 + iy_2) = \max\{x_1, x_2\}. \tag{4.1}$$

Since $|f|^2 = |f_1|^2 + |f_2|^2 = e^{2u_1} + e^{2u_2}$, f is proper into $\mathbb{C}^2$ if and only if $\rho(g(\zeta)) = \max\{u_1(\zeta), u_2(\zeta)\}$ tends to $+\infty$ as $|\zeta| \to 1$. Such map g will be obtained as the limit $g = \lim_{k\to\infty} g_k$ of an inductively constructed sequence g_k, where the inductive step from g_{k-1} to g_k will be furnished by corollary 2.2. To this end we need a suitable family of analytic discs which we now construct.

4.2 Proposition. *Let ρ be the function (4.1). Given a compact set $K \subset\subset \mathbb{C}^2$ and constants $C_0, C_1 \in \mathbb{R}$ such that $C_0 < \rho(z) < C_1$ $(z \in K)$, there is a continuous map $\lambda: K \times \overline{U} \to \mathbb{C}^2$ such that for each $z \in K$ the map $\lambda(z, \cdot): U \to \mathbb{C}^2$ is holomorphic and*
(i) $\rho(\lambda(z, w)) = C_1 \quad (z \in K,\ |w| = 1)$,
(ii) $\rho(\lambda(z, w)) > C_0 \quad (z \in K,\ |w| \leq 1)$.

Proof. We follow the proof of Bochner's tube theorem (see [Hör], p. 41). We first describe the model situation. Write the coordinates on $\mathbb{C}^2$ in the form

$z = x + iy$, with $x, y \in \mathbb{R}^2$, and identify $\mathbb{R}^2$ with $\{y = 0\} \subset \mathbb{C}^2$. Set

$$
\begin{aligned}
k &= \{(x_1, 0): 0 \le x_1 \le 1\} \cup \{(0, x_2): 0 \le x_2 \le 1\} \\
K_\epsilon &= \{x + iy \in \mathbb{C}^2: x \in k,\ |y|^2 \le 1/\epsilon\} \\
co(k) &= \{(x_1, x_2): x_1 \ge 0,\ x_2 \ge 0,\ x_1 + x_2 \le 1\} \\
\gamma_\epsilon &= \{(x_1, x_2) \in co(k): x_1 + x_2 - \epsilon(x_1^2 + x_2^2) = 1 - \epsilon\} \\
\Gamma_\epsilon &= \{(z_1, z_2) \in \mathbb{C}^2: (x_1, x_2) \in co(k),\ z_1 + z_2 - \epsilon(z_1^2 + z_2^2) = 1 - \epsilon\}
\end{aligned}
$$

4.3 Lemma. *(Notation as above) There is an $\epsilon_0 > 0$ such that for each ϵ with $0 < \epsilon < \epsilon_0$ the set Γ_ϵ is a holomorphic disc with boundary contained in K_ϵ, $\Gamma_\epsilon \cap \mathbb{R}^2 = \gamma_\epsilon$, and γ_ϵ is a smooth real-analytic curve contained in the convex hull $co(k)$ of k. The union $\bigcup_{0<\epsilon<\epsilon_0} \gamma_\epsilon$ contains every point in the interior of $co(k)$ and sufficiently close to the open segment $\gamma_0 = \{(x_1, 1 - x_1): 0 < x_1 < 1\}$.*

Proof. Observe that $\gamma_\epsilon = \{F_\epsilon = 0\} \cap co(k)$ where

$$F_\epsilon(x_1, x_2) = x_1 + x_2 - \epsilon(x_1^2 + x_2^2) - 1 + \epsilon.$$

Simple calculations show that for $0 < \epsilon < 1/2$ we have $F_\epsilon(x_1, 0) < 0$ for $0 \le x_1 < 1$, $F_\epsilon(1, 0) = F_\epsilon(0, 1) = 0$, $F(x_1, 1 - x_1) > 0$ when $0 < x_1 < 1$, and $\frac{\partial}{\partial x_2} F_\epsilon(x_1, x_2) = 1 - 2\epsilon x_2 > 0$ for $0 \le x_2 \le 1$. These properties imply that γ_ϵ is a graph $y_1 = h_\epsilon(x_1)$ of a real-analytic function h_ϵ over the segment $0 \le x_1 \le 1$, with with the endpoints $(1, 0)$ and $(0, 1)$. Since $\partial F_\epsilon / \partial \epsilon = 1 - (x_1^2 + x_2^2) \ge 0$ on $co(k)$ we conclude that, as ϵ decreases to 0, the functions h_ϵ increase to $h_0(x_1) = 1 - x_1$. This gives the last claim in lemma 4.3.

