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FAST RECOGNITION  
ALGORITHMS FOR CLASSES OF  
PARTIAL CUBES

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# Fast recognition algorithms for classes of partial cubes

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## Abstract

Isometric subgraphs of hypercubes, or partial cubes as they are also called, are a rich class of graphs that include median graphs, subdivision graphs of complete graphs, and classes of graphs arising in mathematical chemistry and biology. In general one can recognize whether a graph on  $n$  vertices and  $m$  edges is a partial cube in  $O(mn)$  steps, faster recognition algorithms are only known for median graphs. This paper exhibits classes of partial cubes that are not median graphs but can be recognized in  $O(m \log n)$  steps. On the way relevant decomposition theorems for partial cubes are derived, one of them correcting an error in a previous paper [13].

## 1 Introduction

The structure of isometric subgraphs of hypercubes, or partial cubes, as they are also known, is well understood; see the classical papers of Djoković [7], Winkler [24], and Chepoi [4]. Interestingly, they were first introduced in computer science [9, 10]. Here we are concerned with the fundamental problem of determining the recognition

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complexities for such classes of graphs. The fastest general recognition algorithm for partial cubes on  $n$  vertices and  $m$  edges has complexity  $O(mn)$ . The first such algorithm is due to Aurenhammer and Hagauer [1]. It is direct but difficult. For more transparent algorithms see [12, 15]. Unfortunately only the trivial lower bound  $\Omega(m)$  is known for the recognition of partial cubes.

Median graphs [15, 19, 20, 21] are presumably the most important subclass of partial cubes, they include trees, hypercubes, and can also be characterized as the class of retracts of hypercubes. An  $O(mn)$  recognition algorithm for median graphs is given in [17], while a simple algorithm of the same complexity can be found in [13]. In [11] the complexity was reduced to  $O(m\sqrt{n})$ . Currently the recognition complexity is  $O((m \log n)^{1.41})$ ; see [15]. Because of a connection with triangle-free graphs there is strong evidence that median graphs cannot be recognized much faster; cf. [16].

In order to narrow the gap between  $\Omega(m)$  and  $O(mn)$  for the recognition complexity of partial cubes several classes of graphs have been introduced, notably semi-median graphs and almost-median graphs [13]. Although these graphs can be isometrically embedded into hypercubes in  $O(m \log n)$  time once they have been recognized [15, Lemma 7.10], their recognition complexity is still  $O(mn)$ . Here we show that prism-free almost-median graphs and many related classes can be recognized in  $O(m \log n)$  time. These classes are the first examples of easily describable classes of partial cubes that are not median graphs but can be recognized faster than median graphs.

Expansion theorems and related expansion procedures are fundamental tools in designing recognition algorithms for partial cubes and their subclasses. The expansion theorem for median graphs is due to Mulder [20, 21] and the expansion theorem for partial cubes to Chepoi [4]; cf. also [22]. In Section 3 we prove two expansion theorems, one for semi-median graphs and one for almost-median graphs. The first theorem corrects the corresponding expansion result from [13]. In Section 4 we then present an algorithm of complexity  $O(m \log n)$  that recognizes partial cubes in which certain sets  $U_{ab}$  induce isometric trees; such graphs can also be described as prism-free, almost-median graphs. We conclude with generalizations of this result to larger classes of almost-median graphs.

## 2 Preliminaries

For a graph  $G$ , the *distance*  $d_G(u, v)$ , or briefly  $d(u, v)$ , between vertices  $u$  and  $v$  is defined as the number of edges on a shortest  $u, v$ -path. A subgraph  $H$  of  $G$  is called *isometric* if  $d_H(u, v) = d_G(u, v)$  for all  $u, v \in V(H)$ . A subgraph  $H$  of  $G$  is *convex*, if for any  $u, v \in V(H)$ , all shortest  $u, v$ -paths belong to  $H$ .