We will show that for sufficiently small $\epsilon > 0$ there exists a bounded, simply connected region $D_\epsilon \subset \{z_2 = 0\}$ with piecewise smooth boundary such that Γ_ϵ is the graph of a holomorphic function over D_ϵ. The equation for Γ_ϵ is equivalent to

$$
\begin{aligned}
x_1 + x_2 - \epsilon(x_1^2 + x_2^2) + \epsilon(y_1^2 + y_2^2) &= 1 - \epsilon \\
(1 - 2\epsilon x_1)y_1 + (1 - 2\epsilon x_2)y_2 &= 0.
\end{aligned}
\tag{4.2}
$$

When $y_1 = y_2 = 0$ we get the equation for γ_ϵ, and hence $\Gamma_\epsilon \cap \mathbb{R}^2 = \gamma_\epsilon$. On $co(k)$ we have $x_1 + x_2 \ge 0$ and $x_1^2 + x_2^2 \le 1$, with equality only at the points $(1, 0)$ and $(0, 1)$. Rewriting the first equation in (4.2) in the form

$$(x_1 + x_2) + \epsilon(y_1^2 + y_2^2) = 1 - \epsilon(1 - (x_1^2 + x_2^2)) \le 1$$

we see that (4.2) has no solutions for $|y|^2 = y_1^2 + y_2^2 > 1/\epsilon$, and it has no solutions on $\gamma_0 + i\mathbb{R}^2$ (γ_0 was defined in lemma 4.3). Hence the boundary of Γ_ϵ is contained in K_ϵ and therefore $\Gamma_\epsilon \subset co(K_\epsilon)$. From the second equation in (4.2) we get

$$y_2 = -y_1 \frac{1 - 2\epsilon x_1}{1 - 2\epsilon x_2} \tag{4.3}$$

(again this requires $\epsilon < 1/2$ since $0 \le x_1, x_2 \le 1$ on Γ_ϵ). Inserting this into the first equation (4.2) we get

$$G_\epsilon(x_1, y_1, x_2) := x_1 + x_2 - \epsilon(x_1^2 + x_2^2) + \epsilon y_1^2 \left(1 + \frac{(1-2\epsilon x_1)^2}{(1-2\epsilon x_2)^2}\right) - 1 + \epsilon = 0. \quad (4.4)$$

Consider first its restriction to $x_2 = 0$:

$$G_\epsilon(x_1, y_1, 0) = x_1 - \epsilon x_1^2 + \epsilon y_1^2 \left(1 + (1-2\epsilon x_1)^2\right) - 1 + \epsilon = 0.$$

Let $a_\epsilon > 0$ be the solution of the equation $G(0, a_\epsilon, 0) = 2\epsilon a_\epsilon^2 - 1 + \epsilon = 0$. Calculations show that $G_\epsilon(0, y_1, 0) < 0$ for $|y_1| < a_\epsilon$, $G_\epsilon(1, y_1, 0) \ge 0$ (with equality only at $y_1 = 0$), and $\frac{\partial G_\epsilon}{\partial x_1}(x_1, y_1, 0) > 0$ for $0 \le x_1 \le 1$. This shows that the set

$$\sigma_\epsilon = \{x_1 + iy_1 \colon 0 \le x_1 \le 1,\ G_\epsilon(x_1, y_1, 0) = 0\}$$

is a smooth real-analytic curve which can be written as a graph $x_1 = g_\epsilon(y_1)$ over the interval $|y_1| \le a_\epsilon$, and the set

$$\begin{aligned} D_\epsilon &= \{x_1 + iy_1 \in \mathbb{C} \colon 0 < x_1 < 1,\ G_\epsilon(x_1, y_1, 0) < 0\} \\ &= \{x_1 + iy_1 \colon 0 < x_1 < g_\epsilon(y_1),\ |y_1| < a_\epsilon\} \end{aligned}$$

(with piecewise smooth boundary) is conformally equivalent to the disc. A calculation shows that for $\epsilon > 0$ sufficiently small we have $\frac{\partial G_\epsilon}{\partial x_2}(x_1, y_1, x_2) > 0$ on $0 \le x_1 \le 1$ and $y_1^2 \le 1/\epsilon$, and $G_\epsilon(x_1, y_1, 1) > 0$ for $x_1 + iy_1 \in D_\epsilon$. Since $G_\epsilon(x_1, y_1, 0) < 0$ for $x_1 + iy_1 \in D_\epsilon$, it follows that (4.4) has a unique solution $x_2 = \xi_\epsilon(x_1, y_1) \in [0, 1]$ for each $z_1 = x_1 + iy_1 \in \overline{D}_\epsilon$ and it has no solutions for points in $\{0 \le x_1 \le 1\} \backslash \overline{D}_\epsilon$. From (4.3) we also calculate y_2 and thus obtain a unique analytic solution $z_2 = f_\epsilon(z_1)$ ($z_1 \in \overline{D}_\epsilon$) of the system (4.2). This proves that Γ_ϵ is an analytic disc with boundary in K_ϵ. ♠

We continue with the proof of proposition 4.2. For each $y \in \mathbb{R}^2$ and $C \in \mathbb{R}$ we have

$$\{x \in \mathbb{R}^2 \colon \rho(x + iy) = C\} = \{(x_1, C) \colon x_1 \le C\} \cup \{(C, x_2) \colon x_2 \le C\}.$$

For each point $z = x + iy \in \mathbb{C}^2$ with $C_0 < \rho(z) < C_1$ we can choose a line segment $l_z \subset R^2 + iy$ passing through z such that $\rho > C_0$ on l_z and the endpoints of l_z belong to $\{\rho = C_1\}$. We can choose such l_z depending smoothly on z in the region $C_0 < \rho(z) < C_1$. The segment l_z together with the two bounded segments in the level set $\rho = C_1$ (in $\mathbb{R}^2 + iy$) determines a closed triangle $T_z \subset \mathbb{R}^2 + iy$ which corresponds (after a rotation and dilation of coordinates) to the set $co(k)$ in the model case. Lemma 4.3, applied to a slightly larger triangle $\tilde{T}_z \supset T_z$ obtained by a small parallel translation of the segment l_z so as to include the point z in the interior of $\tilde{T}_z$, gives an analytic disc $\Gamma_z \subset \mathbb{C}^2$ passing through z such that $\rho > C_0$ on Γ_z and $\rho = C_1$ on $b\Gamma_z$. We can parametrize Γ_z by a map $\lambda(z, \cdot) \colon \overline{U} \to \Gamma_z$, holomorphic in U and depending continuously on $z \in K$. ♠

Combining proposition 4.2 an corollary 2.2 we obtain

4.4 Corollary. *Let ρ be the function (4.1). Given a continuous map $g_0 \colon \overline{U} \to \mathbb{C}^2$, holomorphic in U, and constants $0 < r < 1$, $C_0, C_1 \in \mathbb{R}$ that $C_0 < \rho(g_0(\zeta)) < C_1$ for $r \le |\zeta| \le 1$, there is for each $\epsilon > 0$ a holomorphic polynomial map $g \colon \overline{U} \to \mathbb{C}^2$ satisfying*

(i) $|\rho(g(\zeta)) - C_1| < \epsilon \quad (|\zeta| = 1)$,

(ii) $\rho(g(\zeta)) > C_0 \quad (r \le |\zeta| \le 1)$, *and*

(iii) $|g(\zeta) - g_0(z)| < \epsilon \quad (|\zeta| \le r)$.