The *Cartesian product*  $G \square H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  in which the vertex  $(a, x)$  is adjacent to the vertex  $(b, y)$  whenever

$ab \in E(G)$  and  $x = y$ , or  $a = b$  and  $xy \in E(H)$ . The graphs  $K_2 \square C_n$ ,  $n \geq 3$ , are called *prisms*, and the graphs  $K_2 \square P_n$  *ladders*. The Cartesian product of  $k$  copies of  $K_2$  is a *hypercube* or *k-cube*  $Q_k$ . Isometric subgraphs of hypercubes are called *partial cubes*. An important subclass of partial cubes are *median graphs*. These are the graphs  $G$  in which there exists a unique vertex  $x$  to every triple of vertices  $u$ ,  $v$ , and  $w$  such that  $x$  lies simultaneously on a shortest  $u, v$ -path, a shortest  $u, w$ -path, and a shortest  $w, v$ -path.

For partial cubes, the following vertex sets play a crucial role. Let  $ab$  be an edge of a connected, bipartite graph  $G = (V, E)$ . Then

$$\begin{aligned} W_{ab} &= \{w \in V \mid d_G(a, w) < d_G(b, w)\}, \\ U_{ab} &= \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\}, \\ F_{ab} &= \{e \in E \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}. \end{aligned}$$

By abuse of language we shall use the same notation for the sets  $W_{ab}$ ,  $U_{ab}$  and the subgraphs induced by them.

Clearly,  $W_{ab}$  and  $W_{ba}$  are disjoint. Moreover, as all graphs considered are bipartite,  $V = W_{ab} \cup W_{ba}$ . Djoković [7] proved that a graph  $G$  is a partial cube if and only if it is bipartite and if for any edge  $ab$  of  $G$  the subgraph  $W_{ab}$  is convex. It follows from results in [3] that median graphs are precisely the bipartite graphs in which all  $U_{ab}$ 's are convex. By this result, the following definitions make sense.

A bipartite graph is a *semi-median* graph if it is a partial cube in which for any edge  $ab$  the subgraph induced by  $U_{ab}$  is connected. Similarly, a bipartite graph is an *almost-median* graph if it is a partial cube such that for any edge  $ab$  the subgraph induced by  $U_{ab}$  is isometric.

Two edges  $e = xy$  and  $f = uv$  of  $G$  are in the Djoković-Winkler [7, 24] relation  $\Theta$  if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

Clearly,  $\Theta$  is reflexive and symmetric. Winkler [24] proved that a bipartite graph is a partial cube if and only if  $\Theta = \Theta^*$ , where  $\Theta^*$  denotes the transitive closure of  $\Theta$ . We will need the following basic property of  $\Theta$ ; see [12]:

**Lemma 1** *Suppose  $P$  is a walk connecting the endpoints of an edge  $e$ . Then  $P$  contains an edge  $f$  with  $e\Theta f$ .*

Another relevant relation defined on the edge set of a graph is  $\delta$ . We say an edge  $e$  is in relation  $\delta$  to an edge  $f$  if  $e$  and  $f$  are opposite edges of a 4-cycle without diagonals in  $G$  or if  $e = f$ . Clearly  $\delta$  is reflexive and symmetric. Moreover, it is contained in  $\Theta$ . Thus its transitive closure  $\delta^*$  is contained in  $\Theta^*$ . In [13] it is shown that a bipartite graph is a semi-median graph if and only if  $\Theta = \delta^*$ , in analogy to Winkler's characterization of partial cubes.

Let  $G$  be a connected graph. A *proper cover* of  $G$  consists of two isometric subgraphs  $G_1$  and  $G_2$  with union  $G$  and nonempty intersection. In symbols,  $G =$

$G_1 \cup G_2$  and  $G_0 = G_1 \cap G_2 \neq \emptyset$ .  $G_0$  is called the *intersection* of the cover. The *expansion* of  $G$  with respect to  $G_1, G_2$  is the graph  $H$  obtained by the following procedures:

- (i) Replacement of each vertex  $v \in G_1 \cap G_2$  by vertices  $v_1, v_2$  and insertion of the edge  $v_1 v_2$ .
- (ii) Insertion of edges between  $v_1$  and the neighbors of  $v$  in  $G_1 \setminus G_2$  as well as between  $v_2$  and the neighbors of  $v$  in  $G_2 \setminus G_1$ .
- (iii) Insertion of the edges  $v_1 u_1$  and  $v_2 u_2$  whenever  $v, u \in G_1 \cap G_2$  are adjacent in  $G$ .