Proof of theorem 4.1. Choose a sequence $\epsilon_k > 0$, $\sum_{k=1}^{\infty} \epsilon_k < 1$. We begin by an arbitrary continuous map $g_1 \colon \overline{U} \to \mathbb{C}^2$ that is holomorphic in U and a number $0 < r_1 < 1$. Choose numbers $M_0, M_1 \in \mathbb{R}$ such that $M_0 < \rho(g_0(\zeta)) < M_1$ for $r_1 \le |\zeta| \le 1$. Choose a number $M_2 \ge M_1 + 1$ and apply corollary 4.4 to get a polynomial map $g_2 \colon \mathbb{C} \to \mathbb{C}^2$ and a number r_2, $r_1 < r_2 < 1$, such that the following hold for $k = 2$:

(a$_k$) $M_{k-1} < \rho(g_k(\zeta)) < M_k \quad (r_k \le |\zeta| \le 1)$,

(b$_k$) $\rho(g_k(\zeta)) > M_{k-2} \quad (r_{k-1} \le |\zeta| \le 1)$, and

(c$_k$) $|g_k(\zeta) - g_{k-1}(\zeta)| < \epsilon_{k-1} \quad (|\zeta| \le r_{k-1})$.

This process can be continued inductively as follows. Suppose that we have already constructed g_{k-1} for some $k \ge 2$. Choose $M_k \ge M_{k-1} + 1$ and apply corollary 4.4 to get a map g_k which satisfies (a$_k$) for $|\zeta| = 1$ and it satisfies (b$_k$) and (c$_k$). By continuity we can choose $r_k < 1$ such that $1 - r_k < (1 - r_{k-1})/2$ and such that (a$_k$) holds for $r_k \le |\zeta| \le 1$.

By construction we have $\lim_{k\to\infty} r_k = 1$, $\lim_{k\to\infty} M_k = +\infty$, and $g = \lim_{k\to\infty} g_k$ exists uniformly on compacts in U by (c$_k$). It remains to show that $\rho(g(\zeta)) \to \infty$ as $|\zeta| \to 1$. Fix $k \ge 2$ and consider points in $A_k = \{\zeta \colon r_{k-1} \le |\zeta| \le r_k\}$. For $l \ge k$ we have $|g_{l+1}(\zeta) - g_l(\zeta)| < \epsilon_l$, so $|g(\zeta) - g_k(\zeta)| \le \sum_{l=k}^{\infty} |g_{l+1}(\zeta) - g_l(\zeta)| < \sum_{l=k}^{\infty} \epsilon_l < 1$. From this and (b$_k$) we get $\rho(g(\zeta)) > \rho(g_k(\zeta)) - 1 > M_{k-2} - 1$ for $\zeta \in A_k$. Since $M_{k-2} \to \infty$ as $k \to \infty$, the result follows. ♠

Remark. One can give an alternative proof of theorem 1.4 as follows. One can construct a family of holomorphic maps $F_p \colon \mathbb{C} \to \mathbb{C}^2$, depending continuouly on $p \in (\mathbb{C}^*)^2$, such that (i) $F_p(0) = p$, (ii) $|F_p(\zeta)| \ge |p| - \epsilon_p$ for all $\zeta \in \mathbb{C}$ (where $\epsilon_p > 0$ can be made independent of p in any compact set $K \subset\subset (\mathbb{C}^*)^2$, (iii) $F_p(\mathbb{C})$ misses $zw = 0$, and (iv) $\lim_{|\zeta|\to\infty} |F_p(\zeta)| = +\infty$. The discs $\zeta \to F(\zeta)$, $|\zeta| \le R$ with R large enough, can be taken as building blocks to construct proper holomorphic discs $U \to \mathbb{C}^2$ whose image avoids both coordinate axes (compare with proposition 4.2). Similar method can used to construct proper holomorphic discs in $\mathbb{C}^2$ avoiding the curve $zw = 1$.

&5. Boundary behavior of proper holomorphic discs.