Suppose  $e = e_1 \delta e_2 \delta \dots \delta e_k = f$  is a sequence of edges by virtue of which  $e$  and  $f$  are in relation  $\delta^*$ . Then the union of the squares that contain  $e_i, e_{i+1}$  for  $i = 1, 2, \dots, k-1$  forms a ladder, that is, the Cartesian product of a path of length  $k-1$  by an edge. We shall frequently use the fact that any two edges  $e, f$  in relation  $\delta^*$  are connected by a “ladder”.

Note that there is a natural projection of  $H$  onto  $G$ . In this projection the edges introduced by (i) are all contracted into single vertices. We call this mapping a *contraction*; it is the inverse of the expansion.

Clearly the edges introduced by (i) form a  $\Theta$ -class in  $H$ . We call it the new  $\Theta$ -class of  $H$ . All the other edges of  $H$  are in relation  $\Theta_H$  if and only if their preimages are in relation  $\Theta_G$ ; cf. the proof of Theorem 2.

If  $G_0$  is connected, isometric, or convex in  $G$ , we speak of a *connected, isometric, or convex expansion*, respectively. If  $H$  can be obtained from the one-vertex graph  $K_1$  by a sequence of expansions of a given type, then we say that  $H$  is obtainable by an *expansion procedure* of that type.

$Q_3^-$  is the 3-cube minus a vertex. By  $a(G)$  we denote the *arboricity* of  $G$ , that is, the minimum number of disjoint spanning forests into which  $G$  can be decomposed.

### 3 Decomposing semi-median and almost-median graphs

In this section we prove two expansion theorems; one for semi-median graphs and one for almost-median graphs.

In [13] it is asserted that a graph is a semi-median graph if and only if it can be obtained by a connected expansion procedure. However, Chepoi [5] showed that this condition is not sufficient. To see this, take the semi-median graph  $Q_3^-$  and expand it with respect to its isometric 6-cycle and with respect to the subgraph induced by the vertices of  $Q_3^-$  minus a vertex of degree 2. This is a connected expansion but the obtained graph is not semi-median; cf. Fig. 1.

We can correct this result by imposing an additional condition on the expansion steps.

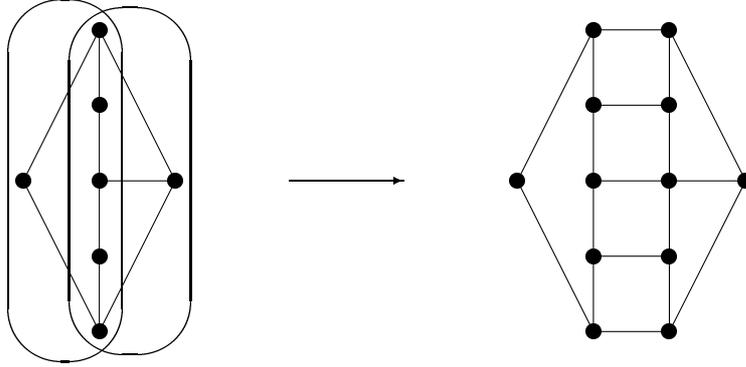


Figure 1: Connected expansion yielding a non semi-median graph

**Theorem 2** *A graph  $H$  is semi-median if and only if it can be obtained by a connected expansion procedure with the following property:*

(P) *If two edges  $e \in G_1$ ,  $f \in G_2$  are in relation  $\delta$  and if each one of them has a vertex outside  $G_0 = G_1 \cap G_2$ , then there exists an edge  $g \in G_0$  with  $e\delta_{G_1}^*g$  and  $f\delta_{G_2}^*g$ .*