In this section we prove theorem 1.5. We begin by recalling some classical results on boundary behavior of meromorphic functions on $U = \{|\zeta| < 1\}$ (see e.g. [CL] and [Pri]). Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{0\}$ denote the Riemann sphere. For $a \in \mathbb{C}$

and $r > 0$ set $D(a;r) = \{\zeta \in \mathbb{C}: |\zeta - a| < r\}$. In what follows let f be a meromorphic function on U. We denote by $C(f, e^{i\theta})$ its **unrestricted cluster set** at $e^{i\theta} \in T$:

$$C(f, e^{i\theta}) = \bigcap_{r>0} \overline{f\big(U \cap D(e^{i\theta}; r)\big)}.$$

Equivalently, $a \in \overline{\mathbb{C}}$ belongs to $C(f, e^{i\theta})$ if and only if there exists a sequence $\zeta_j \in U$ such that $\lim_{j\to\infty} \zeta_j = e^{i\theta}$ and $\lim_{j\to\infty} f(\zeta_j) = a$. If $D \subset U$ is a subset with $e^{i\theta} \in \overline{D}$, we denote by $C_D(f, e^{i\theta})$ the **restricted cluster set** of f at $e^{i\theta}$, defined as the set of limits of f along sequences $\zeta_j \in D$ with $\lim_{j\to\infty} \zeta_j = e^{i\theta}$.

A point $e^{i\theta}$ for which $C(f, e^{i\theta}) = \overline{\mathbb{C}}$ is called a **Weierstrass point** of f, and the set of all such points is the **Weierstrass set** $W(f)$ [CW, p. 149].

For each $\epsilon^{i\theta} \in T$ and $0 < \alpha < 1$ we set

$$\Gamma_\alpha(e^{i\theta}) = \{\zeta \in U: |\Im(1 - \zeta e^{-i\theta})| < \alpha|\zeta - e^{i\theta}|\}.$$

This is an angle in U with vertex at $e^{i\theta}$ and opening $2 \arcsin \alpha$, bisected by the radius that terminates at $e^{i\theta}$. If the limit

$$f^*(e^{i\theta}) = \lim_{\Gamma_\alpha(e^{i\theta}) \ni \zeta \to e^{i\theta}} f(\zeta) \in \overline{\mathbb{C}} \tag{5.1}$$

exists and is independent of α, it is called the *nontangential limit of f at $e^{i\theta}$* and $e^{i\theta}$ is called a **Fatou point** of f. The set of all Fatou points the **Fatou set** $F(f)$ [CW, p. 21].

A point $e^{i\theta} \in T$ is called a **Plessner point** of f if for every angle Γ with vertex $e^{i\theta}$ the partial cluster set $C_\Gamma(f, e^{i\theta})$ equals $\overline{\mathbb{C}}$ (i.e., it is total). The set of all Plessner points is the **Plessner set** $I(f)$ [CW, p. 147]. Clearly $I(f) \subset W(f)$.

The **Nevanlinna characteristic** of a holomorphic function f on U is defined by

$$T(r, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, \frac{dt}{2\pi} \qquad (0 \le r < 1);$$

for meromorphic functions see [CW, p. 39] or [Nev].

The **range** of f, denoted $R(f)$, is the set of all $\alpha \in \overline{\mathbb{C}}$ such that $f(\zeta_j) = \alpha$ for all points in a sequence $\zeta_j \in U$ $(j \in \mathbb{N})$. By restricting the attention only to sequences $z_j \in U$ with $\lim_{j\to\infty} \zeta_j = e^{i\theta}$ we get the range of f at $e^{i\theta}$, denoted $R(f, e^{i\theta})$.

The notion of **logarithmic capacity** of a Borel set $E \subset \mathbb{C}$ can be found in [CL, p. 9]. Such a set is of capacity zero if and only if it is polar, i.e., it is contained in the $-\infty$ level set of a non-constant subharmonic function on $\mathbb{C}$ [Lan, Tsu]. The following summarizes some of the known results which we shall need in the proof of theorem 1.5.

5.1 Theorem: *Let f be a meromorphic function on the disc U.*

(a) If $e^{i\theta} \in T$ is not a Weierstrass point of f then there is an open arc $\gamma \subset T$ containing $e^{i\theta}$ such that almost every point in γ is a Fatou point of f.

(b) *If f has bounded Nevanlinna characteristic on U then almost every point of T is a Fatou point of f.*
(c) *Almost every point in T belongs to $F(f) \cup I(f)$.*
(d) *If $f(U) \subset \overline{\mathbb{C}} \backslash E$ for some set E of positive capacity then f has bounded Nevanlinna characteristic.*
(e) *If $R(f, e^{i\theta})$ omits a set $E \subset \overline{\mathbb{C}}$ of positive capacity then there is an open arc $\gamma \subset T$ containing $e^{i\theta}$ such that almost every point of γ is a Fatou point of f.*

Proof. (a) If $e^{i\theta}$ is not a Weierstrass point of f, there is a disc $D(e^{i\theta}; r)$ such that $f(D(e^{i\theta}; r) \cap U)$ omits a disc $D(a; \delta) \subset \mathbb{C}$. The function $g(\zeta) = 1/(f(\zeta) - a)$ is then bounded holomorphic in $D(e^{i\theta}; r) \cap U$ and hence by Fatou's theorem it has nontangential limit $g^*(e^{it})$ at almost every point $e^{it} \in \gamma = T \cap D(e^{i\theta}; r)$ [CW, p. 21]. The same is then true for f and hence almost every point of γ belongs to the Fatou set $F(f)$. (See also [CW, Theorem 8.4].) Part (b) follows by combining Fatou's theorem with a theorem of R. Nevanlinna to the effect that a meromorphic function with bounded Nevanlinna characteristic on U is the quotient of two bounded holomorphic functions [CW, p. 41]. Part (c) is a classical theorem due to Plessner ([Ple], [Pri, p. 217] or [CW, p. 147]).

Part (d) is due to Frostman [Fro]; the following simple proof was shown to us by D. Marshall. After a fractional linear transformation we may assume that $\infty \in E \subset \{|z| > 1\}$. Let $g_0(z)$ be the Green's function for $\mathbb{C} \backslash E$ with a logarithmic pole at 0 (so $g_0(z) \to 0$ as $z \to E$). Then $\log^+ \frac{1}{|z|} \le g_0(z)$. The function $u(z) := g_0(z) + \log|z|$ is harmonic on $\mathbb{C} \backslash E$ since both summands are harmonic on $\mathbb{C} \backslash (E \cup \{0\})$ and the pole at 0 cancels off. If $f: U \to \mathbb{C} \backslash E$ is a holomorphic function then $u \circ f$ is harmonic on U and we have

$$\begin{aligned}
\int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} &= \int_0^{2\pi} \left(\log^+ \frac{1}{|f(re^{i\theta})|} + \log|f(re^{i\theta})| \right) \frac{d\theta}{2\pi} \\
&\le \int_0^{2\pi} \left(g_0(f(re^{i\theta})) + \log|f(re^{i\theta})| \right) \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} u(f(re^{i\theta})) \frac{d\theta}{2\pi} = u(f(0)).
\end{aligned}$$

Thus $\int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{dt}{2\pi} \le u(f(0))$ for $r \in (0,1)$ which proves (d).

For (e) observe that $R(f, e^{i\theta}) = \bigcap_{n \in \mathbb{N}} f(D_n)$, where $D_n = D(e^{i\theta}; 1/n) \cap U$. The sets $f(D_n)$ are decreasing with n. If $R(f, e^{i\theta})$ omits a set E of positive capacity then $f(D_n)$ omits a set E' of positive capacity for some sufficiently large $n \in \mathbb{N}$. We may assume that $\infty \in E'$. Observe that D_n is conformally equivalent to the disc. From (d) and (b) applied to the holomorphic function $f: D_n \to \mathbb{C} \backslash E'$ it follows that almost every point of the arc $\gamma = D_n \cap T$ is a Fatou point of f. ♠

We shall frequently use the following uniqueness theorem due to Plessner ([Ple], [CL, p. 146]) and to Lusin and Priwalow [Pri, p. 212].

5.2 Theorem. *If a meromorphic function f on U has angular limit equal to zero at each point in a set $E \subset T$ of positive Lebesgue measure then f is the zero function.*

Remark. In theorem 5.2 *we cannot replace angular limits with radial limits*; see examples due to Lusin and Priwalow in [Pri], sec. IV.5.