**Proof.** Let  $H$  be a semi-median graph. We already know that it can be obtained by a connected expansion procedure. We have to prove that all expansions in the sequence satisfy property (P). Suppose, on the contrary, that one of the expansions in the sequence does not satisfy (P). Since every contraction of a semi-median graph is semi-median, we can assume without loss of generality that this happens in the last expansion step by which we obtain  $H$  from a graph  $G$ . Thus  $G$  is semi-median. Let  $G_1, G_2$  be the corresponding proper cover of  $G$ . Since the proper cover does not satisfy property (P), there exist edges  $e \in E(G_1)$  and  $f \in E(G_2)$  with  $e\delta f$ , but no edge  $g$  in  $G_0$  with  $e\delta_{G_1}^*g$  and  $f\delta_{G_2}^*g$ . Then we still have  $e\Theta_H f$ , yet  $e$  and  $f$  clearly cannot be in relation  $\delta_H^*$ . Hence  $H$  is not semi-median.

For the converse, let  $G$  be a semi-median graph, and  $G_1, G_2$  be a proper cover that satisfies the conditions of the theorem. We need to prove that the graph  $H$  obtained by the expansion is semi-median, that is  $\delta_H^* = \Theta_H$ . Clearly, if  $xy$  and  $uv$  are edges of the new  $\Theta$ -class, say  $F_{ab}$ , then  $xy\delta^*uv$ , because the expansion is connected. So let  $xy$  and  $uv$  be the edges of  $G$  that correspond to edges  $xy$  and  $uv$  of  $H$ . As we already noted,  $xy\Theta_H uv$  holds precisely when the corresponding edges are in relation  $\Theta_G$ . Indeed, if  $xy$  and  $uv$  are both edges of  $W_{ab}$  or of  $W_{ba}$  then the distances between endvertices are not changed by the expansion; otherwise, the distances between endvertices increase by one.

Suppose now that  $xy\Theta_H uv$ . Since  $G$  is semi-median,  $xy\delta_G^*uv$  and  $G$  contains a ladder from  $xy$  to  $uv$ . In  $H$  we wish to use the same ladder to show the validity

of  $xy \delta_H^* uv$ . This is possible unless it contains two consecutive edges  $e \in G_1$  and  $f \in G_2$ . In this case we use property (P) to find a ladder from  $e$  to  $f$  in  $H$ . Hence we also obtain  $xy \delta_H^* uv$ .

On the other hand, if  $xy$  and  $uv$  are not in relation  $\Theta$ , then they are not in relation  $\delta^*$  in  $G$ , since  $G$  is semi-median. Thus  $xy$  and  $uv$  cannot be in relation  $\delta^*$  in  $H$ .

We conclude that  $H$  is semi-median. ■

Note that property (P) of the above theorem holds when  $G_0$  is 2-isometric. (A subgraph  $H$  of a graph  $G$  is called 2-isometric if any two vertices  $u, v \in H$  of distance 2 in  $G$  have distance 2 in  $H$  as well.) In particular this holds for median graphs as all  $G_0$  are convex and thus 2-isometric. Moreover, this is also true for almost-median graphs which can be obtained by an isometric expansion procedure.

The problem with an expansion theorem for almost-median graphs is that an isometric expansion of an almost-median graph need not be almost-median. For instance, the graph  $G_{13}$  of Figure 2 is obtained by an isometric expansion from an almost-median graph  $Q_3^-$ . (We expand the outer 6-cycle of  $Q_3^-$ .)