Proof of theorem 1.5. Let $(f_1, f_2)\colon U \to \mathbb{C}^2$ be a proper holomorphic map and let g be any of the functions as in theorem 1.5. It suffices to show that the Fatou set $F(g)$ has measure zero. From theorem 5.1 (a) it will then follow that $W(g) = T$, theorem 5.1 (c) will imply that the Plessner set $I(g)$ has full measure in T, and theorem 5.1 (e) will imply that the complement of the range $R(g, e^{i\theta})$ in $\mathbb{C}$ has capacity zero for each $e^{i\theta} \in T$. Since sets of capacity zero in $\mathbb{C}$ coincide with polar sets ([Tsu], [Lan]), theorem 1.5 (d) follows.

To prove that $F(g)$ has measure zero we consider separately each case.

Case (i). Suppose that f_1 has an angular limit $f_1^*(e^{i\theta}) \in \overline{\mathbb{C}}$ (5.1) at all points $e^{i\theta}$ in a set $A \subset T$. Then A is Lebesgue measurable and can be written as $A = A_1 \cup A_2$, where A_1 is the set of all $e^{i\theta} \in A$ such that $f_1^*(e^{i\theta}) \in \mathbb{C}$ and A_2 is the set of all $e^{i\theta} \in A$ with $f_1^*(e^{i\theta}) = \infty$. Then $1/f_1$ has angular limits zero at each point of A_2. If A_2 is of positive measure, theorem 5.2 implies that $1/f_1$ is identically zero in U, a contradiction. Thus A_2 has measure zero. Consider now A_1. Since $(f_1, f_2)\colon U \to \mathbb{C}^2$ is proper, $\max\{|f_1(\zeta)|, |f_2(\zeta)|\}$ tends to $+\infty$ as $|\zeta| \to 1$. Since f_1 has a finite angular limit at each $e^{i\theta} \in A_1$, $|f_2|$ has an angular limit ∞ at each point of A_1. If A_1 is of positive measure, Plessner's theorem, applied to $1/f_2$, gives a contradiction as before. This shows that A_1 is of measure zero as well, and therefore the Fatou set of f_1 is of measure zero. The same applies to f_2.

Case (ii). Suppose that $g = f_1/f_2$ has an angular limit $g^*(e^{i\theta}) \in \overline{\mathbb{C}}$ (5.1) within an angle Γ_θ at each point $e^{i\theta}$ in a set $A \subset T$. As in part (i) we write $A = A_1 \cup A_2$, where g^* is finite on A_1 and equals ∞ on A_2. Theorem 5.2 shows as above that A_2 must be of measure zero for otherwise g would be constant. If A_1 is of positive measure, there is a set $A_0 \subset A_1$ of positive measure and a number $0 < M < \infty$ such that $|g^*(e^{i\theta})| < M$ for each $e^{i\theta} \in A_0$. Hence there is a disc U_θ centered at $e^{i\theta}$ such that $|f_1(\zeta)/f_2(\zeta)| \le M$ for $\zeta \in \Gamma_\theta \cap U_\theta$. Hence $|f_1(\zeta)| \le M|f_2(\zeta)|$ and therefore

$$\max\{|f_1(\zeta)|, |f_2(\zeta)|\} \le \max\{M|f_2(\zeta)|, |f_2(\zeta)|\} \quad (\zeta \in \Gamma_\theta \cap U_\theta).$$

Since this maximum tends to $+\infty$ as $\zeta \to e^{i\theta}$, it follows that $|f_2(\zeta)| \to \infty$ as $\zeta \to e^{i\theta}$ within Γ_θ. Thus $1/f_2$ has angular limits zero at each point of A_0, a contradiction to theorem 5.2. This proves that A_1 must be of measure zero as well.

Case (iii). This follows from case (i) by observing that for each nonconstant holomorphic polynomial P on $\mathbb{C}^2$ there exists another holomorphic polynomial Q such that $(P, Q)\colon \mathbb{C}^2 \to \mathbb{C}^2$ is a proper map, and hence $P(f_1, f_2)\colon U \to \mathbb{C}^2$ is the first component of a proper map $U \to \mathbb{C}^2$. In fact, we have

5.3 Lemma. *Let P and Q be nonconstant holomorphic polynomials on $\mathbb{C}^2$ whose leading order homogeneous parts P' resp. Q' have no common zero on $\mathbb{C}^2 \backslash \{0\}$. Then $(P,Q)\colon \mathbb{C}^2 \to \mathbb{C}^2$ is a proper map.*

We leave out the simple proof. Observe that the zero set of P' is a finite union of complex lines, so it suffices to choose Q to be a linear function which does not vanish on $P' = 0$ except at the origin; the pair (P,Q) then provides a proper self-map of $\mathbb{C}^2$.

Case (iv). Apply (i) and lemma 5.3 to the map $\big(P(f_1,f_2), Q(f_1,f_2)\big)\colon U \to \mathbb{C}^2$.

Appendix: Crossing a critical level by analytic discs.

Let X be a Stein manifold of dimension at least two and $\rho\colon X \to \mathbb{R}$ a strongly plurisubharmonic Morse exhaustion function. Let $p \in X$ be a critical point of ρ. Choose constants c_0, c_1 such that $c_0 < \rho(p) < c_1$ and p is the only critical point of ρ in $A(c_0,c_1) = \{x \in X\colon c_0 \le \rho(x) \le c_1\}$. Suppose that $f_0\colon \overline{U} \to X$ is a holomorphic map such that $c_0 < \rho(f_0(e^{i\theta})) < \rho(p)$ for each $e^{i\theta} \in T$. In [Glo] the second author showed how to construct a smooth map $f_1\colon \overline{U} \to X$ which is close to being holomorphic on U such that $\rho(p) < \rho(f_1(e^{i\theta})) < c_1$ for $e^{i\theta} \in T$ and such that f_1 approximates f_0 on a smaller disc $|\zeta| \le r < 1$. The map f_1 is obtained by adding to f_0 a small non-holomorphic contribution which can be controlled by the data. Once the boundary curve $f_1(T)$ passes the critical level of ρ at p we can use the procedure described in sect. 2 above (or in [FG]) to continue pushing it higher towards the next critical level of ρ. It was shown in [Glo] that the non-holomorphic contribution made at the initial step may be cancelled off during a later stage of the construction, once the boundary of the disc is sufficiently far above the critical level at p. The reason is that the modification process is a linear one, and we obtain the final solution as the sum of a convergent series. (Here it is convenient to embed X into a Euclidean space $\mathbb{C}^N$.)

Here we wish to point out that the transition from f_0 to f_1 as above can also be accomplished by applying to f_0 the gradient flow θ_t of ρ (in the direction of increasing ρ). Unless a point $x \in A(c_0,c_1)$ belongs to the stable manifold $W^s(p)$ of p (see e.g. Shub [Sh]), we have $\rho(\theta_t(x)) > \rho(p)$ for sufficiently large $t > 0$. Thus, if $f_0(T) \cap W^s(p) = \emptyset$, we can choose a smooth positive function a on $\overline{U}$ such that the map $f_1(\zeta) = \theta_{a(\zeta)}(f_0(\zeta))$ $(|\zeta| \le 1)$ satisfies $\rho(f_1(e^{i\theta})) > \rho(p)$ for $e^{i\theta} \in T$.

If the number c_0 is sufficiently close to $\rho(p)$ as we may assume to be the case, the set $W^s(p) \cap A(c_0,c_1)$ is a closed real submanifold of $A(c_0,c_1)$ whose dimension equals the index $i(p)$ (the number of negative eigenvalues of the Hessian) of ρ at p. Since ρ is strongly plurisubharmonic, we have $i(p) \le \dim_{\mathbb{C}} X$ (see [AF]) and therefore $\dim f_0(T) + \dim W^s(p) \le 1 + \dim_{\mathbb{C}} X < \dim_{\mathbb{R}} X$. By transversality a generic small holomorphic perturbation of f_0 satisfy the required condition $f_0(T) \cap W^s(p) = \emptyset$ which makes it possible to obtain f_1 as above. The rest of the procedure remains as in [Glo].

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