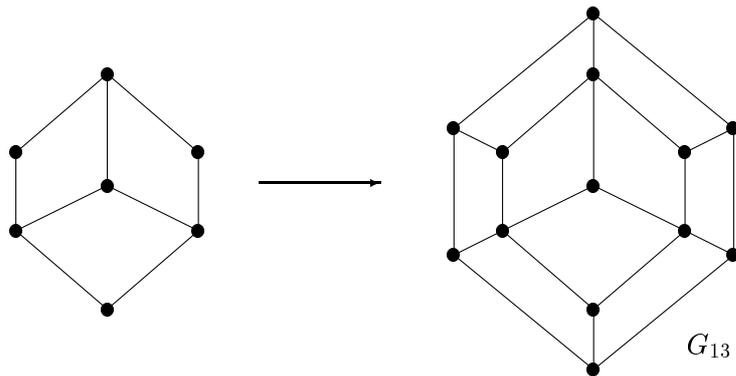


Figure 2: An isometric expansion of  $Q_3^-$

To obtain an expansion theorem for this class we recall the following well-known property of median graphs: the covering sets in convex expansions of median graphs are also median graphs. This observation leads to Theorem 4.

First a lemma. For a fixed edge  $uv$  we will consider sets  $U_{uv}$  and  $F_{uv}$  with respect to different (sub)graphs  $H$ . We will hence use the notation  $U_{uv}(H)$  and  $F_{uv}(H)$  to emphasize that  $U_{uv}$  or  $F_{uv}$  are considered with respect to  $H$ .

**Lemma 3** *Let  $ab$  be an edge of a partial cube  $G$ . If  $uv$  is an edge from  $W_{ab}$  then  $U_{uv}(W_{ab}) = U_{uv}(G) \cap W_{ab}$ .*

**Proof.** Note that  $F_{uv} = F_{u'v'}$  for any  $u'v' \in F_{uv}$  by the transitivity of  $\Theta$ . Since  $W_{ab}$  is convex in  $G$ ,  $\Theta$  with respect to  $W_{ab}$  coincides with  $\Theta$  with respect to  $G$ . Hence  $F_{uv}(W_{ab}) \subseteq F_{uv}(G)$ . Clearly this implies  $F_{uv}(W_{ab}) = F_{uv}(G) \cap W_{ab}$ , which yields the desired result.  $\blacksquare$

**Theorem 4** *A graph is almost-median if and only if it can be obtained by an isometric expansion procedure whose covering sets induce almost-median graphs.*

**Proof.** Let  $H$  be an almost-median graph. We need to prove that both  $W_{ab}$  and  $W_{ba}$  induce almost-median graphs (since then  $H$  can be obtained from a smaller almost-median graph  $G$  with covering sets isomorphic to  $W_{ab}$  and  $W_{ba}$ ).

For the proof that  $W_{ab}$  is almost-median we first note that  $W_{ab}$  is a partial cube. So we need to prove that for any  $uv \in W_{ab}$  the set  $U_{uv}(W_{ab})$  is isometric. Because  $W_{ab}$  is convex, any shortest path between vertices  $x, y$  of  $U_{uv}(W_{ab})$  is in  $W_{ab}$ . By Lemma 3,  $U_{uv}(W_{ab}) = U_{uv}(G) \cap W_{ab}$ . Hence, as  $U_{uv}(G)$  is isometric in  $G$ , there is a shortest  $x, y$ -path that lies in  $U_{uv}(W_{ab})$ .

For the converse let  $H$  be a graph obtained by an isometric expansion procedure in which covering sets induce almost-median graphs. Let  $H$  be obtained by such an expansion from  $G$  with covering sets  $G_1$  and  $G_2$ . Let  $uv$  be any edge of  $H$  and assume without loss of generality that  $uv$  is not in the new  $\Theta$ -class obtained by the expansion. Let  $ab$  be a representative of this new class. Let  $x$  and  $y$  be vertices of  $U_{uv}$ . By Lemma 3 we have  $U_{uv}(W_{ab}) = U_{uv}(G) \cap W_{ab}$  and  $U_{uv}(W_{ba}) = U_{uv}(G) \cap W_{ba}$ . Hence, if  $x$  and  $y$  lie either both in  $W_{ab}$  or both in  $W_{ba}$ , then  $U_{uv}$  already contains a shortest  $x, y$ -path of  $G$  since  $G$  is almost-median. Finally, let  $x \in W_{ab}$  and  $y \in W_{ba}$ . The proof is completed by the observation that the length of any shortest  $x, y$ -path increases by one with respect to such paths in  $G$ . So we can use the isometry of  $U_{uv}$  in  $G$  again.  $\blacksquare$

## 4 Partial cubes whose $U_{ab}$ 's are isometric trees

In this section we present an algorithm of complexity  $O(n \log n)$  that recognizes partial cubes in which the sets  $U_{ab}$  induce isometric trees.

**Lemma 5** *A graph  $G$  is a partial cube in which the sets  $U_{ab}$  induce isometric trees if and only if  $G$  is a prism-free almost-median graph.*

**Proof.** Let  $G$  be a partial cube in which the sets  $U_{ab}$  induce isometric trees and assume that  $G$  contains  $X = K_2 \square C_{2r}$  as an induced subgraph. Since the  $U_{ab}$ 's are isometric (and thus connected),  $G$  is a semi-median graph. Thus  $\delta^* = \Theta$ . It follows that the edges of  $X$  corresponding to the factor  $K_2$  belong to the same  $\Theta$ -class. This in turn implies that the corresponding  $U_{ab}$ 's contain cycles.

The converse is clear.  $\blacksquare$

Partial cubes whose  $U_{ab}$ 's are isometric trees can also be characterized as almost-median graphs whose  $U_{ab}$ 's are trees. To recognize such graphs we first check whether they are bipartite with not too many edges. Then we determine  $\delta^*$ , construct the  $U_{ab}^*$ 's, check whether they induce trees, no two edges of which are in the same  $\delta^*$  class, and whether an  $F_{ab}^*$  induces an isomorphism between  $U_{ab}$  and  $U_{ba}$ . Finally, it suffices to show that  $\Theta = \delta^*$ , because then  $\Theta$  is transitive.

We proceed as follows. Let  $E_1^*, \dots, E_k^*$  be the equivalence classes of  $\delta^*$ , which we also call *color classes with respect to  $\delta^*$* . We first construct  $G$  by joining these classes one by one, checking certain properties which necessarily have to hold if  $\Theta = \delta^*$ . If they are not satisfied,  $G$  cannot be a semi-median graph and is rejected.

### Algorithm A

*Input:* A connected, bipartite graph  $G$  with  $m \leq n \log n$ .

*Output:* TRUE if  $G$  is a prism-free almost-median graph, FALSE otherwise.

1. Determine all squares, compute  $\delta^*$  and the  $U_{ab}^*$ 's. Let  $E_1^*, \dots, E_k^*$  be the equivalence classes of  $\delta^*$ .
2. For each edge  $ab$  check if  $U_{ab}^*$  and  $U_{ba}^*$  induce distinct, isometric trees that are isomorphic. If not, then return FALSE and stop.
3. Let  $G_0 = (V(G), \emptyset)$ . For  $i = 1, \dots, n$  let  $G_i = G_{i-1} \cup E_i^*$ . If an edge of  $E_i^*$  has both endpoints in the same component of  $G_{i-1}$ , then return FALSE and stop.
4. Return TRUE.

**Lemma 6** *Every prism-free almost-median graph is accepted by Algorithm A.*

**Proof.** Let  $G$  be a prism-free almost-median graph. Then  $G$  is a partial cube and  $\delta^* = \Theta^* = \Theta$ . Therefore  $F_{uv}^* = F_{uv}$  and  $U_{uv}^* = U_{uv}$ , and so  $G$  clearly passes Step 2. Concerning Step 3 we infer that if there exists an edge  $e$  of  $E_i^*$  that has both endpoints in the same component of  $G_{i-1}$ , then  $e$  is in relation  $\Theta = \delta^*$  to an edge of the component of  $G_{i-1}$  by Lemma 3.2 of [12]. Since this is not possible, a semi-median graph passes Step 3. ■

**Lemma 7** *Let  $G$  be a graph accepted by Algorithm A. Then  $\delta^* = \Theta$ .*

**Proof.** Suppose this is not the case, then there are two edges  $e, f$  that are either in relation  $\Theta$  or in  $\delta^*$ , but not in both. Let  $e, f$  be such a pair with minimum distance  $d$  in  $G$  and  $\ell$  the smallest index such that both  $e$  and  $f$  are in one and the same component of  $G_\ell$ , we denote it by  $C_\ell^{e,f}$ .

**Claim** *Every shortest path from  $e$  to  $f$  is in  $C_\ell^{e,f}$ .*

**Proof of the Claim.** Let  $P$  be a shortest path from  $e$  to  $f$ . By Lemma 1 no two edges of  $P$  are in relation  $\Theta$ . Hence, by the minimality of  $d$ , we infer that they are also not in  $\delta^*$ .

Suppose  $P$  is not in  $G_\ell$ . Then it must contain edges in a class  $E_j^*$  with  $j > \ell$  and so  $G$  is rejected by Step 3 of the algorithm. ■

We continue with the proof of Lemma 7.

**Case 1.**  $e\Theta f$ .

In this case there are shortest paths  $P, Q$  of length  $d$  in  $G_\ell$  between  $e$  and  $f$  such that  $C = e \cup P \cup f \cup Q$  is a cycle. Note that neither  $e$  nor  $f$  can be in relation  $\delta^*$  with any of the edges of  $P$  or  $Q$  (because of the minimality of  $d$ ).

Assume first that no two edges of the cycle have the same color. Then the algorithm rejects  $G$  in Step 3.

Thus, suppose  $g\delta^*h$ , where  $g \in P$  and  $h \in Q$ . Clearly the distance between  $g$  and  $h$  is at most  $d$ . Let  $R$  be any shortest path between  $g$  and  $h$ . Any two edges of it are in different  $\delta^*$ -classes, and thus  $R$  is in  $G_\ell$  by the Claim. Moreover, consider a ladder between  $g$  and  $h$ . By Step 2 of the algorithm the sides of the ladder consist of paths, no two edges of which are in the same  $\delta^*$ -class. By the same arguments as before, they have to be in  $G_\ell$  and cannot be longer than  $R$  (because both paths have the same colors). Thus, we can assume without loss of generality that  $R$  already is a side of the ladder, let  $R'$  be the other. Let  $S$  be the part of  $C$  with the same endpoints as  $R$  and  $S'$  the one with the same endpoints as  $R'$ . Again without loss of generality, we may assume that the walk  $W = R \cup S$  is not longer than the walk  $R' \cup S'$  and that  $W$  contains  $e$ . Then the length of  $W$  is at most  $2d$ . Clearly any two edges of  $W$  have distance at most  $d-1$ . Thus, by Lemma 1 and the minimality, there is an edge on  $R$  that is in relation  $\delta^*$  with  $e$ .

Now we consider two subcases.

**Subcase 1.1**  $|R| = d$ .

Then  $S, R'$  and  $S'$  also have length  $d$ . This implies that they all have the same  $\delta^*$ -colors. Hence  $e$  must be in relation  $\delta^*$  to an edge of  $S'$ . Since this edge can be on neither  $P$  nor  $Q$ , it must be  $f$ .

**Subcase 1.2**  $|R| < d$ .

Then  $R' \cup g \cup S \cup h$  also has length at most  $2d$ . Hence there is an edge  $e'$  in  $R'$  with  $e\delta^*e'$ . Because of Step 3 we find an edge  $e''$  on  $S'$  with  $e'\delta^*e''$ . By the transitivity of  $\delta^*$  we have  $e'' = f$ .

We have thus shown that  $e$  and  $f$  are in relation  $\delta^*$  if they are in relation  $\Theta$ .

**Case 2.**  $e\delta^*f$ .

By the Claim there is a shortest path  $P$  of length  $d$  between  $e$  and  $f$ , no two edges of which have the same  $\delta^*$ -color. Consider a ladder between  $e$  and  $f$ . As before, it must be in  $G_\ell$  and its sides must be shortest paths in  $G$ . But then  $e\Theta f$ . ■

**Theorem 8** *Algorithm A correctly recognizes prism-free almost-median graphs and can be implemented to run in  $O(m \log n)$  time.*

**Proof.** By Lemma 6 prism-free almost-median graphs are accepted by the algorithm. Suppose now that  $G$  is accepted. Then, by Lemma 7,  $\Theta = \delta^*$  which implies that for any edge  $uv$ ,  $U_{uv}^* = U_{uv}$ . Since the  $U_{uv}^*$ 's are checked for being isometric trees,  $G$  is a prism-free almost-median graph.

It remains to determine the algorithm's complexity.

With the algorithm of [6] we first construct all quadrangles of  $G$ . The complexity of the algorithm is  $O(m a(G))$ . It considers every edge  $uv$  and all edges incident with an endpoint of  $uv$  of degree  $\min\{d_G(u), d_G(v)\}$ . Since

$$\sum_{uv \in E(G)} \min\{d_G(u), d_G(v)\} \leq 2a(G)m,$$

and  $a(G) \leq \log n$  for partial cubes, we can stop the algorithm if

$$\sum_{uv \in E(G)} \min\{d_G(u), d_G(v)\} > m \log n.$$

It follows that  $\delta^*$  can be determined in  $O(m \log n)$  time; cf. Proposition 7.6 (ii) of [15].

We observe next that

$$\sum_{uv \in E(G)} |U_{uv}^{\delta^*}| = \sum_{j=1}^k |E_j| = m.$$

Now to Step 2. By standard MAKE-SET, UNION and FIND-SET operations we can check in  $O(m \log n)$  time whether for each edge  $ab$ ,  $U_{ab}^*$  and  $U_{ba}^*$  induce distinct trees that are isomorphic. To check if the  $U_{ab}^*$ 's are isometric, we only need to check if the edges of a  $U_{ab}^*$  have pairwise different colors. Indeed, if a graph  $G$  is accepted, then  $U_{ab}^* = U_{ab}$ , and  $\delta^* = \Theta$ , so pairwise different colors of the edges of  $U_{ab}$  imply isometry by Lemma 1.

In Step 3 we have to perform two FIND-SET operations and possibly one UNION for every one of the  $m$  edges in the graph. It is well known that these operations can be executed within time complexity  $O(m \log n)$ . ■

## 5 Concluding remarks

Algorithm A thus recognizes prism-free almost-median graphs. Let us briefly call them  $AM(T)$  graphs, as they can be characterized as almost-median graphs whose  $U_{ab}$ 's are trees. One of the main properties of trees  $T$  that are isometrically embedded in a partial cube  $G$  is that removal of a  $\Theta$ -class of  $G$  decomposes  $T$  into

two isometrically embedded trees. We say the class of isometrically embedded trees is invariant under removal of  $\Theta$ -classes. If we replace the class of trees in  $AM(T)$  by a class  $S$  of partial cubes that is invariant under removal of  $\Theta$ -classes, then we obtain a new class of almost-median graphs which we denote by  $AM(S)$ . This class can be obtained by isometric expansion procedures whose covering sets  $G_1, G_2$  are almost-median and whose intersections  $G_1 \cap G_2$  are in  $S$ . Let  $S(m, n)$  denote the recognition complexity of  $AM(S)$ .

By the same arguments as in the proof of Theorem 8 their recognition complexity is the maximum of  $S(m, n)$  and  $O(m \log n)$ . For example,  $S$  could be the set  $\mathcal{C}$  of all cycles and all trees or the class  $\mathcal{H}$  of hypercubes. Since their recognition complexities are linear,  $AM(\mathcal{C})$  and  $AM(\mathcal{H})$  have recognition complexities  $O(m \log n)$ . Both classes contain non-median graphs and the latter all acyclic cubical complexes; cf. [2, 14].

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